OPTIMALITY CONDITIONS FOR THE LINEAR FRACTIONAL/QUADRATIC BILEVEL PROBLEM

Herminia I. Calvete and Carmen Galé

Abstract. Bilevel programs are optimization problems which have a subset of their variables constrained to be an optimal solution of another problem parameterized by the remaining variables. They have been applied to decentralized planning problems involving a decision process with a hierarchical structure.

This paper considers the linear fractional/quadratic bilevel programming (LFQBP) problem, in which the first level objective function is linear fractional, the second level objective function is quadratic and the common constraint region is a polyhedron. For this problem, optimality conditions are derived based on Karush-Kuhn-Tucker conditions and duality theory.

Keywords: bilevel, fractional, quadratic, Karush-Kuhn-Tucker, duality

AMS classification: 90C26, 90C32, 90C20, 90C49

§1. Introduction

Bilevel programming involves two optimization problems where the constraint region of the first level problem is implicitly determined by another optimization problem. In terms of modeling, bilevel problems are programs which have a subset of their variables constrained to be an optimal solution of another problem parameterized by the remaining variables. The second level decision maker optimizes his objective function under the given parameters from the first level decision maker. This one, in return, with complete information on the possible reactions of the second level decision maker, selects the parameters so as to optimize his own objective function.

Bilevel problems can be formulated as:

$$\begin{aligned} \min_{(x_1,x_2)\in S} \quad & f_1(x_1,x_2) \\ \text{where} \quad & x_2 \in \arg\min_{\nu\in S(x_1)} f_2(x_1,\nu) \end{aligned} \tag{1}$$

where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are the variables controlled by the first level and the second level decision maker, respectively; $f_1, f_2 : \mathbb{R}^n \longrightarrow \mathbb{R}, n = n_1 + n_2$; $S \subset \mathbb{R}^n$ defines the common constraint region and $S(x_1) = \{x_2 \in \mathbb{R}^{n_2} : (x_1, x_2) \in S\}$.

Let S_1 be the projection of S onto $\mathbb{R}^{n_1}$. For each $x_1 \in S_1$, the second level decision maker solves the problem (2)

$$\begin{array}{ll} \min & f_2(x_1, x_2) \\ \text{s.t.} & \\ & x_2 \in S(x_1) \end{array} \tag{2}$$

The feasible region of the first level decision maker, called inducible region IR, is implicitly defined by the second level optimization problem

$$\text{IR} = \{(x_1, x_2^*) \,:\, x_1 \in S_1,\;\; x_2^* \in M(x_1)\}$$

where $M(x_1)$ denotes the set of optimal solutions to (2). We assume that S is not empty and that for all decisions taken by the first level decision maker, the second level decision maker has some room to respond, i.e. $M(x_1) \neq \emptyset$.

The bilevel programming problem (1) is a nonconvex optimization problem that has received increasing attention in the literature (see [2, 10, 11, 14, 17] and references therein). One of its main features is that, unlike general mathematical problems, the bilevel problem may not possess a solution even when f_1 and f_2 are continuous and S is compact. In particular, difficulties may arise when $M(x_1)$ is not single-valued for all permissible x_1. Different approaches have been proposed in the literature to make sure that the bilevel problem is well posed. The most common one is to assume that, for each value of the first level variables x_1, there is a unique solution to the second level problem, that is, the set $M(x_1)$ is a singleton for all $x_1 \in S_1$ [2, 3, 4, 10, 17].

In this paper the linear fractional/quadratic bilevel programming (LFQBP) problem is considered in which the first level objective function is linear fractional, the second level objective function is quadratic and the common constraint region S is a polyhedron.

