ANALYSIS OF THE BEAM DECOMPOSITION PROBLEM IN SIGNAL BASED RAY TRACING

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Abstract. The beam decomposition problem is the key problem that needs to be solved in Signal Based Ray Tracing (SBRT). This is an elementary problem which consists in solving the reflection/transmission problem of a ray at a circular interface. In this article, we give in details how it is possible to perform such a decomposition.

Keywords: ray tracing, reflection, transmission, circular interface, caustic, winding number integrals

AMS classification: AMS classification codes

§1. Introduction

Signal Based Ray Tracing is a new geophysical tool designed for the fast computation of traveltimes between a source and an array of receivers. It is based on the geometric optics approximation for the high frequency wave propagation. Its originality is to perform the propagation of rays through a model with a complex geometry of the interfaces by using signal processing concepts. It start by noticing that it is not possible with a band limited signal to determine the exact traveltime. Therefore making an error in the evaluation of the traveltimes is permitted if it stay within the bounds given by the signal characteristics. This possibility is in a first time used in SBRT to simplified the model in which we propagate the rays. Interfaces that are encountered are interpolated with C^1 continuous circle arcs. In a second time, beams, for which all rays that are inside have a traveltime difference less than the fixed error, are propagated instead of individual rays. Consequently, the difficult problem of propagating the rays through a complex model is reduced to a set of elementary problems (reflection/transmission of a beam at a circular interface) which can be solved analytically. This article will concentrate on the solution of these elementary problems. For the sake of simplicity, we will only consider constant velocity media which means that ray trajectories will be straight lines.

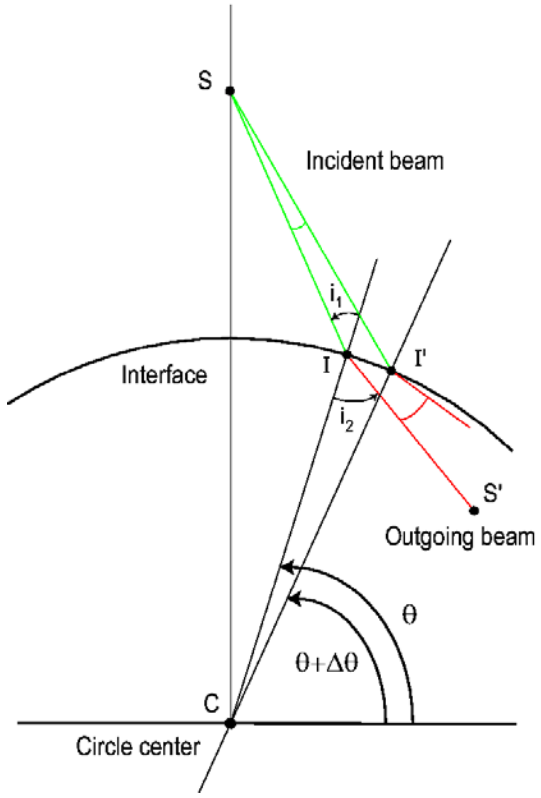

Figure 1: Geometry of the propagation of a ray striking a circular interface

§2. Reflection/Transmission of a beam at a circular interface

This section is devoted to the analysis of the reflection/transmission of a beam at a circular interface. We first establish some preliminary results that are necessary to the next sections. In a first part, we consider the reflection/transmission of an individual ray by means of Fermat's principle of traveltime stationarity extended to higher orders. This corresponds to a local point of view of the propagation process but does not fullfill the needs of SBRT. Therefore the global point of view is investigated in a second part.

2.1. Preliminary results

In this section, we give the basic equations which will be used to solve the problem of beam decomposition. Let first consider three points : **S**, **C** and **I** which respectively represent the location of a point source, the location of the center of the circular interface and the location a point of this interface (see figure 1). Their coordinates are given by : **S** : $\{X_s, Y_s\}$; **C** : $\{X_c, Y_c\}$; **I** : $\{X_c + R\cos\theta, Y_c + R\sin\theta\}$

where R is the radius of the circle. It is real positive number.

