GENERALIZATION OF THE PIECEWISE POLYNOMIAL INTERPOLATION BY FRACTAL FUNCTIONS

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Abstract. The fractal interpolation functions defined by iterated function systems provide new methods of approximation and quantification of experimental data. The polynomial fractal functions can be considered as generalization of the piecewise polynomial interpolants. Assuming some hypotheses on the original function, a bound of the representation of the error for this kind of approximants is obtained here. The results proved guarantee the convergence of the interpolant to any smooth function when the diameter of the partition approaches zero. The property of good fit of the derivatives is also verified if the iterated function system is adequately chosen.

Keywords: Fractal interpolation functions, iterated function systems, piecewise polynomial interpolation

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§1. Introduction

The approximation and quantification of experimental data can be envisaged in the light of fractal interpolation functions defined by iterated functions systems ([1]). In the article of M.F. Barnsley ([2]), the moments of an experimental signal are computed by explicit formulae involving the coefficients of the iterated functions systems defining the function. A moment of any order can be used as an index of the signal, to perform comparisons and quantified measures.

As proved in the paper of M.F. Barnsley & A.N. Harrington ([3]), the polynomial fractal interpolation functions can be integrated indefinitely and smooth functions generalizing splines can be obtained. The main difference with the classic procedures resides in the definition by a functional relation assuming a self-similarity on small scales. In this way, the interpolants are defined as fixed points of maps between spaces of functions. The properties of these correspondences allow to deduce some inequalities that express the sensitivity of the functions and their derivatives to those changes in the parameters defining them ([4]).

In the particular case of polynomial fractal interpolation functions, the method can be considered as generalization of splines of the same kind. Some bounds of the error interpolation by odd degree polynomial fractal interpolation functions are obtained.

If the polynomials are of degree $2m-1$, the bounds range from the function up to the $(2m-2)$th derivative. The degree of regularity required for the function being approximated is lightly superior to the chosen interpolant.

§2. Differentiable fractal interpolation functions

Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N] \subset R$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times R : \ n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n, \ n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1}, \ L_n(t_N) = t_n \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leq l \, |c_1 - c_2| \quad \forall \, c_1, c_2 \, \in I \tag{2}$$

for some $0 \leq l < 1$.

Let $-1 < \alpha_n < 1$; $n = 1, 2, ..., N$, $F = I \times [c, d]$ for some $-\infty < c < d < +\infty$ and N continuous mappings, $F_n : F \to R$ be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \quad n = 1, 2, ..., N \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \leq \alpha_n |x - y|, \quad t \in I, \quad x, y \in R \tag{4}$$

Now define functions

$$w_n(t, x) = (L_n(t), F_n(t, x)), \ \forall \, n = 1, 2, ..., N$$

Theorem 1. *(Barnsley [1]) The iterated function system (IFS)[5]* $\{F, \ w_n \ : n = 1, 2, ..., N\}$ *defined above admits a unique attractor* G*.* G *is the graph of a continuous function* $f : I \to R$ *which obeys* $f(t_n) = x_n$ *for* $n = 0, 1, 2, ..., N$.

The previous function is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$. $f : I \to R$, is the unique function satisfying the functional equation

$$f(L_n(t)) = F_n(t, f(t)), \quad n = 1, 2, ..., N, \ \ t \in I$$

or,

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)), \quad n = 1, 2, ..., N, \ \ t \in I_n = [t_{n-1}, t_n] \tag{5}$$

Let $\mathcal{F}$ be the set of continuous functions $f : [t_0, t_N] \to [c, d]$ such that $f(t_0) = x_0$; $f(t_N) = x_N$. Define a metric on $\mathcal{F}$ by

$$\|f - g\|_\infty = \max \{|f(t) - g(t)| : \ t \in [t_0, t_N]\} \ \ \forall \, f, g \in \mathcal{F}$$

Then $(\mathcal{F}, \| \cdot \|_\infty)$ is a complete metric space.

