ON d-PSEUDO-ORTHOGONALITY OF THE SHEFFER SYSTEMS ASSOCIATED TO A CONVOLUTION SEMIGROUP

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Abstract. We investigate which Sheffer polynomials can be associated to a convolution semigroup of probability measures, usually induced by a stochastic process with stationary and independent increments. From a recent notion of d-pseudo-orthogonality ($d \in \{2, 3, \cdots\}$), we characterize the associated d-pseudo-orthogonal polynomials by the class of generating probability measures, which belongs to the natural exponential family with polynomial variance functions of exact degree $2d-1$. This extends some results of (classical) orthogonality; in particular, some new sets of martingales are then pointed out. For each integer $d \geq 2$ we completely illustrate polynomials with ($2d-1$)-term recurrence relation for the families of positive stable processes.

Keywords: Lévy process, martingale, natural exponential family, polynomial variance function, positive stable process.

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§1. Introduction

Let us specify first what we call a semigroup-Sheffer system. If both functions $a(m)$ and $b(m)$ can be expanded in a formal power series such that $a(0) = 0$, $a'(0) \neq 0$ and $b(0) \neq 0$, then the polynomial sequence $\{Q_n(x); n \in \mathbb{N}\}$ defined by the generating function

$$\sum_{n=0}^{\infty} Q_n(x)\frac{m^n}{n!} = b(m)\exp\{xa(m)\} \tag{1}$$

is a *Sheffer system* [17]. The polynomial $Q_n(x)$ so defined is of exact degree n with $Q_0 \neq 0$.

Following Schoutens and Teugels [16], we now introduce an additional time parameter $\lambda \in \Lambda \subseteq [0, \infty)$ into the polynomials defined in (1) by replacing the function $b(m)$ by $\{b(a(m))\}^{\lambda}$.

Definition 1. Let Λ be a closed additive semigroup of $[0, \infty)$. A polynomial set $\{Q_n(x; \lambda); n \in \mathbb{N}, \lambda \in \Lambda\}$ is called a **semigroup-Sheffer system** if it is defined by a generating function of the form

$$\sum_{n=0}^{\infty} Q_n(x;\lambda)\frac{m^n}{n!} = \{b(a(m))\}^{\lambda}\exp\{xa(m)\}, \tag{2}$$

where: (i) a and b are analytic in the neighborhood of $m = 0$; (ii) $a(0) = 0$, $a'(0) \neq 0$ and $b(0) = 1$; (iii) $1/b(\theta)$ is a Laplace transform.

Note that the quantity λ can be considered to be a (discrete) positive parameter, as such the function $Q_n(x;\lambda)$ will also be polynomial in λ.

If condition (iii) of Definition 1 is satisfied, then there is a convolution semigroup of probability measures $\{\mu_\lambda; \lambda \in \Lambda\}$ defined by

$$L_\mu(\theta) = \int_{\mathbb{R}} \exp\{\theta x\}\mu(dx) = \frac{1}{b(\theta)} \tag{3}$$

through the Laplace transform of $\mu = \mu_1$. That leads necessarily to these inclusions

$$\mathbb{N} \subseteq \Lambda \subseteq [0,\infty). \tag{4}$$

It is convenient to put $\mu_0 = \delta_0$ the Dirac mass at 0, so the set $\Lambda^* = \Lambda\backslash\{0\}$ depends on the generator $\mu = \mu_1$ of $\{\mu_\lambda; \lambda \in \Lambda\}$. Hence, the aim of this paper is to find the correspondence between such a semigroup-Sheffer system and the families of associated distributions μ.

Note however that the calculation of the index set of μ, defined as

$$\Lambda^* = \Lambda^*(\mu) = \{\lambda > 0; \exists \mu_\lambda : L_{\mu_\lambda} = (L_\mu)^\lambda\}, \tag{5}$$

can be quite complicated, even for something as simple as distribution of the sum of two Bernoulli and negative binomial independent random variables (Letac *et al.* [10]). Here are two classical examples of measures such that the inclusions (4) are strict. First, if $\exp\{1+z-z^2/4+z^3+z^4\} = \sum_{n=0}^{\infty} \mu_n z^n$ then the index set of the positive discrete measure $\mu(dx) = \sum_{n=0}^{\infty} \mu_n \delta_n(dx)$ is exactly $\Lambda^*(\mu) = [1/2,\infty)$. Second, the positive measure $\nu(dx) = \sum_{n=0}^{8} \nu_n \delta_n(dx)$ with ($\nu_0 = \nu_8 = 1$, $\nu_1 = \nu_7 = 2$, $\nu_2 = \nu_6 = 1/2$, $\nu_3 = \nu_5 = 3/2$, $\nu_4 = 65/16$) provides $\Lambda^*(\nu) = \{1, 3/2, 2, 5/2, 3, \cdots\}$.

