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THE OSEEN EQUATIONS IN $\mathbb{R}^n$ AND WEIGHTED SOBOLEV SPACES

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Abstract. In this paper, we study the nonhomogeneous Oseen equations in $\mathbb{R}^n$. We prove an existence and uniqueness result in weighted Sobolev spaces. As the main tool, we prove an existence and uniqueness theorem of a scalar model of those equations.

Keywords: Oseen equations, weights, Sobolev spaces

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§1. Introduction

Let Ω' be an open bounded set of $\mathbb{R}^n$ and $\Omega = \mathbb{R}^n \setminus \overline{\Omega'}$. In Ω, the Navier-Stokes equations describe the flow of a viscous and incompressible fluid past the obstacle $\overline{\Omega'}$. The problem consists in looking for a velocity field $\boldsymbol{u}$ and a pressure function π satisfying the following equations

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \boldsymbol{u}.\nabla\boldsymbol{u} + \nabla\pi &= \boldsymbol{f} \text{ in } \Omega \\ \text{div}\,\boldsymbol{u} &= 0 \text{ in } \Omega \\ \boldsymbol{u} &= \boldsymbol{u}_0 \text{ on } \partial\Omega \\ \lim \boldsymbol{u}(\boldsymbol{x}) &= \boldsymbol{u}_\infty, \text{ when } |\boldsymbol{x}| \to \infty. \end{aligned} \tag{1}$$

In System (1), ν is the viscosity of the fluid, $\boldsymbol{u}_0$ is the boundary value, $\boldsymbol{f}$ is the external forces acting on the fluid and $\boldsymbol{u}_\infty$ is a constant vector that, after a change of coordinate, we may assume $\boldsymbol{u}_\infty = h\boldsymbol{e}_1$, with $h > 0$ and $\boldsymbol{e}_1$ is the first vector of the canonical basis of $\mathbb{R}^n$. Linearizing (1) around the constant vector $\boldsymbol{u}_\infty$, we lead to the Oseen system (see [15]):

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + k\frac{\partial\boldsymbol{u}}{\partial x_1} + \nabla\pi &= \boldsymbol{f} \text{ in } \Omega \\ \text{div}\,\boldsymbol{u} &= 0 \text{ in } \Omega \\ \boldsymbol{u} &= \boldsymbol{u}_0 \text{ on } \partial\Omega \\ \lim \boldsymbol{u}(\boldsymbol{x}) &= 0, \text{ when } |\boldsymbol{x}| \to \infty, \end{aligned} \tag{2}$$

where $k > 0$. One of the first work devoted to System (2) is due to Finn [10] where the case $n = 3$ is considered and where the existence of solutions is based on the Galerkin's method. In [11], Galdi investigated (2) by setting the problem in homogeneous Sobolev spaces. The

Lizorkin's Multiplier Theorem is then used to prove existence results. In [8] Farwig studied (2), with $\Omega \subset \mathbb{R}^3$, in anisotropically weighted L^2 spaces, where the weight function $\eta_\beta^\alpha(\boldsymbol{x}) = (1+|\boldsymbol{x}|)^\alpha(1+|\boldsymbol{x}|-x_1)^\beta$ reflects the decay properties of the Oseen fundamental solution $(\boldsymbol{\mathcal{O}}, \boldsymbol{\mathcal{P}})$ which is defined by

$$\boldsymbol{\mathcal{O}}_{ij}(\boldsymbol{x}) = \left(\delta_{ij}\Delta - \frac{\partial}{\partial x_i \partial x_j}\right)\Phi(\boldsymbol{x}),\ \boldsymbol{\mathcal{P}}_i(\boldsymbol{x}) = -\frac{\partial}{\partial x_i}\left(\frac{1}{4\pi|\boldsymbol{x}|}\right), \tag{3}$$

where

$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi k}\int_0^{k(|\boldsymbol{x}|-x_1)/2\nu} \frac{1-e^{-t}}{t} dt. \tag{4}$$