Define a mapping $T : \mathcal{F} \to \mathcal{F}$ by:

$$(Tf)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \quad \forall \, t \in [t_{n-1}, t_n], \ \ n = 1, 2, ..., N$$

Using (1)-(4), it can be proved that $(Tf)(t)$ is continuous on the interval $[t_{n-1}, t_n]$ for $n = 1, 2, ..., N$ and at each of the points $t_1, t_2, ..., t_{N-1}$. T is a contraction mapping on the metric space $(\mathcal{F}, d)$

$$\|Tf - Tg\|_\infty \leq |\boldsymbol{\alpha}|_\infty \|f - g\|_\infty \tag{6}$$

where $|\boldsymbol{\alpha}|_\infty = \max\{|\alpha_n|;\ n = 1, 2, ..., N\}$. Since $|\boldsymbol{\alpha}|_\infty < 1$, T possesses a unique fixed point on $\mathcal{F}$, that is to say, there is $f \in \mathcal{F}$ such that $(Tf)(t) = f(t)\ \ \forall\, t \in [t_0, t_N]$. This function is the FIF corresponding to w_n.

The most widely studied fractal interpolation functions so far are defined by the IFS

$$L_n(t) = a_n t + b_n \tag{7}$$

$$F_n(t, x) = \alpha_n x + q_n(t) \tag{8}$$

where $q_n(t)$ is an affine map [1, 6]. α_n is called a vertical scaling factor of the transformation w_n. We deal here with the case where q_n is a polynomial of odd-degree, that can be considered a generalization of polynomial spline functions.

The following theorem assures the existence of differentiable FIF.

Theorem 2. *(Barnsley and Harrington [3]) Let $t_0 < t_1 < t_2 < ... < t_N$ and $L_n(t)$, $n = 1, 2, ..., N$, the affine function $L_n(t) = a_n t + b_n$ satisfying (1)-(2). Let $a_n = L_n'(t) = \frac{t_n - t_{n-1}}{t_N - t_0}$ and $F_n(t, x) = \alpha_n x + q_n(t)$, $n = 1, 2, ..., N$ verifying Eqs. (3)-(4). Suppose for some integer $p \geq 0$, $|\alpha_n| < a_n^p$ and $q_n \in C^p[t_0, t_N]$; $n = 1, 2, ..., N$. Let*

$$F_{nk}(t, x) = \frac{\alpha_n x + q_n^{(k)}(t)}{a_n^k} \qquad k = 1, 2, ..., p \tag{9}$$

$$x_{0,k} = \frac{q_1^{(k)}(t_0)}{a_1^k - \alpha_1} \quad x_{N,k} = \frac{q_N^{(k)}(t_N)}{a_N^k - \alpha_N} \qquad k = 1, 2, ..., p$$

If

$$F_{n-1,k}(t_N, x_{N,k}) = F_{nk}(t_0, x_{0,k}) \tag{10}$$

with $n = 2, 3, ..., N$ and $k = 1, 2, ..., p$, then $\{(L_n(t), F_n(t, x))\}_{n=1}^N$ determines a FIF $f \in C^p[t_0, t_N]$ and $f^{(k)}$ is the FIF determined by $\{(L_n(t), F_{nk}(t, x))\}_{n=1}^N$, for $k = 1, 2, ..., p$.

In the present paper, q_n are polynomials of degree $2m - 1$. According to the previous theorem, consider $p = 2m - 2$, $f \in C^{2m-2}$. The vertical scaling factor must satisfy $|\alpha_n| < a_n^{2m-2}$, $n = 1, 2, ..., N$. If $\alpha_n = 0\ \ \forall\, n = 1, 2, ..., N$, $f(t) = q_n \circ L_n^{-1}(t)\ \ \forall\, t \in I_n$ (FIF) is a piecewise odd degree polynomial and $f \in C^{2m-2}$, therefore is a polynomial spline ([7]). In this sense, we refer to this kind of functions as spline fractal interpolation functions (SFIF).

§3. Error bounds for the interpolation by odd degree polynomial fractal functions

In this paragraph, the existence of a fractal interpolation function defined by a IFS of type (9) where $p = 2m - 2$ and $q_n(t)$ is a polynomial of degree $2m - 1$ verifying the hypotheses of the theorem of Barnsley $\&$ Harrington is assumed. $2m - 2$ end conditions are specified at the extremes.

The equality of all the vertical factors is supposed for the sake of simplicity. In the first place the error committed in the substitution of the function $x(t)$ by the SFIF $f_\alpha(t)$ with factor

α will be bounded. A result concerning polynomial spline functions due to M.H. Schultz [8] is used.