Before giving the outline of the paper, we recall here a basic application (see Schoutens [15]) in both discrete and continuous cases of Λ via (4). If $\Lambda = \mathbb{N} = \{0,1,2,\cdots\}$ is discrete, we can then associate i.i.d. random variables $X_1, X_2, \cdots$ defined by its Laplace transform $L_{X_i}(\theta) = 1/b(\theta)$ as in (3); and, finally, we obtain the following *martingale equality*

$$\mathbb{E}[Q_n(S_l;l)|S_k] = Q_n(S_k;k), \quad 0 \le k \le l, \;\; n \in \mathbb{N}, \tag{6}$$

where $S_k = X_1 + \cdots + X_k$, $k \in \mathbb{N}$. For continuous time $\Lambda = \mathbb{R}_+ = [0,\infty)$, the Laplace transform in (3) is therefore *infinitely divisible* (Sato [14]); and, it is also shown that

$$\mathbb{E}[Q_n(X_t;t)|X_s] = Q_n(X_s;s), \quad 0 \le s \le t, \;\; n \in \mathbb{N}, \tag{7}$$

where $\{X_t; t \in \mathbb{R}_+\}$ is the Lévy process (i.e., stationary process with independent increments) with the associated distributions $\{\mu_t; t \in \mathbb{R}_+\}$. See Küchler and Sørensen [8] for exponential families of stochastic processes, applicable here.

In Section 2 we provide the construction of semigroup-Sheffer system with the theory of exponential families. In Section 3 we first associate the convolution semigroup of probability measures to any semigroup-Sheffer system. Then, we give some characterizations of semigroup-Sheffer systems via orthogonality and its extension. Section 4 is devoted to some concluding remarks and to interesting examples of positive stable processes.

§2. Construction of semigroup-Sheffer systems

Let μ be a σ-finite positive measure on $\mathbb{R}$ (not necessarily a probability), define the *cumulant function* K_μ by

$$K_\mu(\theta) = \ln \int_{\mathbb{R}} \exp\{\theta x\}\mu(dx) = \ln L_\mu(\theta)$$

on its (canonical parameter) domain $\Theta = \{\theta \in \mathbb{R} : K_\mu(\theta) < \infty\}$. Assuming that both μ and Θ are not degenerate (i.e., μ is not concentrated at one point and the interior of Θ is not empty), and hence K_μ is known to be strictly convex on intΘ. We denote by ψ_μ the inverse of the first derivative function K'_μ. The *natural exponential family* (NEF) $F = F(\mu)$ generated by μ is the set

$$F = \{P(m, F); m \in M_F = K'_\mu(\text{int}\Theta)\},$$

where each $P(m, F)$ is a probability distribution with mean m such that its density with respect to μ can be written as

$$f_\mu(x; m) = \exp\{x\psi_\mu(m) - K_\mu(\psi_\mu(m))\}. \tag{8}$$

For more details, the reader can be referred to Jørgensen [4] or to the contribution of Muriel Casalis in Kotz *et al.* [6, Chapter 54]. Let us recall here the following notion.

Definition 2. Two NEFs $F(\mu)$ and $F(\nu)$ are said to be of the same **type** if there exist an affinity φ and $\lambda \in \Lambda^*(\mu)$ as in (5) such that $\nu = \varphi(\mu^{*\lambda}) = \varphi(\mu_\lambda)$, where $*$ denotes the convolution product.

Thus, to a family of probability measures we associate a family of polynomials from the Taylor expansion of the function $f_{\mu_\lambda}(x; m)$ of the form (8). The following result provides the construction of semigroup-Sheffer systems from a convolution semigroup of probability measures $\{\mu_\lambda; \lambda \in \Lambda\}$.