Fractional programming and quadratic programming when there exists only one level of decision have received remarkable attention in the literature [1, 12, 16]. It is worth mentioning that objective functions which are ratios frequently appear, for instance, when an efficiency measure of a system is to be optimized or when approaching a stochastic programming problem. On the other hand, quadratic problems arise directly in such applications as least-squares regression with bounds or linear constraints, robust data fitting, or portfolio optimization. They also arise as subproblems in optimization algorithms for nonlinear programming and in stochastic optimization. Fractional bilevel problems have been considered in [5, 6, 7, 8, 9]. Quadratic bilevel problems have been addressed in [13, 15, 18, 19, 20].

In this paper, following the approach taken in [9, 20], we use Karush-Kuhn-Tucker optimality conditions to rewrite the bilevel problem as a single level problem and derive optimality conditions for the LFQBP problem by applying duality theory. These results extend optimality conditions developed in [20] for the linear/quadratic bilevel programming problem. The paper is organized as follows. In Section 2 the LFQBP problem is formulated and some preliminary properties are obtained. Section 3 provides the main theoretical results on optimality conditions. Finally, Section 4 concludes the paper with final conclusions and future work.

§2. Problem formulation

Using the common notation in bilevel programming, the LFQBP problem can be written as follows:

$$\min \quad \frac{c_{11}x_1 + c_{12}x_2 + \alpha}{d_{11}x_1 + d_{12}x_2 + \beta}, \qquad \text{where } x_2 \text{ solves}$$

$$\min \quad c_{21}x_1 + c_{22}x_2 + (x_1, x_2)^t Q(x_1, x_2)$$

$$\text{s.t.}$$

$$A_1x_1 + A_2x_2 \leq b$$

where c_{11}, c_{21} and d_{11} are n_1-vectors; c_{12}, c_{22} and d_{12} are n_2-vectors; α and β are scalars; A_1 is an $m \times n_1$ matrix; A_2 is an $m \times n_2$ matrix; b is an m-vector and Q is an $(n_1 + n_2) \times (n_1 + n_2)$ symmetric matrix with

$$Q = \begin{pmatrix} Q_3 & Q_2^t \\ Q_2 & Q_1 \end{pmatrix}$$

where Q_1, Q_2 and Q_3 are matrices of conformal dimensions.

We assume that the polyhedron $S = \{(x_1, x_2) : A_1x_1 + A_2x_2 \leq b\}$ is non-empty and bounded. In addition, it is also assumed that $d_{11}x_1 + d_{12}x_2 + \beta > 0, \ \forall (x_1, x_2) \in S$. If this is not so, it suffices to consider the linear fractional objective function as $-(c_{11}x_1 + c_{12}x_2 + \alpha)/-(d_{11}x_1 + d_{12}x_2 + \beta)$.

For each value of $x_1 \in S_1$, the second level decision maker solves the following quadratic programming problem:

$$\min \quad c_{21}x_1 + c_{22}x_2 + (x_1, x_2)^t Q(x_1, x_2)$$

$$\text{s.t.}$$

$$A_2x_2 \leq b - A_1x_1$$

Bearing in mind that $c_{21}x_1 + x_1^t Q_3 x_1$ is a constant term, this problem is equivalent to:

$$(P_{x_1}): \qquad \begin{array}{ll} \min & (c_{22} + 2x_1^t Q_2^t)x_2 + x_2^t Q_1 x_2 \\ \text{s.t.} & \\ & A_2x_2 \leq b - A_1x_1 \end{array} \tag{3}$$

We assume that Q_1 is positively definite so as there will be a unique optimal solution to the second level problem. That is to say, $M(x_1)$ is a singleton for all $x_1 \in S_1$ and the LFQBP problem is well posed.