Indeed, we can notice that $\boldsymbol{\mathcal{O}}$ has the following decay properties:

$$\boldsymbol{\mathcal{O}}(\boldsymbol{x}) = O(\eta_{-1}^{-1}(\boldsymbol{x})),\ \nabla\boldsymbol{\mathcal{O}}(\boldsymbol{x}) = O(\eta_{-3/2}^{-3/2}(\boldsymbol{x})), \frac{\partial^2\boldsymbol{\mathcal{O}}(\boldsymbol{x})}{\partial x_i \partial x_j} = O(\eta_{-2}^{-2}(\boldsymbol{x})), \frac{\partial \boldsymbol{\mathcal{O}}(\boldsymbol{x})}{\partial x_1} = O(\eta_{-1}^{-2}(\boldsymbol{x})).$$

To solve (2), Farwig used convolutions with the fundamental solution and the results he obtained on the scalar model

$$\begin{aligned} -\nu\Delta u + k\frac{\partial u}{\partial x_1} &= f \text{ in } \Omega \\ u &= u_0 \text{ on } \partial\Omega. \end{aligned} \tag{5}$$

For the particular case $\Omega = \mathbb{R}^3$, we studied in [6], System (2) in anisotropically weighted L^p spaces, $1 < p < \infty$, with the help of the results obtained on the scalar model in [4] and the study of convolutions with the fundamental solution done in [13]. The aim of this paper is to consider the following nonhomogeneous Oseen problem: given a vector field $\boldsymbol{f}$ and a function g, we look for a solution $(\boldsymbol{u}, \pi)$ satisfying

$$\begin{aligned} -\Delta\boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\pi &= \boldsymbol{f} \text{ in } \mathbb{R}^n \\ \operatorname{div} \boldsymbol{u} &= g \text{ in } \mathbb{R}^n. \end{aligned} \tag{6}$$

We supposed, without loss of generality, $\nu = k = 1$. We set the problem in weighted Sobolev spaces with the radial weight function $\eta_0^\alpha(\boldsymbol{x}) = (1+|\boldsymbol{x}|)^\alpha$ and we proceed as in [6]. More precisely, we first prove an existence result for the scalar model (5) when $\Omega = \mathbb{R}^n$ which was announced in [3]. This will be done in Section 3. Then in Section 4, the result will be used to prove the existence of a solution of System (6). In Section 2, we introduce the weighted Sobolev spaces and some of their basic properties that we use in the sequel.

§2. Notations and functional framework

2.1. Notations

In this paper, $n \geq 3$ is an integer, p is a real number in the interval $]1, +\infty[$. The dual exponent of p denoted p' is defined by the relation $1/p + 1/p' = 1$. In the sequel, for any space B of scalar-value distributions, $\mathbf{B}$ denotes B^n, and $\mathbf{F}$ will denote the vector field $\mathbf{F} = (F_1, F_2, ..., F_n)$. A point in $\mathbb{R}^n$ is denoted by $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ and its distance to the origine by

$$r = |\boldsymbol{x}| = (x_1^2 + x_2^2 + ... + x_n^2)^{1/2}.$$

We denote by $[k]$ the integer part of k. For any $j \in \mathbb{Z}$, $\mathbb{P}_{\mathrm{j}}$ stands for the space of polynomials of degree lower than j and $\mathbb{P}_{\mathrm{j}}^{\Delta}$ the harmonic polynomials of $\mathbb{P}_{\mathrm{j}}$. If j is a negative integer, we set by convention $\mathbb{P}_{\mathrm{j}} = \{0\}$. We recall that $\mathcal{D}'(\mathbb{R}^n)$ is the well-known space of distributions and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. Given a Banach space B, with dual space B' and a closed subspace X of B, we denote by $B'\bot X$ the subspace of B' orthogonal to X, *i.e*,

$$B'\bot X = X^{\bot} = \{f \in B', \forall v \in X, < f, v >= 0\} = (B/X)'.$$

Finally, as usual, $C > 0$ denotes a generic constant the value for which may change from line to line.