We next consider the following vectors : $\overrightarrow{\mathbf{IC}}$, $\overrightarrow{\mathbf{IS}}$ and $\overrightarrow{\mathbf{T}}$ the tangent vector to the interface at point **I**. Hence, we have :

$$\overrightarrow{\mathbf{IS}} : \{X - R\cos\theta,\, Y - R\sin\theta\}; \quad \overrightarrow{\mathbf{IC}} : \{-R\cos\theta,\, -R\sin\theta\}; \quad \overrightarrow{\mathbf{T}} : \{-R\sin\theta,\, R\cos\theta\}$$

with $X = X_s - X_c$ and $Y = Y_s - Y_c$

The scalar product of these vectors can be also expressed in the following form by introducing i_1 the angle of incidence.

$$\overrightarrow{\mathbf{IS}}.\overrightarrow{\mathbf{IC}} = R_1 L_1 \cos i_1$$
$$\overrightarrow{\mathbf{IS}}.\overrightarrow{\mathbf{T}} = -R_1 L_1 \sin i_1$$

where R_1 is the signed radius of the circle and L_1 is the signed curvature of the incoming wavefront. The sign conventions which are used are the classical sign conventions of optics. L_1 and R_1 are respectively the oriented segment $\overline{\mathbf{IS}}$ and $\overline{\mathbf{IC}}$ with the convention that it is positive if it is oriented in the direction of propagation. These conventions are used to ensure that the sign of the cosinus of the angle of incidence is positive.

From the two ways of expressing the scalar product of the abovementioned vectors, we get:

$$X \cos\theta + Y \sin\theta = sgn(R_1)\left[R_1 - L_1 \cos i_1\right] \tag{1}$$
$$X \sin\theta - Y \cos\theta = sgn(R_1) L_1 \sin i_1 \tag{2}$$

In the same way, we get for another point $\mathbf{I}'$ of the interface corresponding to angle $\theta + \Delta\theta$:

$$X \cos(\theta + \Delta\theta) + Y \sin(\theta + \Delta\theta) = sgn(R_1)\left[R_1 - L_1' \cos i_1'\right] \tag{3}$$
$$X \sin(\theta + \Delta\theta) - Y \cos(\theta + \Delta\theta) = sgn(R_1) L_1' \sin i_1' \tag{4}$$

Then, from equations (3) and (4), we can obtain an expression of $\cos i_1'$ and $\sin i_1'$:

$$L_1' \cos i_1' = R_1 (1 - \cos\Delta\theta) + L_1 \cos(i_1 - \Delta\theta) \tag{5}$$
$$L_1' \sin i_1' = R_1 \sin\Delta\theta + L_1 \sin(i_1 - \Delta\theta) \tag{6}$$

2.2. Local point of view of ray propagation (Fermat's principle)

The reflection/transmission of ray at an interface is governed by Fermat's principle. Fermat's principle states that the path followed by a ray to connect a source to another point is the path for which the time taken has a stationnary value with respect to an infinitesimal variation. Taking into account the sign conventions, traveltime between point $\mathbf{S}$ and $\mathbf{S}'$ is :

$$T = \frac{\overline{\mathbf{SI}}}{V_1} + \frac{\overline{\mathbf{IS}'}}{V_2} = -\frac{L_1}{V_1} + \frac{L_2}{V_2} \tag{7}$$

With V_1 and V_2 the wave velocities in medium 1 and 2. In the case of reflection, we have, $V_1 = V_2$ and $R_1 = -R_2$. The expression of L_1 is :

$$L_1 = \pm\sqrt{(X - R\cos\theta)^2 + (Y - R\sin\theta)^2} \tag{8}$$

Since the circular interface can be a parametrized as a function of the angle θ, an infinitesimal variation of the path corresponds to a derivation with respect to θ.