As a consequence, the LFQBP problem is equivalent to the following bilevel problem, which will be considered in the sequel:

$$\begin{array}{ll} \min & \dfrac{c_{11}x_1 + c_{12}x_2 + \alpha}{d_{11}x_1 + d_{12}x_2 + \beta}, \qquad \text{where } x_2 \text{ solves} \\ \min & (c_{22} + 2x_1^t Q_2^t)x_2 + x_2^t Q_1 x_2 \\ \text{s.t.} & \\ & A_1x_1 + A_2x_2 \leq b \end{array} \tag{4}$$

Remark 1. Notice that, for any fixed $(x_1, x_2) \in$ IR, x_2 is an optimal solution to (P_{x_1}). Hence, by applying Karush-Kuhn-Tucker necessary and sufficient conditions, there exists $w \in \mathbb{R}^m$ such that (x_1, x_2, w) satisfies

$$A_1x_1 + A_2x_2 \leq b \tag{5}$$

$$w^t(A_1x_1 + A_2x_2 - b) = 0 \tag{6}$$

$$2Q_2x_1 + 2Q_1x_2 + A_2^t w = -c_{22}^t \tag{7}$$

$$w \geq 0 \tag{8}$$

Similarly, if (x_1, x_2, w) satisfies (5)-(8) then $(x_1, x_2) \in$ IR.

§3. Optimality conditions for the LFQBP problem

Theorem 1. (x_1^*, x_2^*) *is an optimal solution to the LFQBP problem if and only if there exists* $w^* \in \mathbb{R}^m$ *such that* (x_1^*, x_2^*, w^*) *is an optimal solution to the following one level nonlinear programming problem*

$$(NLP): \quad \min_{(x_1,x_2,w)} \quad F(x_1, x_2) = \frac{c_{11}x_1 + c_{12}x_2 + \alpha}{d_{11}x_1 + d_{12}x_2 + \beta}$$
$$\text{s.t.} \quad (5) - (8).$$

Proof. Let (x_1^*, x_2^*) an optimal solution to the LFQBP problem. Taking into account remark 1, since $(x_1^*, x_2^*) \in$ IR, there exists $w^* \in \mathbb{R}^m$ such that (x_1^*, x_2^*, w^*) satisfies (5)-(8), that is to say it is a feasible solution to the problem (NLP).

If (x_1^*, x_2^*, w^*) was not an optimal solution to (NLP), there would exist $(\hat{x}_1, \hat{x}_2, \hat{w})$ satisfying (5)-(8), thus $(\hat{x}_1, \hat{x}_2) \in$ IR, such that

$$\frac{c_{11}\hat{x}_1 + c_{12}\hat{x}_2 + \alpha}{d_{11}\hat{x}_1 + d_{12}\hat{x}_2 + \beta} < \frac{c_{11}x_1^* + c_{12}x_2^* + \alpha}{d_{11}x_1^* + d_{12}x_2^* + \beta}$$

This, together with the fact that $(\hat{x}_1, \hat{x}_2) \in$ IR, contradicts the optimality of (x_1^*, x_2^*). Therefore, (x_1^*, x_2^*, w^*) solves the problem (NLP).

Conversely, let (x_1^*, x_2^*, w^*) be an optimal solution to the problem (NLP). Since (x_1^*, x_2^*, w^*) satisfies conditions (5)-(8), we conclude that $(x_1^*, x_2^*) \in$ IR. On the other hand, for any fixed $(x_1, x_2) \in$ IR, there exists $w \in \mathbb{R}^m$ such that (x_1, x_2, w) is a feasible solution to the problem (NLP). Moreover, since (x_1^*, x_2^*, w^*) is an optimal solution to the problem (NLP) then $F(x_1^*, x_2^*) \leq F(x_1, x_2)$. Therefore, (x_1^*, x_2^*) solves the LFQBP problem. □

The problem (NLP) can be reformulated as

$$\min_{w \geq 0} \quad \min_{(x_1,x_2) \in S[w]} \quad F(x_1, x_2) \tag{9}$$

where $S[w] = \{(x_1, x_2) \in \mathbb{R}^{n_1+n_2} : (x_1, x_2) \text{ satisfies } (5) - (7)\}$. By convention, when $S[w] = \emptyset$, we define $\min\{F(x_1, x_2) : (x_1, x_2) \in S[w]\} = \infty$.