By $\mathcal{H}^m(a,b)$ we mean the class of all functions $f(x)$ defined on $[a,b]$ which possess an absolutely continuous $(m-1)$th derivative on $[a,b]$ and whose mth derivative is in $L^2(a,b)$.

From here $m > 1$, $m \in \mathbb{N}$, $N \geq 1$; $h = t_n - t_{n-1}$. The following constants will be used in the next theorem.

If $m-1 \leq z \leq 2m-2$ and $0 \leq j \leq m$:

$$K_{m,m,z,j} = \begin{cases} 1 & if \;\; m-1 \leq z \leq 2m-2, \;\; j=m \\ (\frac{1}{\pi})^{m-j} & if \;\; m-1 = z, \;\; 0 \leq j \leq m-1 \\ \frac{(z+2-m)!}{\pi^{m-j}} & if \;\; m-1 \leq z \leq 2m-2, \;\; 0 \leq j \leq 2m-2-z \\ \frac{(z+2-m)!}{j! \; \pi^{m-j}} & if \;\; m-1 \leq z \leq 2m-2, \; 2m-2-z \leq j \leq m-1 \end{cases}$$

If $m-1 \leq z \leq 2m-2$ and $0 \leq j \leq m$:

$$K_{m,2m,z,j} = K_{m,m,z,j} \; K_{m,m,z,0}$$

If $m < p < 2m$, $4m-2p-1 \leq z \leq 2m-2$ and $0 \leq j \leq m$:

$$K_{m,p,z,j} = K_{p,p,2m-1,j} + K_{m,2m,z,j} \; 2^{\frac{1}{2}(2m-p)} \left(\frac{p!}{(2p-2m)!}\right)^2 \left(\frac{\|\Delta\|}{\underline{\Delta}}\right)^{2m-p}$$

with $\|\Delta\| = \max_{0 \leq i \leq N-1} (t_{i+1} - t_i)$, $\quad \underline{\Delta} = \min_{0 \leq i \leq N-1} (t_{i+1} - t_i)$.

If $m < p \leq 2m$, $4m-2p-1 \leq z \leq 2m-2$ and $m < j \leq p$:

$$K_{m,p,z,j} = K_{p,p,p,j} + (K_{m,p,z,m} + K_{p,p,p,m}) \; 2^{\frac{j-m}{2}} \left(\frac{(2p+m)!}{(2p-j)!}\right)^2 \left(\frac{\|\Delta\|}{\underline{\Delta}}\right)^{j-m}$$

If $m-1 \leq z \leq 2m-2$ and $0 \leq j \leq m-1$:

$$K^{\infty}_{m,m,z,j} = \begin{cases} K_{m,m,z,j+1} & if \;\; m-1 = z, \;\; 0 \leq j \leq m-1 \\ K_{m,m,z,j+1} & if \;\; m-1 < z \leq 2m-2, \;\; 0 \leq j \leq 2m-2-z \\ (j-2m+3+z)^{1/2} \; K_{m,m,z,j+1} & if \;\; m-1 < z \leq 2m-2, \\ & 2m-2-z < j \leq m-1 \end{cases}$$

If $m-1 \leq z \leq 2m-2$ and $0 \leq j \leq m-1$:

$$K^{\infty}_{m,2m,z,j+1} = \begin{cases} K_{m,2m,z,j+1} & if \;\; m-1 = z, \;\; 0 < j \leq m-1 \\ K_{m,2m,z,j+1} & if \;\; m-1 < z \leq 2m-2, \;\; 0 \leq j \leq 2m-2-z \\ (j-2m+3+z)^{1/2} \; K_{m,2m,z,j+1} & if \;\; m-1 < z \leq 2m-2 \\ & if \;\; 2m-2-z < j \leq m-1 \end{cases}$$

If $m < p < 2m,\ 4m-2p-1 \leq z \leq 2m-2,\ 0 \leq j \leq m-1$:

$$K^{\infty}_{m,p,z,j} = K^{\infty}_{p,p,2m-1,j} + K^{\infty}_{m,2m,z,j}\ 2^{\frac{2m-p}{2}} \left(\frac{p!}{(2p-2m)!}\right)^2 \left(\frac{\|\Delta\|}{\underline{\Delta}}\right)^{2m-p} \tag{11}$$