2.2.1. First order stationarity

According to equation (8), the first order derivative of L_1 with respect to θ is :

$$\frac{dL_1}{d\theta} = \frac{1}{2}\frac{2R\sin\theta\,(X - R\cos\theta) - 2R\cos\theta\,(Y - R\sin\theta)}{L_1} = \frac{R\,(X\sin\theta - Y\cos\theta)}{L_1} \tag{9}$$

By using equation 2, we have :

$$\frac{dL_1}{d\theta} = R_1\ \sin i_1 \tag{10}$$

And in the same way, we get :

$$\frac{dL_2}{d\theta} = R_2\ \sin i_2 \tag{11}$$

Consequently stationarity of traveltime through first order leads to the well-known Snell's law :

$$\frac{R_1 \sin i_1}{V_1} = \frac{R_2 \sin i_2}{V_2} \tag{12}$$

This equality defines a new parameter that will denoted by α. Note that it is not common to express Snell's law in this form. The reason for this is the sign convention we use. The change of direction of the reflected ray is taken into account with the change of sign of R instead of considering a negative wave velocity as usually done. For a given direction of the incident ray, Snell's law indicates the direction of the outgoing ray.

2.2.2. Second order stationarity

In order to obtain the second order derivative with respect to θ, we need to express the first order derivative of angle i_1. This quantity is obtained by deriving equation (2) and using equation (1). Some straightforward calculations leads to :

$$\frac{di_1}{d\theta} = \frac{R_1}{L_1}\cos i_1 - 1 \tag{13}$$

The same type of equation is obtained for medium 2 and the second order derivative of traveltime T with respect to θ is :

$$\begin{aligned}\frac{d^2T(\theta)}{d\theta^2} &= \frac{d}{d\theta}\left(\frac{R_1\sin i_1}{V_1} - \frac{R_2\sin i_2}{V_2}\right)\\ &= R_1\frac{\cos i_1}{V_1}\frac{di_1}{d\theta} - R_2\frac{\cos i_2}{V_2}\frac{di_2}{d\theta}\\ &= R_1\frac{\cos i_1}{V_1}\left(1 - \frac{R}{L_1}\cos i_1\right) - R_2\frac{\cos i_2}{V_2}\left(1 - \frac{R}{L_2}\cos i_2\right)\end{aligned}$$

The nullification of the second order leads to the well-known equation known in optics as the conjugaison formula :

$$\frac{R_1}{V_1}\left(\cos i_1 - \frac{R_1}{L_1}\cos^2 i_1\right) = \frac{R_2}{V_2}\left(\cos i_2 - \frac{R_2}{L_2}\cos^2 i_2\right) \tag{14}$$

This equality defines a new parameter denoted by β. Equation 14 gives the location of $\mathbf{S}'$ i.e. the value of L_2. $\mathbf{S}'$ is the location of the center of curvature of the outgoing wavefront. This new point source can be either real or virtual depending on the sign of L_2. This is a local equation. If the angle of incidence of the ray is modified then the new point source position moves to another location. The curve which corresponds the locus of the successive center of curvature is called the caustic (see Figure 2). The outgoing wavefront is circular only if the caustic reduces to a point which is not the case in general.

2.2.3. Third order stationarity

In the same way, it is straightforward to get for the third order derivative of traveltime T :

$$\frac{d^3T(\theta)}{d\theta^3} = R\frac{\sin i_1}{V_1}\left(1 - 3\frac{R}{L_1}\cos i_1\left(1 - \frac{R}{L_1}\cos i_1\right)\right)$$
$$-R\frac{\sin i_2}{V_2}\left(1 - 3\frac{R}{L_2}\cos i_2\left(1 - \frac{R}{L_2}\cos i_2\right)\right)$$

This equation takes the following form when considering the two constants α and β, we introduced in sections 2.2.1 and 2.2.2.

$$\frac{d^3T(\theta)}{d\theta^3} = 3R\alpha\beta\left(\frac{V_1}{L_1} - \frac{V_2}{L_2}\right) \tag{15}$$

It is then easy to see that the nullifcation of the third order derivative occurs only in particular configurations.

2.2.4. Particular configurations

Stationarity beyond all orders occurs for :

$$\alpha = 0$$
$$\beta = 0$$
$$\left(\frac{V_1}{L_1} - \frac{V_2}{L_2}\right) = 0$$

The last equation is a special case in which the two sources $\mathbf{S}$ and $\mathbf{S}'$ are called Weierstrass points.