2.2. Functional framework

We shall now introduce the weighted Sobolev spaces which will be the functional framework. Let ρ be the weight function $\rho = 1 + r$. For $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ we define

$$k = k(m,n,p,\alpha) = \begin{cases} -1 & \text{if } \dfrac{n}{p} + \alpha \notin \{1, ..., m\} \\ m - \dfrac{n}{p} - \alpha & \text{if } \dfrac{n}{p} + \alpha \in \{1, ..., m\} \end{cases}$$

and we define the following weighted space

$$\begin{aligned} W_{\alpha}^{m,p}(\mathbb{R}^n) = \{u \in \mathcal{D}'(\mathbb{R}^n); \forall \lambda \in \mathbb{N}^n, & \\ & 0 \leq |\lambda| \leq k,\ \rho^{\alpha-m+|\lambda|}(\ln(1+\rho))^{-1}\partial^{\lambda}u \in L^p(\mathbb{R}^n), \\ & k+1 \leq |\lambda| \leq m,\ \rho^{\alpha-m+|\lambda|}\partial^{\lambda}u \in L^p(\mathbb{R}^n)\}, \end{aligned}$$

which is a Banach space equipped with its natural norm given by

$$\|u\|_{W_{\alpha}^{m,p}(\mathbb{R}^n)} =$$

$$\left(\sum_{0\leq|\lambda|\leq k} \|\rho^{\alpha-m+|\lambda|}(\ln(1+\rho))^{-1}\partial^{\lambda}u\|_{L^p(\mathbb{R}^n)}^p + \sum_{k+1\leq|\lambda|\leq m} \|\rho^{\alpha-m+|\lambda|}\partial^{\lambda}u\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p}.$$

We define the seminorm

$$|u|_{W_{\alpha}^{m,p}(\mathbb{R}^n)} = \left(\sum_{|\lambda|=m} \|\rho^{\alpha}\partial^{\lambda}u\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p}.$$

Let us give an example of such space. Let $m = 1$, $\alpha = 0$, then the space $W_0^{1,p}(\mathbb{R}^n)$ is defined as follow:

$$\begin{aligned} W_0^{1,p}(\mathbb{R}^n) &= \{u \in \mathcal{D}'(\mathbb{R}^n),\ \rho^{-1}u \in L^p(\mathbb{R}^n), \nabla u \in \mathbf{L}^p(\mathbb{R}^n)\},\ \text{ if } p \neq n, \\ W_0^{1,n}(\mathbb{R}^n) &= \{u \in \mathcal{D}'(\mathbb{R}^n)),\ (\ln(1+\rho))^{-1}\rho^{-1}u \in L^n(\mathbb{R}^n), \nabla u \in \mathbf{L}^n(\mathbb{R}^n)\}. \end{aligned}$$

The logarithmic weight function introduced in the definition of the space $W_{\alpha}^{m,p}(\mathbb{R}^n)$ only appears for the values of m, p, α such that $n/p + \alpha \in \{1, ..., m\}$ and, for such values, it allows to obtain Hardy-type inequalities (see below). A detailed study of the space $W_{\alpha}^{m,p}(\mathbb{R}^n)$

can be found in [1, 12, 14]. We recall some of its properties. All the local properties of $W_\alpha^{m,p}(\mathbb{R}^n)$ coincide with those the classical Sobolev space $W^{m,p}(\mathbb{R}^n)$. The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $W_\alpha^{m,p}(\mathbb{R}^n)$. Thus the dual space, denoted by, $W_{-\alpha}^{-m,p'}(\mathbb{R}^n)$ is a space of distributions. If $n/p+\alpha\notin\{1,...,m\}$, we have the following algebraic and topological inclusions

$$W_\alpha^{m,p}(\mathbb{R}^n)\subset W_{\alpha-1}^{m-1,p}(\mathbb{R}^n)\subset ...\subset W_{\alpha-m}^{0,p}(\mathbb{R}^n). \tag{7}$$