If $m < p \leq 2m,\ 4m-2p-1 \leq z \leq 2m-2,\ m \leq j \leq p-1$:

$$K^{\infty}_{m,p,z,j} = K^{\infty}_{p,p,p,j} + (K^{\infty}_{m,p,z,m-1} + K^{\infty}_{p,p,p,j})2^{j-m+1}\left(\frac{(2p-m)!}{(2p-j-1)!}\right)^2 \left(\frac{\|\Delta\|}{\underline{\Delta}}\right)^{j-m+1}$$

Theorem 3. *([8]) Let $x(t)$ be in $\mathcal{H}^{2m-1}(a,b)$ and let $\Delta : a = t_0 < t_1 < ... < t_N = b$ be a mesh of the interval. Let $S(x,t)$ be a spline of degree $(2m-1)$ to $x(t)$ on Δ satisfying $S^{(k)}(x,t_0) = x^{(k)}(t_0)$, $S^{(k)}(x,t_N) = x^{(k)}(t_N)$ for $0 \leq k \leq m-1$. Then we have*

$$\|x^{(k)}(t) - S^{(k)}(x,t)\|_\infty \leq K^{\infty}_{m,2m-1,2m-2,k}\ \|x^{(2m-1)}\|_2\ \|\Delta\|^{2m-\frac{3}{2}-k} \tag{12}$$

for $0 \leq k \leq 2m-2$, with

$$\|x^{(2m-1)}\|_2 = \left(\int_a^b (x^{(2m-1)}(t))^2 dt\right)^{1/2}$$

Consider the mapping

$$T : J \times \mathcal{F} \to \mathcal{F}$$

$$(\alpha, f) \to T_\alpha f$$

with $J = [0,r]$; $0 \leq r < 1$; r fixed and $[t_0, t_N] = I$. For $t \in I_n = [t_{n-1}, t_n]$ define

$$T_\alpha f(t) = F^\alpha_n(L^{-1}_n(t), f \circ L^{-1}_n(t)) = \alpha f \circ L^{-1}_n(t) + q^\alpha_n \circ L^{-1}_n(t) \tag{13}$$

The superscript α represents the dependence regarding the vertical scaling factor. The polynomial $q_n(t)$ is expressed in terms of α, that is to say, $q^\alpha_n(t) = q_n(\alpha,t)$. As previously asserted, the fixed point of T_α is the FIF (Barnsley's theorem).

Proposition 4. *Let $f \in \mathcal{F}$ and let $q_n(\alpha,t)$ be differentiable and such that $\exists\ D_0 \geq 0$ verifying $|\frac{\partial q_n}{\partial \alpha}(\xi,t)| \leq D_0\ \forall\ (\xi,t) \in J \times I$ and $\forall\ n = 1,2,...,N$. Then:*

$$\|T_\alpha f - T_\beta f\|_\infty \leq |\alpha - \beta|\ (\|f\|_\infty + D_0)$$

holds.

Proof. Let $f \in \mathcal{F}$, for each value $t\ \in I_n$:

$$|T_\alpha f(t) - T_\beta f(t)| = |\alpha f \circ L^{-1}_n(t) + q^\alpha_n \circ L^{-1}_n(t) - \beta f \circ L^{-1}_n(t) - q^\beta_n \circ L^{-1}_n(t)| \leq$$

$$|\alpha f \circ L^{-1}_n(t) - \beta f \circ L^{-1}_n(t)| + |q^\alpha_n \circ L^{-1}_n(t) - q^\beta_n \circ L^{-1}_n(t)|$$

The first term verifies the inequality:

$$|\alpha f \circ L^{-1}_n(t) - \beta f \circ L^{-1}_n(t)| \leq |\alpha - \beta|\ |f \circ L^{-1}_n(t)| \leq |\alpha - \beta|\ \|f\|_\infty \tag{14}$$

To bound the second term, the mean-value theorem for functions of several variables is applied. With the enunciated hypotheses $\exists\, \xi \in\; J$ such that

$$q_n(\alpha,\tilde{t}) - q_n(\beta,\tilde{t}) = \frac{\partial q_n}{\partial \alpha}(\xi,\tilde{t})(\alpha-\beta)$$

and therefore,

$$|q_n^{\alpha} \circ L_n^{-1}(t) - q_n^{\beta} \circ L_n^{-1}(t)| \leq D_0|\alpha-\beta| \tag{15}$$