Theorem 1. *Let $n \in \mathbb{N}$ and let $\{\mu_\lambda; \lambda \in \Lambda\}$ be a convolution semigroup of probability measures generating a type of NEF $F = F(\mu)$. Define*

$$f^{(n)}_{\mu_\lambda}(x; m_\lambda) = \partial^n f_{\mu_\lambda}(x; m)/\partial m^n|_{m=m_\lambda}.$$

Then $\{Q_n(x; \lambda) = \lambda^n f^{(n)}_{\mu_\lambda}(x; m_\lambda); n \in \mathbb{N}, \lambda \in \Lambda\}$ form a semigroup-Sheffer system.

Proof. By induction on n, $f^{(n)}_{\mu_\lambda}(x; m_\lambda)$ is a polynomial in x of exact degree n; and, hence, $Q_n(x; \lambda)$ is also one. Since $K_{\mu_\lambda} = \lambda K_\mu$ and ψ_{μ_λ} are analytic, it follows that, for all m in the neighborhood of m_λ, we have

$$\begin{aligned}\sum_{n=0}^{\infty} Q_n(x; \lambda)\frac{m^n}{n!} &= f_{\mu_\lambda}(x; \lambda m + m_\lambda) \\ &= \exp\{x\psi_{\mu_\lambda}(\lambda m + m_\lambda) - \lambda K_\mu(\psi_{\mu_\lambda}(\lambda m + m_\lambda))\}.\end{aligned}$$

>From $K_{\mu_\lambda} = \lambda K_\mu$, it is easily seen that $m_\lambda = \lambda m_1$ and $\psi_{\mu_\lambda}(m) = \psi_\mu(m/\lambda)$. Hence (2) occurs with $a(m) = \psi_\mu(m + m_1)$ and $b(\theta) = \exp\{-K_\mu(\theta)\}$. □

Let us conclude this section by recalling briefly the notion of variance function for a NEF. The variance V_F of $P(m,F)$ is considered as a function of $m = K'_\mu(\theta)$ and is satisfied

$$V_F(m) = K''_\mu(\theta) = 1/\psi'_\mu(m) = \int_{\mathbb{R}} (x-m)^2 f_\mu(x;m)\mu(dx).$$

Note also that L_μ, K_μ, ψ_μ and V_F are analytic functions in their domains and, for fixed $\lambda \in \Lambda^*$,

$$V_{F(\mu_\lambda)}(m) = \lambda V_{F(\mu)}(m/\lambda) \tag{9}$$

for $m \in M_{F(\mu_\lambda)} = \lambda M_{F(\mu)}$.

Together with the mean domain M_F, the variance function V_F characterizes the family F within the class of all NEFs [11]; it does not depend on a particular generating measure, and it presents an expression simpler than the density of $P(m,F)$. Among the different forms of V_F in literature, the most basic is of course the *polynomial variance function* with degree $\delta \in \mathbb{N}$ that, to make short, we denote by NEFs with δ-PVF:

$$V_F(m) = \sum_{k=0}^{\delta} \alpha_k m^k, \qquad \alpha_k \in \mathbb{R}.$$

Morris [11] characterized all NEFs on $\mathbb{R}$ with quadratic variance functions (i.e., $\delta = 0,1,2$) in six types (normal, Poisson, gamma, binomial, negative binomial and cosine hyperbolic), which are associated to six orthogonal Sheffer systems (Hermite, Charlier, Laguerre, Krawtchouk, Meixner-type I and Pollaczek, respectively). All these types are infinitely divisible, except the binomial one, whose index set is $\Lambda^* = \mathbb{N}^*$. The characterization of these orthogonal polynomials have been obtained by different manners (e.g., Feinsilver [3]). But for an account of particular theory of semigroup-Sheffer systems as Lévy-Sheffer (7) and i.i.d.-Sheffer (6) systems, we refer to Schoutens and Teugels [16] for univariate cases and to Pommeret [12] for multivariate cases. When the degree of PVF is greater than or equal to 3, we cannot characterize by the classical orthogonal Sheffer systems; see Kokonendji [5] for univariate case with the following extension of orthogonality and Pommeret [13] for multivariate cases with only some terms recurrence relations.