For any $\lambda\in\mathbb{N}^n$, the mapping

$$u\in W_\alpha^{m,p}(\mathbb{R}^n)\to\partial^\lambda u\in W_\alpha^{m-|\lambda|,p}(\mathbb{R}^n) \tag{8}$$

is continuous. For any $\lambda\in\mathbb{N}^n$ and $\gamma\in\mathbb{R}$, we have

$$|\partial^\lambda(\rho^\gamma)|\leq C\rho^{\gamma-|\lambda|}, \tag{9}$$

which implies that, if $n/p+\alpha\notin\{1,...,m\}$, the mapping

$$u\in W_\alpha^{m,p}(\mathbb{R}^n)\to\rho^\gamma u\in W_{\alpha-\gamma}^{m,p}(\mathbb{R}^n) \tag{10}$$

is an isomorphism.
Let j be an integer, then $\mathbb{P}_{\mathrm{j}}$ is included in $W_\alpha^{m,p}(\mathbb{R}^n)$ with

$$\begin{aligned} j&=\left[m-\frac{n}{p}-\alpha\right] && \text{if } \frac{n}{p}+\alpha\notin\mathbb{Z}^-\\ j&=m-1-\frac{n}{p}-\alpha && \text{otherwise.}\end{aligned} \tag{11}$$

The Hardy-type inequalities are one of the main properties of the space $W_\alpha^{m,p}(\mathbb{R}^n)$ (see [1]). Indeed, let $m\geq 1$ and $\alpha\in\mathbb{R}$. Then there exists a constant $C=C(m,p,\alpha,n)$ such that

$$\forall u\in W_\alpha^{m,p}(\mathbb{R}^n),\quad \inf_{\lambda\in\mathbb{P}_{\mathrm{j}'}}\|u+\lambda\|_{W_\alpha^{m,p}(\mathbb{R}^n)}\leq C|u|_{W_\alpha^{m,p}(\mathbb{R}^n)}, \tag{12}$$

where $j'=min(j,0)$ and j is the highest degree of polynomials contained in $W_\alpha^{m,p}(\mathbb{R}^n)$. Inequality (12) also allows to have some isomorphism results on the gradient and the divergence operators. For instance, denote $\mathbf{H}_p=\{\boldsymbol{v}\in\mathbf{L}^p(\mathbb{R}^n),\mathrm{div}\,\boldsymbol{v}=0\}$, then the following operator

$$\mathrm{div}\;:\mathbf{L}^p(\mathbb{R}^n)/\mathbf{H}_p\to W_0^{-1,p}(\mathbb{R}^n)\perp\mathbb{P}_{[1-\mathrm{n/p'}]} \tag{13}$$

is an isomorphism.
Another consequence of Inequality (12) is the following property (see [1]): Let $m\geq 1$ be an integer and let $u\in\mathcal{D}'(\mathbb{R}^n)$ be such that

$$\forall\lambda\in\mathbb{N}^n\;:\;|\lambda|=m,\;\;\partial^\lambda u\in L^p(\mathbb{R}^n).$$

Then, there exists a polynomial $Q\in\mathbb{P}_{\mathrm{m}-1}$, depending on u such that $u+Q\in W_0^{m,p}(\mathbb{R}^n)$ and

$$\inf_{\mu\in\mathbb{P}_{[\mathrm{m-n/p}]}}\|u+Q+\mu\|_{W_0^{m,p}(\mathbb{R}^n)}\leq C|u|_{W_0^{m,p}(\mathbb{R}^n)}, \tag{14}$$

Let us now introduce the anisotropically weighted space

$$\widetilde{W}_0^{1,p}(\mathbb{R}^n) = \left\{ v \in W_0^{1,p}(\mathbb{R}^n), \frac{\partial v}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^n) \right\}$$

which is a Banach space for the following norm

$$\|v\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^n)} = \|\rho^{-1} v\|_{L^p(\mathbb{R}^n)} + \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^p(\mathbb{R}^n)} + \left\| \frac{\partial v}{\partial x_1} \right\|_{W_0^{-1,p}(\mathbb{R}^n)}.$$

This norm, which is equivalent to the natural one, allows to prove the density of $\mathcal{D}(\mathbb{R}^n)$ in $\widetilde{W}_0^{1,p}(\mathbb{R}^n)$. This property is proved in [5]. Note that if $p = n$, then the weight function ρ in the definition of the norm is replaced by $\rho \ln(1+\rho)$.