2.3. Global point of view of ray propagation

Instead of working with infinitesimal variation of the position of a point on the interface, we consider two points $\mathbf{I}$ and $\mathbf{I}'$ on the circle which correspond to angles θ and $\theta + \Delta\theta$ of the parametrisation of the circle. The variation of the distance between the point source $\mathbf{S}$ and the two points $\mathbf{I}$ and $\mathbf{I}'$ is $\Delta L_1 = L_1 - L_1'$. L_1' is the oriented segment $\overline{\mathbf{I'S}}$:

$$L_1' = \pm\sqrt{\left(R\cos\left(\theta + \Delta\theta\right) - X_s\right)^2 + \left(R\sin\left(\theta + \Delta\theta\right) - Y_s\right)^2}$$

The square of this distance is :

$$L_1'^2 = R^2 + X^2 + Y^2 - 2R\left[X\cos(\theta+\Delta\theta) + Y\sin(\theta+\Delta\theta)\right]$$

Combining equations (1) and (2), we have :

$$X\cos(\theta+\Delta\theta) + Y\sin(\theta+\Delta\theta) = R\cos\Delta\theta - L_1\cos(i_1+\Delta\theta)$$

As a consequence, we get :

$$\begin{aligned} L_1'^2 &= R^2 + X^2 + Y^2 - 2R\left[R\cos\Delta\theta - L_1\cos(i_1+\Delta\theta)\right] \\ L_1^2 &= R^2 + X^2 + Y^2 - 2R\left[R - L_1\cos i_1\right] \end{aligned}$$

The calculus of the difference of the two previous equations leads to the solution of the following second order equation :

$$(\Delta L_1)^2 + 2L_1\Delta L_1 - 2R\left[R(1-\cos\Delta\theta) + L_1(\cos(i_1+\Delta\theta) - \cos i_1)\right] = 0$$

The root which is of interest is the one which tends to zero as $\Delta\theta$ tends to zero :

$$\Delta L_1(\Delta(\theta)) = -L_1 + sgn(L_1)L_1\sqrt{1 + \frac{2R}{L_1}\left(\cos(i_1+\Delta\theta) - \cos i_1\right) + \frac{2R^2}{L_1^2}(1-\cos\Delta\theta)} \tag{16}$$

This is the expression of the traveltime difference for the incident medium. For the other term, we need to calculate the intersection of the two extremal rays of the outgoing beam. The directions of these two rays is connected to the angle of incidence i_2 in the following way : $\alpha = \theta + i_2 \qquad \alpha' = \theta + \Delta\theta + i_2'$.

After some straightforward calculations, we obtain :

$$\begin{aligned} t &= 2R\sin\left(\frac{\Delta\theta}{2}\right)\frac{\cos\left(\frac{\Delta\theta}{2} + i_2'\right)}{\sin(i_2 - i_2' - \Delta\theta)} \\ t' &= 2R\sin\left(\frac{\Delta\theta}{2}\right)\frac{\cos\left(\frac{\Delta\theta}{2} + i_2\right)}{\sin(i_2 - i_2' - \Delta\theta)} \end{aligned}$$

where t and t' represent the distance between **I** and **J**, and between **I**$'$ and **J**. **J** being the intersection point of the two rays. The second term of the traveltime difference is :

$$t - t' = \frac{4R\sin\left(\frac{\Delta\theta}{2}\right)}{\sin(i_2 - i_2' - \Delta\theta)}\ \sin\left(\frac{\Delta\theta + i_2 + i_2'}{2}\right)\sin\left(\frac{i_2 - i_2'}{2}\right)$$

And finally, we have :

$$\Delta T(\Delta\theta) = \frac{\Delta L_1}{V_1} + \frac{t' - t}{V_2} \tag{17}$$

If $\Delta T(\Delta\theta) < \epsilon$ then **J** can be the new point source for the outgoing beam. This process can be iterated. Therefore, beam decomposition is equivalent to sample the caustic or approximating the wavefront with circle arcs.