§3. Main results with d-pseudo-orthogonality

In this section we link all semigroup-Sheffer systems to a convolution semigroup of probability measures following the classical orthogonality. Then we investigate the family of distributions with respect to our extension of orthogonality [5], which is not the common notion of d-orthogonality (e.g., Douak [1]).

Definition 3. Let $d \in \{2,3,\cdots\}$. A sequence of real polynomials $(P_n)_{n\in\mathbb{N}}$ is said d-pseudo-orthogonal with respect to a probability measure μ (denoted $\mu - d$-pseudo-orthogonal) if

$$\int_{\mathbb{R}} P_n(x)P_q(x)\mu(dx) = 0 \quad \text{for} \quad n \geq dq, \;\; q \in \mathbb{N}, \quad \text{and}$$

$$\exists q \geq 0, n \in [(q+1)/d; dq-1] \cap \mathbb{N} \;\text{ such that }\; \int_{\mathbb{R}} P_n(x)P_q(x)\mu(dx) \neq 0.$$

As discussed in [5], Definition 3 extended to $d = 1$ corresponds to the quasi-othogonality of order 1. Recall that the classical μ-orthogonality is defined by $\int P_n(x)P_q(x)\mu(dx) = 0$ when $n \neq q$, and it is the quasi-orthogonality of order 0. The following theorem extends the characterization of Lévy-Sheffer and i.i.d.-Sheffer systems of [16] to semigroup-Sheffer systems by the classical orthogonality.

Theorem 2. *Let $\{Q_n(x;\lambda); n \in \mathbb{N}, \lambda \in \Lambda\}$ be a semigroup-Sheffer system. Then there exists a unique convolution semigroup of probability measures $\{\mu_\lambda; \lambda \in \Lambda\}$ such that $\{Q_n(x;\lambda); n \in \mathbb{N}\}$ is μ_λ-orthogonal, for all $\lambda \in \Lambda$.*

Proof. By taking generating functions in

$$\int_{\mathbb{R}} Q_n(x;\lambda)Q_p(x;\lambda)\mu_\lambda(dx) = \delta_{np}c_n$$

($\delta_{np} = 1$ when $n = p$ and 0 for $n \neq p$) and setting $n = 0$ we obtain

$$\int_{\mathbb{R}} \{b(a(m))\}^\lambda \exp\{xa(m)\}\mu_\lambda(dx) = c_0 = 1.$$

Putting $a(m) = \theta$ we have the Laplace transform of μ_λ

$$\int_{\mathbb{R}} \exp\{\theta x\}\mu_\lambda(dx) = \{b(\theta)\}^{-\lambda},$$

which characterizes of unique manner each μ_λ of $\{\mu_\lambda; \lambda \in \Lambda\}$. $\square$

Obviously we have the same six systems obtained in [16], which are connected to six types of NEFs with quadratic variance functions [11].

Before showing the two main theorems related to $\mu{-}d$-pseudo-orthogonality of the semigroup-Sheffer systems, we need here the basic and useful result of [5, Theorem 3] extending the Feinsilver [3] characterization . It also provides the $(2d-1)$-term recurrence relation of the associated polynomials.

Lemma 3 (Kokonendji [5]). *Let F be a NEF on $\mathbb{R}$ and μ an element of F with mean m_0. Consider the polynomials $(P_n)_{n\in\mathbb{N}}$ associated to F and defined by $P_n(x) = f_\mu^{(n)}(x;m_0)$. Then, for all $d \in \{2,3,\cdots\}$, the three following statements are equivalent: (i) $(P_n)_{n\in\mathbb{N}}$ are $\mu - d$-pseudo-orthogonal polynomials; (ii) F is a NEF with $(2d-1)$-PVF; (iii) there exist real numbers $(a_k)_{k=0,1,\cdots,2d-1}$ such that, for all $n \geq 2$,*

$$a_0P_{n+1}(x) = [x - (na_1 + m_0)]P_n(x) - n[(n-1)a_2 + 1]P_{n-1}(x) - \sum_{k=2}^{2(d-1)} a_{k+1}A_n^{k+1}P_{n-k}(x)$$

with $A_n^k = n(n-1)\cdots(n-k+1)$ and $A_n^0 = 1$. In this case, we have: $V_F(m) = \sum_{k=0}^{2d-1} a_k(m - m_0)^k$.