§3. The scalar Oseen equation in $\mathbb{R}^n$

We consider the scalar model of the Oseen equations: given f, we look for a function u satisfying

$$-\Delta u + \frac{\partial u}{\partial x_1} = f \text{ in } \mathbb{R}^n. \tag{15}$$

In this section, we shall solve (15) in weighted Sobolev spaces. Let us first recall an existence result of Equation (15) when $f \in L^p(\mathbb{R}^n)$. The result states that, in this case, the scalar Oseen equation (15) has a solution $u \in L^p_{\text{loc}}(\mathbb{R}^n)$ such that $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\mathbb{R}^n)$, $i, j = 1, ..., n$ and $\frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^n)$ also satisfying

$$\left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^n)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.$$

A proof of this result can be found in [9, 11]. It uses Fourier transform and the multiplier theorem of Lizorkin. We also recall that if $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$-\Delta u + \frac{\partial u}{\partial x_1} = 0,$$

then u is a polynomial (see [4, 9]). We shall now look for a solution which belongs to $\widetilde{W}_0^{1,p}(\mathbb{R}^n)$. Note that in this case, from (8), we have $-\Delta u + \frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^n)$. Moreover, for any $\lambda \in \widetilde{W}_0^{1,p'}(\mathbb{R}^n)$, due to the density of $\mathcal{D}(\mathbb{R}^n)$ in $\widetilde{W}_0^{1,p'}(\mathbb{R}^n)$, we can easily prove that

$$< -\Delta u + \frac{\partial u}{\partial x_1}, \lambda >_{W_0^{-1,p}(\mathbb{R}^n) \times W_0^{1,p'}(\mathbb{R}^n)} = < u, -\Delta \lambda - \frac{\partial \lambda}{\partial x_1} >_{W_0^{1,p}(\mathbb{R}^n) \times W_0^{-1,p'}(\mathbb{R}^n)}.$$

Thus, we have the following theorem:

Theorem 1. *Assume that $f \in W_0^{-1,p}(\mathbb{R}^n)$ and satisfies the compatibility condition*

$$\forall \lambda \in \mathbb{P}_{[1-n/p']}, \quad < f, \lambda >_{W_0^{-1,p}(\mathbb{R}^n) \times W_0^{1,p'}(\mathbb{R}^n)} = 0. \tag{16}$$

Then the scalar Oseen equation (15) *has a solution $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^n)$, unique up to a polynomial of $\mathbb{P}_{[1-n/p]}$, also satisfying*

$$\inf_{\lambda \in \mathbb{P}_{[1-n/p]}} \|u + \lambda\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^n)} \leq C \|f\|_{W_0^{-1,p}(\mathbb{R}^n)}. \tag{17}$$

Proof. From the uniqueness result, one can easily see that if $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^n)$ satisfies (15) with $f = 0$, then u is a polynomial of $\mathbb{P}_{[1-\mathrm{n/p}]}$. Let us prove existence. Since $f \in W_0^{-1,p}(\mathbb{R}^n)$ and satisfies (16), from (13), there exists $\mathbf{F} \in \mathbf{L}^p(\mathbb{R}^n)$ such that div $\mathbf{F} = f$ and

$$\|\mathbf{F}\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^n)},$$

where the constant $C > 0$ does not depend on $\mathbf{F}$. Now, using the existence result given above, for any $i = 1, ..., n$, there exists $v_i \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that, for $j, k = 1, ..., n$