For a given $w \geq 0$, since $w^t(A_1x_1 + A_2x_2 - b) \leq 0$, the inner problem in (9) can be written as:

$$(P_w): \quad \min_{(x_1,x_2)} \quad F(x_1,x_2) = \frac{c_{11}x_1 + c_{12}x_2 + \alpha}{d_{11}x_1 + d_{12}x_2 + \beta}$$

s.t.

$$A_1x_1 + A_2x_2 \leq b$$

$$w^tA_1x_1 + w^tA_2x_2 \geq w^tb$$

$$2Q_2x_1 + 2Q_1x_2 = -c_{22}^t - A_2^tw$$

Since w is fixed, this is a linear fractional programming problem whose dual problem is:

$$(D_w): \quad \max_{(u_1,u_2,u_3,u_4)} \quad G(u_1,u_2,u_3,u_4) = u_4$$

s.t.

$$(u_2w - u_1)^tA_1 + 2u_3^tQ_2 + u_4d_{11} = c_{11} \tag{10}$$

$$(u_2w - u_1)^tA_2 + 2u_3^tQ_1 + u_4d_{12} = c_{12} \tag{11}$$

$$-(u_2w - u_1)^tb + u_3^t(c_{22}^t + A_2^tw) + u_4\beta = \alpha \tag{12}$$

$$u_1 \geq 0, \ u_2 \geq 0 \tag{13}$$

where $u_1 \in \mathbb{R}^m, u_2 \in \mathbb{R}, u_3 \in \mathbb{R}^{n_2}$ and $u_4 \in \mathbb{R}$.

For a given $w \geq 0$, let

$$S\{w\} = \{(u_1,u_2,u_3,u_4) \in R^{m+n_2+2} : (u_1,u_2,u_3,u_4) \text{ satisfies } (10)-(13)\}.$$

By convention, if $S\{w\} = \emptyset$ we define $\max\{G(u_1,u_2,u_3,u_4) : (u_1,u_2,u_3,u_4) \in S\{w\}\} = \infty$.

Let (DP) be the following min-max problem:

$$\min_{w\geq 0} \quad \max_{(u_1,u_2,u_3,u_4)\in S\{w\}} \quad G(u_1,u_2,u_3,u_4) \tag{14}$$

Theorem 2. *If $(\hat{x}_1,\hat{x}_2,\hat{w})$ and $(\hat{u}_1,\hat{u}_2,\hat{u}_3,\hat{u}_4,\hat{w})$ are feasible solutions to the problems (NLP) and (DP), respectively, then*

$$\hat{u}_4 \ \leq \ \frac{c_{11}\hat{x}_1 + c_{12}\hat{x}_2 + \alpha}{d_{11}\hat{x}_1 + d_{12}\hat{x}_2 + \beta}$$

Proof. Since $(\hat{u}_1,\hat{u}_2,\hat{u}_3,\hat{u}_4) \in S\{\hat{w}\}$, $\hat{u}_1,\hat{u}_2,\hat{u}_3,\hat{u}_4$ and $\hat{w}$ satisfy the corresponding constraints (10)-(13). Post-multiplying (10) by $\hat{x}_1$, (11) by $\hat{x}_2$ and then adding them together with (12), we get

$$(\hat{u}_2\hat{w} - \hat{u}_1)^t(A_1\hat{x}_1 + A_2\hat{x}_2 - b) + \hat{u}_3^t(2Q_2\hat{x}_1 + 2Q_1\hat{x}_2 + c_{22}^t + A_2^t\hat{w})$$
$$+\hat{u}_4(d_{11}\hat{x}_1 + d_{12}\hat{x}_2 + \beta) = c_{11}\hat{x}_1 + c_{12}\hat{x}_2 + \alpha$$

