MULTIINDEXING POINTS FOR POISED SUBPROBLEMS IN MULTIVARIATE POLYNOMIAL INTERPOLATION

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Abstract. In this paper it is shown that any set of points that allows for unique interpolation by the vector space of all polynomials of at most a certain total degree can be equipped with multiindices in such a way that all the natural subproblems of lower degree are uniquely solvable as well.

Keywords: Multivariate polynomial interpolation, multiindexing interpolation points, term order

AMS classification: 65D05, 41A05

§1. Introduction

The polynomial interpolation problem is easily stated in the following way:

> *Given distinct points $x_1, \dots, x_N$ and an N–dimensional space $\mathcal{P}$ of polynomials find, for given values $y_1, \dots, y_N$, a unique polynomial $f \in \mathcal{P}$ such that*

$$f(x_j) = y_j, \qquad j = 1, \dots, N.$$

Clearly, the solvability or *poisedness* or *correctness* of the interpolation problem depends on the relationship between the set $\mathcal{X} = \{x_1, \dots, x_N\}$ and the space $\mathcal{P}$. In the univariate case there is a canonical choice for $\mathcal{P}$, namely the space Π_{N-1} of all polynomials of degree $\le N-1$ which is well–known to be poised for *any* choice of N distinct points. This is a classical issue, covered in many textbooks on Numerical Analysis.

In $d \ge 2$ variables, however, the situation is totally different. Here poisedness of the interpolation problem for the space Π_n of all polynomials of *total* degree $\le n$ is an intricate condition on the set $\mathcal{X}$ of interpolation points. This is not so much due to the requirement that the cardinality of $\mathcal{X}$ must coincide with the dimension $\binom{n+d}{d}$ of Π_n, but stems from the fact that poisedness relative to Π_n is equivalent to the intricate condition that the points must not lie on a nontrivial algebraic hypersurface of degree $\le n$, a condition that is hard to check and can only be ensured by very special constructions. For more information on multivariate polynomial interpolation and the history thereof, the reader is referred to the surveys [9, 10].

In this paper, we focus on another property that is trivially satisfied by univariate polynomial interpolation problems and that forms the basis for recursive schemes to compute the values of interpolation polynomials at a given point by recursing to subproblems of lower degree. Here we refer to the Aitken–Neville scheme introduced by Aitken [1] and modified a short while later by Neville in [11], both with the intention of providing means for a simpler and easier handling of tabulated values of functions. These schemes are based on repeated *linear interpolation/extrapolation* and, though they are neither the most efficient way to compute the value of the interpolation polynomial, cf. [7], nor a very stable way to so, they gained popularity because the scheme is very simple to perform. The process of linear interpolation can be interpreted as repeated *affine* or *barycentric combinations* of the intermediate data. This geometric aspect has been generalized to several variables in [17], but it turned out that the point configurations that permit such a recursive evaluation scheme must be of a very specific structure.

On the other hand, these restrictive configurations, relying on *multiindexed* interpolation points had a very appealing property: not only the full problem was poised but also all subproblems that arose from natural subsets of multiindices. The goal of this paper is to show that and how *any* interpolation problem that is poised with respect to Π_n can be rewritten in such a way that all those natural subproblems are poised as well. This will be done by a closer look at the process of *putting points into blocks* introduced in [16] as the basic building block for a multivariate Newton approach and the derivation of error formulae for polynomial interpolation.

We will revise this process after establishing the necessary notation in Section 2 and then give a precise statement of the problem together with some examples in Section 3. Finally, Section 4 will give a constructive, even algorithmic proof of the result by showing a particular way to compute the multiindexing and verifying that is has the desired properties.