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} \in L^p(\mathbb{R}^n) \text{ and } \frac{\partial v_i}{\partial x_1} \in L^p(\mathbb{R}^n),$$

solution of

$$-\Delta v_i + \frac{\partial v_i}{\partial x_1} = F_i.$$

Moreover, for any $i = 1, ..., n$, we have the estimate

$$\left\|\frac{\partial^2 v_i}{\partial x_j \partial x_k}\right\|_{L^p(\mathbb{R}^n)} + \left\|\frac{\partial v_i}{\partial x_1}\right\|_{L^p(\mathbb{R}^n)} \leq C\|F_i\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^n)}.$$

As $\frac{\partial^2 v_i}{\partial x_j \partial x_k} \in L^p(\mathbb{R}^n)$, from (14), there exists a polynomial $q_i \in \mathbb{P}_1$ such that $v_i + q_i \in W_0^{2,p}(\mathbb{R}^n)$ also satisfying

$$\inf_{\mu \in \mathbb{P}_{[2-\mathrm{n/p}]}} \|v_i + q_i + \mu\|_{W_0^{2,p}(\mathbb{R}^n)} \leq C \left\|\frac{\partial^2 v_i}{\partial x_j \partial x_k}\right\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^n)}. \tag{18}$$

Now, setting $u = \mathrm{div}\,(\mathbf{v} + \mathbf{q})$, it follows from (8) that, $u \in W_0^{1,p}(\mathbb{R}^n)$ and satisfies (15). From (8) and (18), we get the estimate

$$\inf_{\lambda \in \mathbb{P}_{[1-\mathrm{n/p}]}} \|u + \lambda\|_{W_0^{1,p}(\mathbb{R}^n)} \leq C\|f\|_{W_0^{-1,p}(\mathbb{R}^n)} \tag{19}$$

Finally, since $\Delta u \in W_0^{-1,p}(\mathbb{R}^n)$, from differential equation, we deduce $\frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^n)$, which in turn gives $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^n)$. From (15) and (19), we obtain (17). ■

Note that, the particular case $p = 2$ and $n = 3$ of the previous theorem is proved in [2]. The case $1 < p < \infty$ and $n = 3$ is proved in [4] in a slightly different way.

§4. The Oseen problem in $\mathbb{R}^n$

In this section, we consider the nonhomogeneous Oseen problem: given a vector field $\boldsymbol{f}$ and a function g, we look for a solution $(\boldsymbol{u}, \pi)$ satisfying

$$\begin{aligned} -\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi = \boldsymbol{f} \text{ in } \mathbb{R}^n, \\ \mathrm{div}\, \boldsymbol{u} = g \text{ in } \mathbb{R}^n. \end{aligned} \tag{20}$$

We recall that if $\boldsymbol{f} \in \mathbf{L}^p(\mathbb{R}^n)$ and $g \in W^{1,p}(\mathbb{R}^n)$, System (20) has a solution $(\boldsymbol{u}, \pi) \in \mathbf{L}^p_{\text{loc}}(\mathbb{R}^n) \times \mathbf{L}^p_{\text{loc}}(\mathbb{R}^n)$ such that $\frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j} \in L^p(\mathbb{R}^n)$, $i, j = 1, ..., n$, $\frac{\partial \boldsymbol{u}}{\partial x_1} \in \mathbf{L}^p(\mathbb{R}^n)$ and $\nabla \pi \in \mathbf{L}^p(\mathbb{R}^n)$ with the estimate

$$\left\| \frac{\partial^2 \boldsymbol{u}}{\partial x_i \partial x_j} \right\|_{\mathbf{L}^p(\mathbb{R}^n)} + \left\| \frac{\partial \boldsymbol{u}}{\partial x_1} \right\|_{\mathbf{L}^p(\mathbb{R}^n)} + \|\nabla \pi\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{L}^p(\mathbb{R}^n)} + \|g\|_{W^{1,p}(\mathbb{R}^n)} \right).$$