Since $(\hat{x}_1,\hat{x}_2) \in S[\hat{w}]$, taking into account the corresponding constraints (6) and (7), the above equality can be rewritten as

$$-\hat{u}_1^t(A_1\hat{x}_1 + A_2\hat{x}_2 - b) + \hat{u}_4(d_{11}\hat{x}_1 + d_{12}\hat{x}_2 + \beta) = c_{11}\hat{x}_1 + c_{12}\hat{x}_2 + \alpha$$

Since $d_{11}\hat{x}_1 + d_{12}\hat{x}_2 + \beta > 0$, $\hat{u}_1 \geq 0$ and $A_1\hat{x}_1 + A_2\hat{x}_2 - b \leq 0$, it is derived that

$$\hat{u}_4 \leq \frac{c_{11}\hat{x}_1 + c_{12}\hat{x}_2 + \alpha}{d_{11}\hat{x}_1 + d_{12}\hat{x}_2 + \beta}$$

and the proof is completed. □

Remark 2. Notice that for a given $w \geq 0$, if $S[w] \neq \emptyset$ then the problem (P_w) is a linear fractional programming problem on a nonempty and bounded polyhedron, so it reaches an optimal solution and so its dual problem (D_w). Bearing in mind primal-dual relationships, their optimal objective values are equal, i.e.,

$$\min_{(x_1,x_2)\in S[w]} F(x_1,x_2) = \max_{(u_1,u_2,u_3,u_4)\in S\{w\}} G(u_1,u_2,u_3,u_4)$$

Moreover, if $S[w] = \emptyset$, by convention

$$\min_{(x_1,x_2)\in S[w]} F(x_1,x_2) = \max_{(u_1,u_2,u_3,u_4)\in S\{w\}} G(u_1,u_2,u_3,u_4) = \infty$$

Hence,

$$\min_{w\geq 0} \min_{(x_1,x_2)\in S[w]} F(x_1,x_2) = \min_{w\geq 0} \max_{(u_1,u_2,u_3,u_4)\in S\{w\}} G(u_1,u_2,u_3,u_4) \tag{15}$$

In other words, the optimal objective values of the problems (NLP) and (DP) are equal.

Theorem 3. *Let* (x_1^*, x_2^*, w^*) *and* $(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ *be feasible solutions to the problems* (NLP) *and* (DP)*, respectively. Then they are respectively optimal if and only if*

$$\frac{c_{11}x_1^* + c_{12}x_2^* + \alpha}{d_{11}x_1^* + d_{12}x_2^* + \beta} = u_4^* \tag{16}$$

and, for all $(x_1, x_2) \in$ *IR,*

$$(u_2^*w^* - u_1^*)^t(A_1x_1 + A_2x_2 - b) + u_3^{*t}(2Q_2x_1 + 2Q_1x_2 + c_{22}^t + A_2^t w^*) \geq 0 \tag{17}$$

Proof. If (x_1^*, x_2^*, w^*) and $(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ are optimal solutions to the problems (NLP) and (DP), respectively, then (16) holds as a consequence of remark 2.

On the other hand, let $(x_1, x_2) \in$ IR. Bearing in mind remark 1, there exists $w \in \mathbb{R}^m$ such that (x_1, x_2, w) is a feasible solution to the problem (NLP).

Since $(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ is a feasible solution to the problem (DP), post-multiplying the corresponding constraints (10) by x_1, (11) by x_2 and then adding them together with (12) we get

$$(u_2^*w^* - u_1^*)^t(A_1x_1 + A_2x_2 - b) + u_3^{*t}(2Q_2x_1 + 2Q_1x_2 + c_{22}^t + A_2^t w^*)$$
$$+u_4^*(d_{11}x_1 + d_{12}x_2 + \beta) = c_{11}x_1 + c_{12}x_2 + \alpha$$