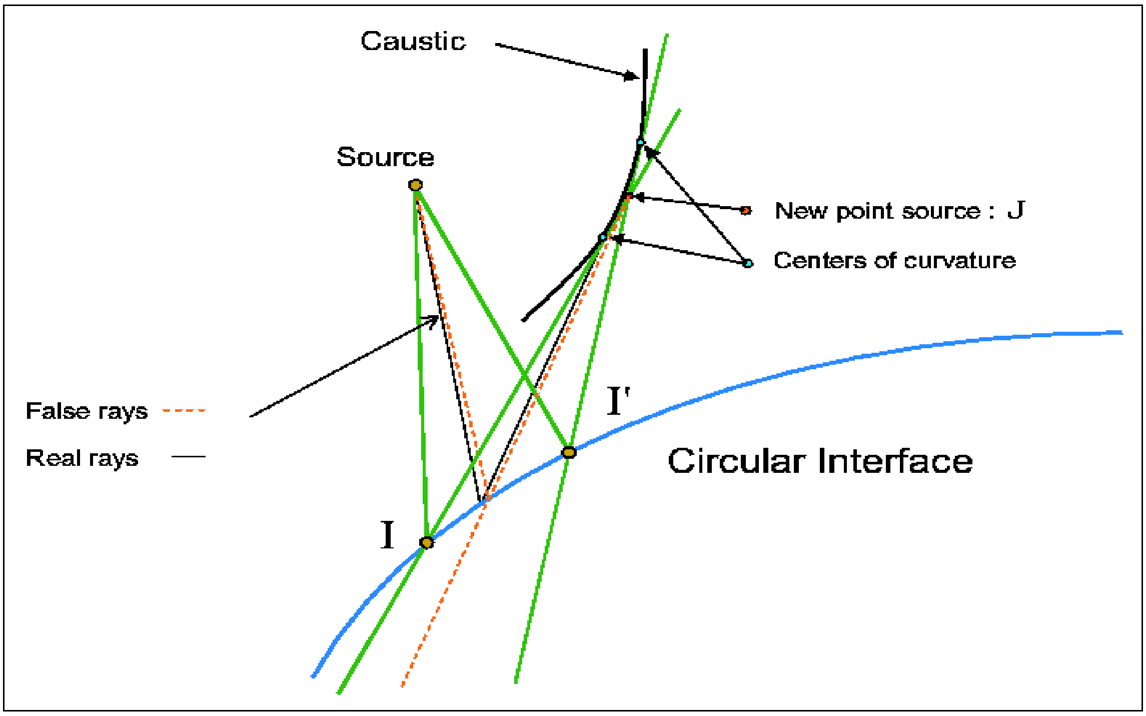

Figure 2: Beam decomposition

§3. Beam decomposition

In the previous section, we express the traveltime difference between two point sources as a function of $\Delta\theta$ the variation of angle along a circle arc. It corresponds to the error we can commit but this error must be controlled and less to a fixed value. As a consequence, we need to solve the following equation :

$$\Delta T(\Delta\theta) = \epsilon \tag{18}$$

As seen previously, this equation may have a solution or not depending on the configuration (see section 2.2.4). For this reason, we chose to use a particular method to solve equation (18) because conventional methods have difficulties in such a situation. It is based on winding number and has been used with success in other context [1, 2]. Its main advantage is that it determines in a first step if there is or not a solution to the equation that is to be solved. In addition to all the advantages of this method, it avoids to take into consideration all particular cases where this equation has no solution.

3.1. Principles of the technique

This section closely follows the article published by Davies [3] and is just a summary of the basic principles. For a more complete analysis of the method the reader may refer to the original paper.

Consider $f(z)$, an analytic function over a region in the complex plane except for n_q poles, φ an analytic function and C a closed contour on which f has no zeros or poles, then the winding number integral is defined by :

$$\frac{1}{2\pi \mathrm{i}} \int_C \frac{f'(z)}{f(z)} \varphi(z) dz = \sum_{k=1}^{p} n_k \varphi(a_k) - \sum_{l=1}^{q} n_l \varphi(b_l)$$

where a_k et b_l are the zeros and poles of f inside the contour C with the multiplicity orders n_k and n_l. This result also called the argument principle is a consequence of the residue theorem.