As for the scalar case, the proof of this existence result is based on the Fourier transform and the multiplier theorem of Lizorkin (see [9, 11]). We also recall that if $(\boldsymbol{u}, \pi) \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ satsifies (20) with $\boldsymbol{f} = g = 0$, then $\boldsymbol{u}$ and π are polynomials. The proof of this uniqueness result can be found in [6, 9]. We introduce the space

$$\boldsymbol{\mathcal{N}}_k = \left\{ (\boldsymbol{\lambda}, \mu) \in \boldsymbol{\mathbb{P}}_k \times \mathbb{P}^{\Delta}_{k-1}, -\Delta \boldsymbol{\lambda} + \frac{\partial \boldsymbol{\lambda}}{\partial x_1} + \nabla \mu = 0,\ \operatorname{div} \boldsymbol{\lambda} = 0 \right\}.$$

We shall now look for a solution $(\boldsymbol{u}, \pi)$ of (20) belonging to $\widetilde{\mathbf{W}}^{1,p}_0(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$. To reach this goal, we shall use Theorem 1. For the existence of the pressure π, we need a result on the Laplace operator: The following mapping

$$\Delta : L^p(\mathbb{R}^n) \longrightarrow W^{-2,p}_0(\mathbb{R}^n) \perp \mathbb{P}_{[2-n/p']}, \tag{21}$$

is an isomorphism (see [1] for the proof).

Theorem 2. *Let* $\boldsymbol{f} \in \mathbf{W}^{-1,p}_0(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n) \cap W^{-1,p}_0(\mathbb{R}^n)$ *satisfy*

$$\forall \boldsymbol{\lambda} \in \boldsymbol{\mathbb{P}}_{[1-n/p']}, \quad <\boldsymbol{f}, \boldsymbol{\lambda}>_{\mathbf{W}^{-1,p}_0(\mathbb{R}^n) \times \mathbf{W}^{1,p'}_0(\mathbb{R}^n)} = 0 \tag{22}$$

and

$$\forall \lambda \in \mathbb{P}_{[1-n/p']}, \quad < g, \lambda >_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = 0. \tag{23}$$

Then the Oseen system (20) *has a solution* $(\boldsymbol{u}, \pi) \in \widetilde{\mathbf{W}}^{1,p}_0(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$, *unique up to an element of* $\boldsymbol{\mathcal{N}}_{[1-n/p]}$, *also satisfying*

$$\inf_{\boldsymbol{\lambda} \in \boldsymbol{\mathbb{P}}_{[1-n/p]}} \|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\widetilde{\mathbf{W}}^{1,p}_0(\mathbb{R}^n)} + \|\pi\|_{L^p(\mathbb{R}^n)} \leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}^{-1,p}_0(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} + \|g\|_{W^{-1,p}_0(\mathbb{R}^n)} \right). \tag{24}$$

Proof. Let us first notice that if $(\boldsymbol{u}, \pi) \in \widetilde{\mathbf{W}}^{1,p}_0(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ and satisfies (20) with $\boldsymbol{f} = g = 0$, then $(\boldsymbol{u}, \pi) \in \boldsymbol{\mathcal{N}}_{[1-n/p]}$ which proves the uniqueness. Let us now prove existence. Let $\boldsymbol{f} \in \mathbf{W}^{-1,p}_0(\mathbb{R}^n)$ satisfies (22) and $g \in L^p(\mathbb{R}^n) \cap W^{-1,p}_0(\mathbb{R}^n)$ satisfies (23). Then, from (8), we get $\operatorname{div} \boldsymbol{f} + \Delta g - \frac{\partial g}{\partial x_1} \in W^{-2,p}_0(\mathbb{R}^n)$. Note that the polynomials of $\mathbb{P}_{[2-n/p']}$ are at most polynomials of degree less than one. Then, from (22) and (23), for any $\mu \in \mathbb{P}_{[2-n/p']}$, we have

$$\begin{aligned} &< \operatorname{div} \boldsymbol{f} + \Delta g - \frac{\partial g}{\partial x_1}, \mu >_{W^{-2,p}_0(\mathbb{R}^n) \times W^{2,p'}_0(\mathbb{R}^n)} \\ &= - < \boldsymbol{f}, \nabla \mu >_{\mathbf{W}^{-1,p}_0(\mathbb{R}^n) \times \mathbf{W}^{1,p'}_0(\mathbb{R}^n)} + < g, \frac{\partial \mu}{\partial x_1} >_{W^{-1,p}_0(\mathbb{R}^n) \times W^{1,p'}_0(\mathbb{R}^n)} = 0. \end{aligned}$$