Taking into account that $d_{11}x_1 + d_{12}x_2 + \beta \neq 0$ and the fact that (x_1^*, x_2^*, w^*) is an optimal solution to the problem (NLP), we can write

$$u_4^* + \frac{1}{d_{11}x_1 + d_{12}x_2 + \beta}\Big\{(u_2^*w^* - u_1^*)^t(A_1x_1 + A_2x_2 - b) + u_3^{*t}(2Q_2x_1 + 2Q_1x_2 + c_{22}^t + A_2^t w^*)\Big\}$$

$$= \frac{c_{11}x_1 + c_{12}x_2 + \alpha}{d_{11}x_1 + d_{12}x_2 + \beta} \geq \frac{c_{11}x_1^* + c_{12}x_2^* + \alpha}{d_{11}x_1^* + d_{12}x_2^* + \beta} = u_4^*$$

Since $d_{11}x_1 + d_{12}x_2 + \beta > 0$, we conclude that

$$(u_2^* w^* - u_1^*)^t (A_1x_1 + A_2x_2 - b) + u_3^{*t}(2Q_2x_1 + 2Q_1x_2 + c_{22}^t + A_2^t w^*) \geq 0$$

which proves (17).

Conversely, let (x_1, x_2, w) be a feasible solution to the problem (NLP). As a consequence of remark 1, $(x_1, x_2) \in$ IR. Since $(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ is a feasible solution to the problem (DP), in the same way as we have done previously, post-multiplying the corresponding constraints (10) by x_1, (11) by x_2 and then adding them together with (12) we get

$$\begin{aligned} \frac{c_{11}x_1 + c_{12}x_2 + \alpha}{d_{11}x_1 + d_{12}x_2 + \beta} = &\; u_4^* + \\ \frac{1}{d_{11}x_1 + d_{12}x_2 + \beta} & \Big\{ (u_2^* w^* - u_1^*)^t (A_1x_1 + A_2x_2 - b) + u_3^{*t}(2Q_2x_1 + 2Q_1x_2 + c_{22}^t + A_2^t w^*) \Big\} \end{aligned}$$

By applying conditions (16) and (17) and bearing in mind that $d_{11}x_1 + d_{12}x_2 + \beta > 0$, we get

$$\frac{c_{11}x_1 + c_{12}x_2 + \alpha}{d_{11}x_1 + d_{12}x_2 + \beta} \;\; \geq \;\; \frac{c_{11}x_1^* + c_{12}x_2^* + \alpha}{d_{11}x_1^* + d_{12}x_2^* + \beta}$$

and so we conclude that (x_1^*, x_2^*, w^*) is an optimal solution to the problem (NLP).

Finally, since (x_1^*, x_2^*, w^*) solves the problem (NLP), taking into account remark 2, from (16) directly follows that $(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ is an optimal solution to the problem (DP). □

Theorem 4. $(x_1^*, x_2^*) \in S$ *is an optimal solution to the LFQBP problem if and only if there exist* $w^* \in \mathbb{R}^m, u_1^* \in R^m, u_2^* \in \mathbb{R}, u_3^* \in \mathbb{R}^{n_2}$ *and* $u_4^* \in \mathbb{R}$ *satisfying* $w^* \geq 0$, $u_1^* \geq 0$, $u_2^* \geq 0$, *such that*

$$w^{*t}(A_1x_1^* + A_2x_2^* - b) = 0 \tag{18}$$

$$2Q_2x_1^* + 2Q_1x_2^* + A_2^t w^* = -c_{22}^t \tag{19}$$

$$(u_2^* w^* - u_1^*)^t A_1 + 2u_3^{*t} Q_2 + u_4^* d_{11} = c_{11} \tag{20}$$

$$(u_2^* w^* - u_1^*)^t A_2 + 2u_3^{*t} Q_1 + u_4^* d_{12} = c_{12} \tag{21}$$

$$-(u_2^* w^* - u_1^*)^t b + u_3^{*t}(c_{22}^t + A_2^t w^*) + u_4^* \beta = \alpha \tag{22}$$

$$u_1^{*t}(A_1x_1^* + A_2x_2^* - b) = 0 \tag{23}$$

$$\begin{aligned} &\forall (x_1, x_2) \in \textit{IR} \\ &(u_2^* w^* - u_1^*)^t (A_1x_1 + A_2x_2 - b) + u_3^{*t}(2Q_2x_1 + 2Q_1x_2 + c_{22}^t + A_2^t w^*) \geq 0 \end{aligned} \tag{24}$$