Under the assumption of $\mu - d$-pseudo-orthogonality, the following theorem shows an intrinsic construction of the semigroup-Sheffer systems.

Theorem 4. *For all $\lambda \in \Lambda$, let the polynomial sequence $\{P_n(x;\lambda); n \in \mathbb{N}\}$ be a $\mu_\lambda - d$-pseudo-orthogonal. Then the two following statements are equivalent: (i) $\{P_n(x;\lambda); n \in \mathbb{N}\}$ form a semigroup-Sheffer system; (ii) there exists $\gamma \in \mathbb{R}^*$ such that $P_n(x;\lambda) = (\gamma\lambda)^n f_{\mu_\lambda}^{(n)}(x; m_\lambda)$, for all $(n,\lambda) \in \mathbb{N} \times \Lambda$.*

Proof. $(i) \Leftarrow (ii)$ Easy from Theorem 1.

$(i) \Rightarrow (ii)$ For all $\lambda \in \Lambda$, the generating function of $(P_n(x;\lambda))_{n\geq 0}$ is "exponential"; that is

$$\sum_{n\geq 0} \frac{m^n}{n!} P_n(x;\lambda) = \exp\{xA_\lambda(m) + B_\lambda(m)\},$$

where both functions $A_\lambda(m)$ and $B_\lambda(m)$ are suitable as in (2). Hence, the desired result is obtained in the same spirit as Theorem 4 ($i \Rightarrow ii$) of Kokonendji [5]. $\square$

Now we are interested in finding out for which (class of) real measures μ_λ the d-pseudo-orthogonality ($d \in \{2, 3, \cdots\}$) of the semigroup-Sheffer polynomials occurs.

Theorem 5. *Let $d \in \{2, 3, \cdots\}$ and let F be a NEF on $\mathbb{R}$ and μ an element of F with mean m. Let $\{P_n(x;\lambda); n \in \mathbb{N}, \lambda \in \Lambda\}$ be a semigroup-Sheffer system associated to μ. Then, the d-pseudo-orthogonality of the semigroup-Sheffer system occurs if and only if $F(\mu)$ is a NEF with $(2d-1)$-PVF and there exists an affinity φ such that $a(m) = \psi_\mu(\varphi(m))$ in (2).*

Proof. Assume the polynomials $P_n(x;\lambda)$ are $\mu_\lambda - d$-pseudo-orthogonal. From Theorem 4 there exist $\gamma \in \mathbb{R}^*$ such that $P_n(x;\lambda) = (\gamma\lambda)^n f_{\mu_\lambda}^{(n)}(x; m_\lambda)$ and $a(m) = \psi_\mu(\varphi(m))$ and $\varphi(m) = \gamma m + m_1$. To show that F is $(2d-1)$-PVF we may fixe $\lambda = 1$, because the general case is obtained by the notion of type given in Definition 2; see also (9) for the power of convolution. Thus the remainder is easily deduced from Lemma 3.

Conversely, from the effect of the power of convolution (9) we may also use Lemma 3 to show that polynomials $P_n(x) = P_n(x;1)$ are $\mu - d$-pseudo-orthogonal for fixed $\lambda = 1$. $\square$

§4. Concluding remarks and examples

The characterization of the d-pseudo-orthogonal ($d \in \{2, 3, \cdots\}$) semigroup-Sheffer systems is done globaly in term of class (or set of types) of distributions. Similar results yield for multivariate families by introducing this notion of d-pseudo-orthogonality in the work of Pommeret [13]. Note that for $d = 1$ (quasi-orthogonality of order 1) we can only associate the Poisson distribution (or process) and then the Charlier polynomial. As consequence of martingale equality we easily obtain the following convolution relation:

$$Q_n(x;\lambda) = \int_{\mathbb{R}} Q_n(x+y;t)\mu_{t-\lambda}(dy), \quad 0 \leq \lambda \leq t \in \Lambda,$$

for all $n \in \mathbb{N}$. When $\Lambda = [0;\infty)$ it can be used in stochastic integration theory or Itô integrals (see Schoutens [15, Chapter 5] for orthogonality).