It follows that $\operatorname{div} \boldsymbol{f} + \Delta g - \frac{\partial g}{\partial x_1} \in W_0^{-2,p}(\mathbb{R}^n) \perp \mathbb{P}_{[2-\mathrm{n}/\mathrm{p}']}$. Thanks to the isomorphism of the Laplace operator (21), there exists a unique function $\pi \in L^p(\mathbb{R}^n)$, such that

$$\Delta \pi = \operatorname{div} \boldsymbol{f} + \Delta g - \frac{\partial g}{\partial x_1} \tag{25}$$

also satisfying

$$\begin{aligned} \|\pi\|_{L^p(\mathbb{R}^n)} &\leq C \left(\|\operatorname{div} \boldsymbol{f}\|_{W_0^{-2,p}(\mathbb{R}^n)} + \|\Delta g\|_{W_0^{-2,p}(\mathbb{R}^n)} + \left\| \frac{\partial g}{\partial x_1} \right\|_{W_0^{-2,p}(\mathbb{R}^n)} \right) \\ &\leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} + \|g\|_{W_0^{-1,p}(\mathbb{R}^n)} \right). \end{aligned} \tag{26}$$

Next, we deduce that $\boldsymbol{f} - \nabla \pi \in \mathbf{W}_0^{-1,p}(\mathbb{R}^3)$. Moreover, recalling that the polynomials of $\mathbf{P}_{[1-n/p]}$ are at most constants and from (22), we see that

$$\forall \boldsymbol{\lambda} \in \mathbf{P}_{[1-n/p']}, \quad < \boldsymbol{f} - \nabla \pi, \boldsymbol{\lambda} >_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n) \times \mathbf{W}_0^{1,p'}(\mathbb{R}^n)} = 0.$$

Using Theorem 1, there exists a vector field $\boldsymbol{u} \in \widetilde{\mathbf{W}}_0^{1,p}(\mathbb{R}^n)$ satisfying

$$-\Delta \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{f} - \nabla \pi.$$

with the estimate

$$\begin{aligned} \inf_{\boldsymbol{\lambda} \in \mathbf{P}_{[1-n/p]}} \|\boldsymbol{u} + \boldsymbol{\lambda}\|_{\widetilde{\mathbf{W}}_0^{1,p}(\mathbb{R}^n)} &\leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n)} + \|\nabla \pi\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n)} \right) \\ &\leq C \left(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n)} + \|g\|_{L^p(\mathbb{R}^n)} + \|g\|_{W_0^{-1,p}(\mathbb{R}^n)} \right). \end{aligned} \tag{27}$$

From (26) and (27), we easily obtain (24). Let us now prove $\operatorname{div} \boldsymbol{f} = g$. From (25), we can observe that

$$-\Delta(\operatorname{div} \boldsymbol{u} - g) + \frac{\partial}{\partial x_1}(\operatorname{div} \boldsymbol{u} - g) = 0.$$

Thanks to the uniqueness result of the scalar Oseen equation (15), we deduce that $\operatorname{div} \boldsymbol{u} - g$ is a polynomial. But, we have $\operatorname{div} \boldsymbol{u} - g \in L^p(\mathbb{R}^n)$ which implies that $\operatorname{div} \boldsymbol{u} - g$ is a polynomial of $L^p(\mathbb{R}^n)$. Thus $\operatorname{div} \boldsymbol{u} - g = 0$ which ends the proof. ■

This result is proved in [6] for the particular case $n = 3$.

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