The result is obtained from inequalities (14)-(15). □

Proposition 5. *Let f_α, f_β be fractal interpolation functions with vertical scaling factors α and β. Under the hypotheses of proposition 4, the following inequality holds:*

$$\|f_\alpha - f_\beta\|_\infty \leq \frac{1}{1-|\alpha|}\; |\alpha-\beta|\; (\|f_\beta\|_\infty + D_0)$$

Proof. By definition f_α, f_β are fixed points of T_α and T_β, respectively. Therefore $T_\alpha(f_\alpha) = f_\alpha$, $T_\beta(f_\beta) = f_\beta$. Applying the inequality (6) and the proposition 4:

$$\|f_\alpha - f_\beta\|_\infty = \|T_\alpha f_\alpha - T_\alpha f_\beta + T_\alpha f_\beta - T_\beta f_\beta\|_\infty \leq$$
$$\leq \|T_\alpha f_\alpha - T_\alpha f_\beta\|_\infty + \|T_\alpha f_\beta - T_\beta f_\beta\|_\infty \leq$$
$$\leq |\alpha|\; \|f_\alpha - f_\beta\|_\infty + |\alpha-\beta|(\|f_\beta\|_\infty + D_0)$$

From here:

$$\|f_\alpha - f_\beta\|_\infty \leq \frac{1}{1-|\alpha|}\; |\alpha-\beta|\; (\|f_\beta\|_\infty + D_0) \tag{16}$$

□

Remark 1. Setting $\beta = 0$ in (16)

$$\|f_\alpha - f_0\|_\infty \leq \frac{1}{1-|\alpha|}\; |\alpha|\; (\|f_0\|_\infty + D_0) \tag{17}$$

As previously explained, f_0 is a polynomial spline of degree $2m-1$ that interpolates the data points: $f_0 = S(x,t)$. One can bound $\|f_0\|_\infty$ applying the theorem of Schultz [8].
Denoting $C_0 = K^{\infty}_{m,2m-1,2m-2,0}\; \|x^{(2m-1)}\|_2$, and setting $k=0$ in (12):

$$\|f_0\|_\infty \leq C_0 h^{2m-\frac{3}{2}} + \|x\|_\infty \tag{18}$$

If $\|x\|_\infty = L_0$, from (17),(18)

$$\|f_\alpha - f_0\|_\infty \leq \frac{1}{1-|\alpha|}\; |\alpha|\; (C_0 h^{2m-\frac{3}{2}} + L_0 + D_0) \tag{19}$$

Theorem 6. *Interpolation error bound. Let $x(t)$ be a function verifying $x(t) \in \mathcal{H}^{2m-1}(t_0,t_N)$. Let $q_n(\alpha,t)$ be differentiable and such that $\exists\, D_0 \geq 0$ with $|\frac{\partial q_n}{\partial \alpha}(\xi,t)| \leq D_0\; \forall\, (\xi,t) \in J \times I$, $\forall\, n = 1,2,...,N$. Let $|\alpha| < \frac{1}{N^{2m-2}}$. Then*

$$\|x - f_\alpha\|_\infty \leq \frac{N^{2m-2}}{N^{2m-2}-1}[C_0 h^{2m-\frac{3}{2}} + \frac{(L_0+D_0)}{T^{2m-2}} h^{2m-2}]$$

being $L_0 = \|x\|_\infty, T = t_N - t_0$.

Proof.

$$\|x - f_\alpha\|_\infty \leq \|x - f_0\|_\infty + \|f_0 - f_\alpha\|_\infty$$

The first adding can be bounded applying the theorem of Schultz with $k = 0$:

$$\|x - f_0\|_\infty \leq C_0 h^{2m-\frac{3}{2}} \tag{20}$$

In the second term the remark of the proposition 5 is used (19):

$$\|f_0 - f_\alpha\|_\infty \leq \frac{1}{1-|\alpha|}\,|\alpha|\,(C_0 h^{2m-\frac{3}{2}} + L_0 + D_0) \tag{21}$$