Of course there exist many NEFs with polynomial variance functions of degree $2d-1$, for $d \in \{2, 3, \cdots\}$. For example to build such variance functions, we consider the Bar-Lev criterion described in Letac and Mora [9, Corollary 3.3] as following:

$$V_F(m) = m\Delta(m^2)$$

on $M_F = (0; r)$, where Δ is a polynomial with non-negative coefficients and $r \in (0; \infty]$. In this case the NEF F is infinitely divisible and, hence, a generator of a Lévy process. However, it seems hard indeed even impossible, to obtain "explicitly" their densities or cumulant functions when the degree of V_F is greater than or equal to 3 [9]. Note that the cumulant function K_μ is necessary to the computation of our sequence polynomials, and the density or generating measure μ is important to precise the d-pseudo-orthogonality with respect to this μ. The calculation of the sequence polynomials $P_n(x; m_\lambda)$ can be done by means of the Faà di Bruno formula [7] as follows:

$$P_n(x;\lambda) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1!\cdots k_n!} \prod_{j=1}^{n} \left(\frac{\partial^j}{\partial m^j} [x\psi_{\mu_\lambda}(m) - K_{\mu_\lambda}(\psi_{\mu_\lambda}(m))]|_{m=m_\lambda} \right)^{k_j}. \tag{10}$$

We conclude this paper by explaining the interesting cases from positive stable processes, which are Lévy processes generated by probability measures

$$\mu_{\alpha,t}(dx) = \frac{dx}{\pi x} \sum_{k=1}^{\infty} \frac{(\alpha-1)^k \Gamma(1+\alpha k)}{k!\alpha^k t^{-k}} \left(\frac{-1}{(\alpha-1)x} \right)^{\alpha k} \sin(-k\pi\alpha), \qquad x > 0,$$

where $0 < \alpha < 1$ and $t > 0$; see Feller [2] for basic properties. Note that for $\alpha \in [1; 2]$ it is defined a family of stable distributions concentrated on the real line $\mathbb{R}$ where special cases are Gaussian ($\alpha = 2$) and Cauchy ($\alpha = 1$) distributions. Instead of the stability index α, it is convenient to introduce the "power" parameter p, defined by

$$(p-1)(1-\alpha) = 1;$$

that means $p > 2$ for $0 < \alpha < 1$. The well-known special case of positive stable families is the Lévy or inverse Gaussian distribution, which corresponds to $p = 3$ or $\alpha = 1/2$, with

$$\mu_{1/2,t}(dx) = \frac{dx}{\sqrt{2\pi x^3}} t \exp\{-t^2/(2x)\}, \qquad x > 0.$$

For fixed $p > 2$ (or $0 < \alpha < 1$) and $t > 0$, the infinitely divisible NEF $F_{p,t} = F(\mu_{\alpha,t})$ generated by $\mu_{\alpha,t}$ is such that $\Theta(\mu_{\alpha,t}) = (-\infty; 0]$, $K_{\mu_{\alpha,t}}(\theta) = t(\alpha-1)[\theta/(\alpha-1)]^\alpha/\alpha$, $\psi_{\mu_{\alpha,t}}(m) = (\alpha-1)(m/t)^{1/(\alpha-1)} = (m/t)^{1-p}/(1-p)$, and $V_{F_{p,t}}(m) = m^p t^{1-p}$ on $M_{F_{p,t}} = (0; \infty)$; see Jørgensen [4, Chapter 4] for the complete classification with $p \in \mathbb{R}$.

Hence, for all $p = 2d-1$ with $d \in \{2, 3, \cdots\}$, we associate to the NEF $F_{p,t} = F(\mu_{\alpha,t})$ the sequence of $\mu_{\alpha,t} - d$-pseudo-orthogonal polynomials $(P_n(x; m_t))_{n \geq 0}$, which the general expression is obtained from (10) as

$$P_n(x;t) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!t^{n(2d-2)}}{k_1!\cdots k_n!} \prod_{j=1}^{n} \left(x\frac{m_t^{2-2d-j} A^j_{2-2d}}{2-2d} - \frac{m_t^{3-2d-j} A^j_{3-2d}}{3-2d} \right)^{k_j},$$

for any $m_t > 0$, where $A^j_k = k(k-1)\cdots(k-j+1)$ with $A^0_k = 1$.