Let suppose that f has no poles inside C, that all the zeros are simple and take as function φ the following function : $\varphi(z) = z^n$. Under this hypothesis, the winding number integral will lead to the following relation :

$$S_n = \frac{1}{2\pi \mathrm{i}} \int_C z^n \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{N} a_k^n$$

This means that the latter equation provides the knowledge of the number of zeros, the sum of the roots, the sum of the squares of the roots and etc... It is then possible to use Newton's relation to determine the coefficients of the polynomial which have these roots. The problem of finding the zeros of function f is then reduced to the canonical problem of finding the roots of a polynomial which can be solved by standard techniques.

$$P_N = \prod_{j=1}^{N} (z - a_j) = \sum_{j=0}^{N} A_j z^{N-j}$$

with the A_j solutions of the following system :

$$\left\{ \begin{array}{l} S_1 + A_1 = 0 \\ S_2 + A_1 S_1 + 2A_2 = 0 \\ \cdots \\ \cdots \\ S_n + A_1 S_{n-1} + \cdots + A_{n-1} S_1 + n A_n = 0 \end{array} \right.$$

3.2. Application of beam decomposition

The previous method may be applied to equation (18) with circles as contours in the complex plane but one must take care of the square root function of equation (16). This is done by first determining the branch point associated to the square root function and then defining the radius of the circle such that the branch point is outside the contour. In this way, the winding number integral method can be used to perform the beam decomposition of SBRT.

§4. Examples

Two examples of the decomposition of a beam are presented. The aperture of the beam is only limited by the bounds of the model. In figure 3, the case of the reflection of a beam at a circular interface is considered. It can be seen that the enveloppe of the outgoing rays is the caustic.

The case of the transmission is presented in figure 4. In this configuration, the aperture of the beam is limited because of the existence of a critical angle.