From (20)-(21):

$$\|x - f_\alpha\|_\infty \leq \frac{1}{1-|\alpha|}\,[C_0 h^{2m-\frac{3}{2}} + |\alpha|(L_0 + D_0)]$$

By the hypotheses of the theorem of differentiability of fractal interpolation functions ([3]): $|\alpha| < \frac{1}{N^{2m-2}} = \frac{h^{2m-2}}{T^{2m-2}}$ and, therefore, $\frac{1}{1-|\alpha|} \leq \frac{N^{2m-2}}{N^{2m-2}-1}$, so the inequality above is transformed in:

$$\|x - f_\alpha\|_\infty \leq \frac{N^{2m-2}}{N^{2m-2}-1}[C_0 h^{2m-\frac{3}{2}} + \frac{(L_0 + D_0)}{T^{2m-2}} h^{2m-2}] \tag{22}$$

□

Following the theorem of Barnsley & Harrington, the derivatives $f^{(k)}$ of f are FIF corresponding to the IFS $\{(L_n(t), F_{nk}(t,x))\}_{n=1}^N$ with

$$F_{nk}(t,x) = N^k \alpha x + N^k q_n^{(k)}(t)$$

Consequently, the results above can be generalized to the first derivatives of f.

Proposition 7. *Let $f_\alpha^{(k)}$, $f_\beta^{(k)}$ be the $k-$th derivatives $(k = 0, 1, ..., 2m-2)$ of f_α and f_β respectively and let $\frac{\partial^k q_n}{\partial t^k}(\alpha, t)$ be differentiable and such that $\exists\ D_k \geq 0$ with $|\frac{\partial^{k+1} q_n}{\partial \alpha \partial t^k}(\xi, t)| \leq D_k$ $\forall\ (\xi, t) \in J \times I$ and $\forall\ n = 1, 2, ..., N$. Then:*

$$\|f_\alpha^{(k)} - f_\beta^{(k)}\|_\infty \leq \frac{N^k|\alpha - \beta|}{1 - N^k|\alpha|}\,(\|f_\beta^{(k)}\|_\infty + D_k)$$

holds.

Proof. Analogous to the proposition 5. □

Theorem 8. *Derivatives interpolation error bounds. Let $x(t)$ be a function verifying $x(t) \in \mathcal{H}^{2m-1}(t_0, t_N)$. Let $\frac{\partial^k q_n}{\partial t^k}(\alpha, t)$ be differentiable and $\exists\ D_k \geq 0$ such that $|\frac{\partial^{k+1} q_n}{\partial \alpha \partial t^k}(\xi, t)| \leq D_k$ $\forall\ (\xi, t) \in J \times I$ and $\forall\ n = 1, 2, ..., N$. Let $s = s(N)$ such that $0 < s < 1$ and $|\alpha| \leq \frac{1}{N^{2m-2+s}}$. Then:*

$$\|x^{(k)} - f_\alpha^{(k)}\|_\infty \leq \frac{N^{2m-2+s-k}}{N^{2m-2+s-k}-1}[C_k h^{2m-\frac{3}{2}-k} + \frac{(L_k + D_k)}{T^{2m-2+s-k}} h^{2m-2+s-k}]$$

for $k = 0, 1, ..., 2m-2$, being $L_k = \|x^{(k)}\|_\infty$, $h = t_n - t_{n-1}$, $T = t_N - t_0$ and $C_k = K_{m,2m-1,2m-2,k}^\infty\,\|x^{(2m-1)}\|_2$

Proof. By hypothesis $|\alpha| < \frac{1}{N^{2m-2}}$. Since $\frac{1}{N^{2m-2+x}} \to \frac{1}{N^{2m-2}}$ as $x \to 0^+$, there exists $s = s(N)$ such that $0 < s < 1$ and $|\alpha| \leq \frac{1}{N^{2m-2+s}}$. The rest is analogous to the theorem 6. □

§4. Conclusions

The bounds of error in the approximation by polinomial splines are generalized to differentiable polynomial Barnsley-Harrington functions. The error obtained is comparable to other precision procedures, as the interpolation by piecewise polynomials. The property of good fit of the derivatives is also verified here. The possible loss of precision is counterbalanced with the generality of the method, as the fractal interpolants contain the odd degree polynomial spline functions as a particular case. That extension is verified under preservation of the smoothness of the function (in the sense of continuity of the derivatives).

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