Proof. If (x_1^*, x_2^*) is an optimal solution to the LFQBP problem, by Theorem 1, there exists $w^* \in \mathbb{R}^m$, $w^* \geq 0$, such that (x_1^*, x_2^*, w^*) is an optimal solution to the problem (NLP). Hence, it is clear that (18) and (19) are satisfied.

Since (x_1^*, x_2^*) is an optimal solution to (P_{w^*}), then there exist $u_1^* \in \mathbb{R}^m, u_2^* \in \mathbb{R}, u_3^* \in \mathbb{R}^{n_2}$ and $u_4^* \in \mathbb{R}$ such that $u_1^* \geq 0$, $u_2^* \geq 0$ and $(u_1^*, u_2^*, u_3^*, u_4^*)$ is an optimal solution to (D_{w^*}).

Hence, it is clear that (20), (21) and (22) are satisfied. In addition, their optimal objective values are equal, i.e.,

$$\frac{c_{11}x_1^* + c_{12}x_2^* + \alpha}{d_{11}x_1^* + d_{12}x_2^* + \beta} \;=\; u_4^* \tag{25}$$

Moreover, post-multiplying (20) by x_1^*, (21) by x_2^* and then adding them together with (22), while applying (18) and (19), we get

$$-u_1^{*t}(A_1x_1^* + A_2x_2^* - b) + u_4^*(d_{11}x_1^* + d_{12}x_2^* + \beta) = c_{11}x_1^* + c_{12}x_2^* + \alpha$$

Taking into account (25), we conclude that $u_1^{*t}(A_1x_1^* + A_2x_2^* - b) = 0$, which proves (23).

Finally, since $(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ is a feasible solution to the problem (DP) which satisfies (25) and (x_1^*, x_2^*, w^*) is an optimal solution to the problem (NLP), by applying remark 2 we conclude that $(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ is an optimal solution to the problem (DP). As a result of Theorem 3, (24) is satisfied.

Conversely, let $(x_1^*, x_2^*) \in S$. As a consequence of (18) and (19), (x_1^*, x_2^*, w^*) is a feasible solution to the problem (NLP). Similarly, since (20), (21) and (22) hold $(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ is a feasible solution to the problem (DP).

Moreover, post-multiplying (20) by x_1^*, (21) by x_2^* and then adding them together with (22), while applying (18), (19) and (23), we get

$$u_4^*(d_{11}x_1^* + d_{12}x_2^* + \beta) = c_{11}x_1^* + c_{12}x_2^* + \alpha$$

From this condition and (24), as a consequence of Theorem 3, we conclude that (x_1^*, x_2^*, w^*) and
$(u_1^*, u_2^*, u_3^*, u_4^*, w^*)$ are optimal solutions to the (NLP) and (DP) problems, respectively . Hence, from Theorem 1 we get that (x_1^*, x_2^*) is an optimal solution to the LFQBP problem. □

§4. Conclusions

We have introduced necessary and sufficient optimality conditions for a particular class of bilevel programming problems. The main concepts involved were Karush-Kuhn-Tucker conditions and duality relationships.

Our future work includes the use of these conditions to develop an algorithm which solves the LFQBP problem by solving only linear fractional programming problems.

References

[1] M. Avriel, W.E. Diewert, S. Schaible, and I. Zang. *Generalized concavity*. Plenum Press, New York, London, 1988.