Taking simply $m_t = 1 = t$ and then, for all $d \in \{2, 3, \cdots\}$,

$$V_{F_{2d-1}}(m) = m^{2d-1} = \sum_{k=0}^{2d-1} \frac{A^k_{2d-1}}{k!} (m-1)^k,$$

the $(2d-1)$-order recurrence relation (Lemma 3 (iii)) is given by

$$\begin{aligned} P_{k+2d-1}(x;1) &= (x - A^1_{k+2d-2}A^1_{2d-1} - 1)P_{k+2d-2}(x;1) - \\ & \quad (k+2d-2+A^2_{k+2d-2}A^2_{2d-1}/2)P_{k+2d-3}(x;1) \\ & \quad -\sum_{\tau=0}^{2d-4} \frac{A^{\tau+3}_{k+2d-2}A^{\tau+3}_{2d-1}}{(\tau+3)!}P_{k+2d-4-\tau}(x;1), \qquad k \geq 0, \end{aligned}$$

with the initial conditions:

$$\begin{aligned} P_0(x;1) &= 1,\ P_1(x;1) = x-1,\ P_2(x;1) = x^2-5x+3, \quad \text{and} \\ P_j(x;1) &= (x - A^1_{j-1}A^1_{2d-1} - 1)P_{j-1}(x;1) - (j-1+A^2_{j-1}A^2_{2d-1}/2)P_{j-2}(x;1) \\ & \quad -\sum_{\tau=0}^{j-3} \frac{A^{\tau+3}_{j-2-\tau}A^{\tau+3}_{2d-1}}{(\tau+3)!}P_{j-3-\tau}(x;1), \quad j = 3,\cdots,2d-2. \end{aligned}$$

When $d=2$, we have the results corresponding to the Lévy case.

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References

[1] DOUAK, K. *Contributions à l'Étude des Polynômes d-Orthogonaux*. Habilitation à diriger des recherches, Université Paris VI, 2003.

[2] FELLER, W. *An Introduction to Probability Theory and its Applications, Vol. II, Second edition*. Wiley, New York, 1971.

[3] FEINSILVER, P. Some classes of orthogonal polynomials associated with martingale. *Proc. Amer. Math. Soc.*, 98 (1986), 298-302.

[4] JØRGENSEN, B. *The Theory of Dipersion Models*. Chapman & Hall, London, 1997.

[5] KOKONENDJI, C.C. On d-orthogonality of polynomials associated to a natural exponential family. Preprint LMA - University of Pau, submitted to Adv. Appl. Math., 2003.

[6] KOTZ, S., BALAKRISHNAN, N., AND JOHNSON, N.L. *Continuous Multivariate Distributions, Vol. 1: Models and Applications, Second edition*. Wiley, New York, 2000.

[7] KNUTH, D.E. *The Art of Computer Programming, Vol. 1: Fundamental Algorithms*. Addison-Wesley/Readings, Mass, 1981.

[8] KÜCHLER, U., AND SØRENSEN, M. *Exponential Families of Stochastic Processes*. Springer-Verlag, New York, 1997.

[9] LETAC, G., AND MORA, M. Natural real exponential families with cubic variance functions. *Ann. Statist.*, 18 (1990), 1-37.

[10] LETAC, G., MALOUCHE, D., AND MAURER, S. The real powers of the convolution of a negative binomial distribution and a Bernoulli distribution. *Proc. Amer. Math. Soc.*, 130 (7) (2002), 2107-2114.

[11] MORRIS, C.N. Natural exponential family with quadratic variance functions. *Ann. Statist.*, 10 (1982), 65-82.

[12] POMMERET, D. Orthogonality of the Sheffer system associated to a Lévy process. *J. Statist. Plann. Inference*, 86 (2000), 1-10.

[13] POMMERET, D. K terms recurrence relations and polynomial variance functions of the Kth degree. *J. Comput. Appl. Math.*, 133 (2001), 555-565.

[14] SATO, K. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge, 1999.

[15] SCHOUTENS, W. *Stochastic Processes and Orthogonal Polynomials*. Lecture Notes in Statistics No. 146, Springer, New-York, 2000.

[16] SCHOUTENS, W., AND TEUGELS, J.L. Lévy Processes, polynomials and martingales. *Commun. Statist.-Stochastic Models*, 14 (1998), 335-349.

[17] SHEFFER, I.M. Concerning Appell sets and associated linear functional equations. *Duke Math. J.*, 3 (1937), 593-609.

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