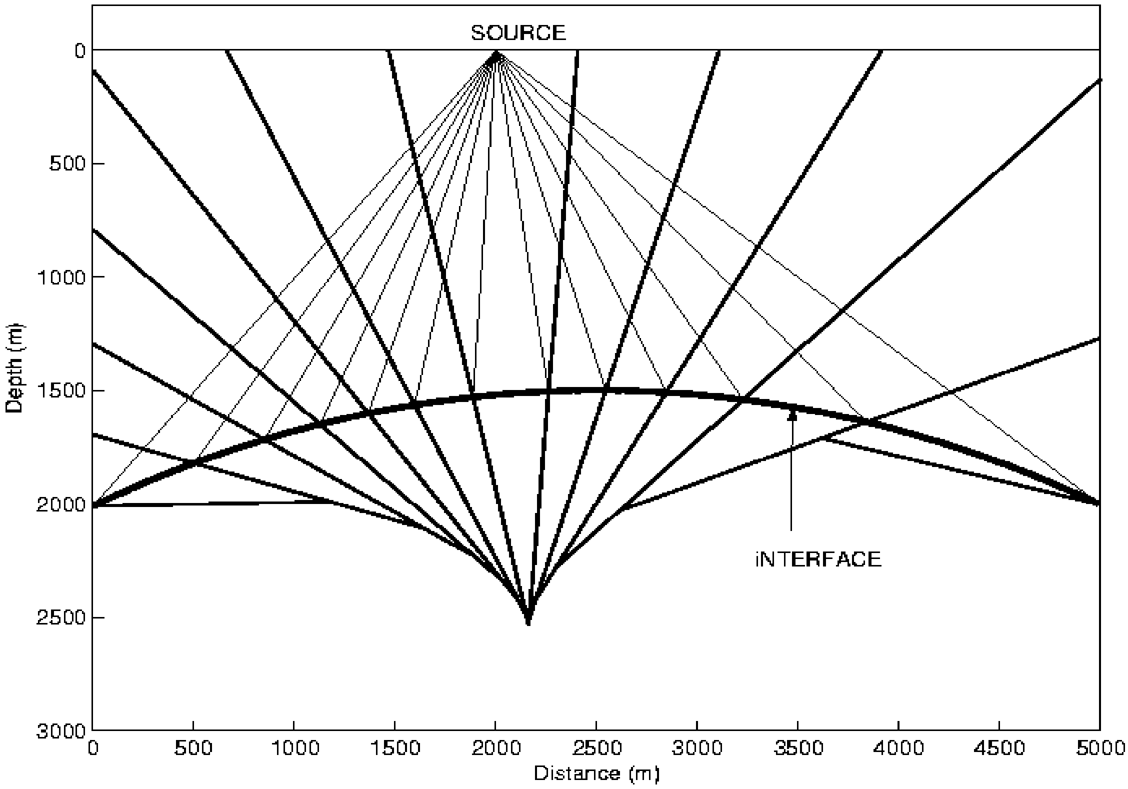

Figure 3: Decomposition of a reflected beam

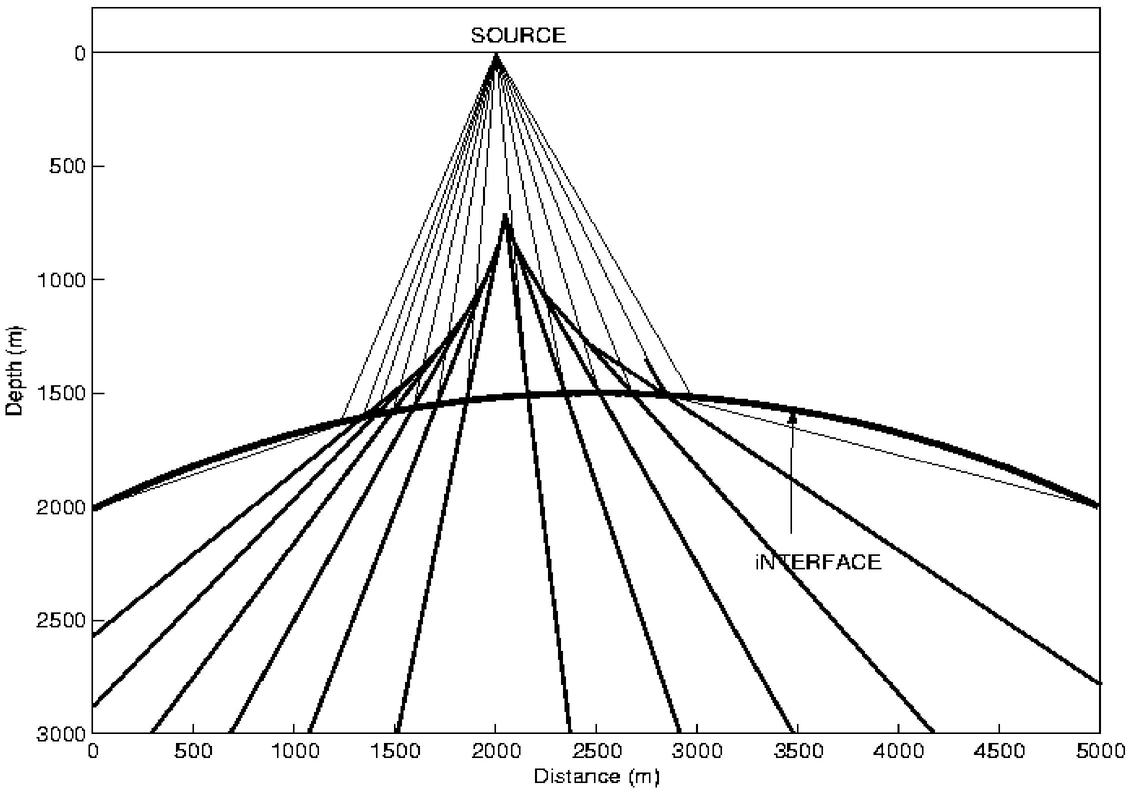

Figure 4: Decomposition of a transmitted beam

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