[2] J.F. Bard. *Practical bilevel optimization. Algorithms and applications*. Kluwer Academic Publishers, Dordrecht, Boston, London, 1998.

[3] J.F. Bard and J.E. Falk. An explicit solution to the multilevel programming problem. *Computers and Operations Research*, 9(1):77–100, 1982.

[4] W.F. Bialas and M.H. Karwan. Two-level linear programming. *Management Science*, 30:1004–1024, 1984.

[5] H.I. Calvete and C. Galé. The bilevel linear/linear fractional programming problem. *European J. of Operational Research*, 114(1):188–197, 1999.

[6] H.I. Calvete and C. Galé. Bilevel fractional programming. In C.A. Floudas and P.M. Pardalos, editors, *Encyclopedia of Optimization*, pages 135–137. Kluwer Academic Publishers, Dordrecht, 2001.

[7] H.I. Calvete and C. Galé. A note on ‘bi-level linear fractional programming problem’. *European J. of Operational Research*, 152(1):296–299, 2004(a).

[8] H.I. Calvete and C. Galé. Solving linear fractional bilevel programs. *Operations Research Letters*, 32(2):143–151, 2004(b).

[9] H.I. Calvete and C. Galé. Note on ’optimality conditions for linear fractional bilevel programs’. Submitted, 2004(c).

[10] S. Dempe. *Foundations of Bilevel Programming*. Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[11] S. Dempe. Annotated bibliography on bilevel programming and mathematical programs with equilibrium constraints. *Optimization*, 52:333–359, 2003.

[12] C.A. Floudas and V. Visweswaran. Quadratic optimization. In R. Horst and P.M. Pardalos, editors, *Handbook of Global Optimization*, pages 217–269. Kluwer Academic Publishers, Dordrecht, 1995.

[13] J. Júdice and A.M. Faustino. The linear-quadratic bilevel programming problem. *INFOR*, 32:87–98, 1994.

[14] A. Migdalas, P.M. Pardalos, and P. Värbrand, editors. *Multilevel optimization: Algorithms and applications*. Kluwer Academic Publishers, Dordrecht, Boston, London, 1998.

[15] L.D. Muu and N.V. Quy. A global optimization method for solving convex quadratic bilevel programming problems. *J. of Global Optimization*, 26:199–219, 2003.

[16] S. Schaible. Fractional programming. In R. Horst and P.M. Pardalos, editors, *Handbook of Global Optimization*, pages 495–608. Kluwer Academic Publishers, Dordrecht, 1995.

[17] K. Shimizu, Y. Ishizuka, and J.F. Bard. *Nondifferentiable and Two-level Mathematical Programming*. Kluwer Academic Publishers, Boston, London, Dordrecht, 1997.

[18] L.N. Vicente and P.H. Calamai. Geometry and local optimality conditions for bilevel programs with quadratic strictly convex lower levels. In D. Du and P.M. Pardalos, editors, *Minimax and Applications, Nonconvex Optimization and its Applications*, pages 141–151. Kluwer Academic Publishers, Dordrecht, 1995.

[19] L.N. Vicente, G. Savard, and J. Judice. Descent approaches for quadratic bilevel programming. *J. of Optimization Theory and Applications*, 81:379–399, 1994.

[20] S. Wang, Q. Wang, and S. Romano-Rodríguez. Optimality conditions and an algorithm for linear-quadratic bilevel programs. *Optimization*, 31:127–139, 1994.

Herminia I. Calvete
Dpto. de Métodos
Estadísticos
Facultad de Ciencias
Edificio B
Universidad de Zaragoza
Pedro Cerbuna, 12
50009 Zaragoza
España
`herminia@unizar.es`

Carmen Galé
Dpto. de Métodos Estadísticos
Centro Politécnico Superior
Edificio Torres Quevedo
Universidad de Zaragoza
María de Luna, 3
50015 Zaragoza
España
`cgale@unizar.es`