§2. Notation and basics

We use standard multiindex notation for polynomials in d variables. Indeed, to a *multiindex* $\alpha = (\alpha_1, \dots, \alpha_d) \in \Gamma := \mathbb{N}_0^d$ the length

$$|\alpha| = \sum_{j=1}^{d} \alpha_j$$

is associated and by Γ_n we denote the set of all multiindices of length $\le n$, i.e.,

$$\Gamma_n := \{\alpha \in \Gamma \ : \ |\alpha| \le n\}, \qquad n \in \mathbb{N}_0.$$

By $\Pi = \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_d]$ we denote the ring of polynomials in d variables with real coefficients, that is, all expressions of the forms

$$f(x) = \sum_{\alpha \in \Gamma} f_\alpha \, x^\alpha, \qquad x^\alpha = \prod_{j=1}^{d} x_j^{\alpha_j},$$

where only *finitely many* of the coefficients f_α are different from zero. Moreover, we write

$$\Pi_n := \left\{ f \in \Pi \ : \ f(x) = \sum_{\alpha \in \Gamma_n} f_\alpha \, x^\alpha \right\}$$

for the vector space of all polynomials of *total degree* $\leq n$, $n \in \mathbb{N}_0$.

Let $\mathcal{X} = \{x_1, \ldots, x_N\}$, $N = \dim \Pi_n = \#\Gamma_n = \binom{n+d}{d}$ be a finite subset of $\mathbb{R}^d$ of proper cardinality. We say that *the interpolation problem based on* $\mathcal{X}$ *is poised with respect to* Π_n or simply that $\mathcal{X}$ is *poised for* Π_n if the interpolation is (uniquely) solvable for any prescribed values at the points in $\mathcal{X}$, that is, if for any $y_1, \ldots, y_N \in \mathbb{R}$ there exists a (unique) polynomial $f \in \Pi_n$ such that

$$f(x_j) = y_j, \qquad j = 1, \ldots, N.$$

Equivalently, this is the case if the *Vandermonde matrix*

$$V_n(\mathcal{X}) := \left[x_j^\alpha \; : \; j = 1, \ldots, N, \; \alpha \in \Gamma_n\right] \tag{1}$$

is nonsingular, i.e., has nonzero determinant. By the requirement on N the matrix $V_n(\mathcal{X})$ in (1) is a square one though it looks quite strange as rows and columns are indexed by different structural entities. To avoid this effect, one can either linearly index the multiindices or multiindex the points. Both approaches are more than cosmetic modifications, the first one leading to the use of *term orders* for multiindices or the associated monomials, respectively, while the second one leads to the concept of *putting points into blocks*, by rewriting $\mathcal{X}$ in the re–indexed or *multiindexed* way $\mathcal{X} = \{x_\alpha \; : \; \alpha \in \Gamma_n\}$. There are, of course, many ways to do the multiindexing, but it is important to note that this can be done in such a way that the induced total degree subproblems are poised as well as the following result from [16] shows.

Theorem 1. *If* $\mathcal{X}$ *is poised with respect to* Π_n*, it can be multiindexed in such a way that for* $k = 0, \ldots, n$ *the subsets* $\mathcal{X}_k := \{x_\alpha \; : \; \alpha \in \Gamma_k\}$ *are poised with respect to* Π_k.

Recall that this result leads to a Newton approach for the computation of the interpolation polynomial and that this process not only be done algorithmically, but has been implemented and tested as well, see [12] for more information.

§3. The main result

We now consider an extension of Theorem 1 to a wider variety of subsets. This is motivated by considering the Aitken–Neville scheme once more. In fact, the univariate Aitken–Neville scheme uses combinations of the interpolation polynomials of degree k with respect to the points $x_j, \ldots, x_{j+k}$ and $x_{j+1}, \ldots, x_{j+k+1}$ to compute (the value of) the interpolation polynomial of degree $k+1$ with respect to the points $x_j, \ldots, x_{j+k+1}$, $j + k \leq n - 1$. Since *all* the points were assumed to be distinct, so are these subsets as well and consequently the interpolation subproblems are trivially poised.

In the multivariate Aitken–Neville scheme from [17], on the other hand, the subproblems to be considered are the ones based on the index sets

$$\alpha + \Gamma_k := \{\alpha + \beta \; : \; \beta \in \Gamma_k\}, \qquad |\alpha| + k \leq n,$$

and these sets are obviously of a more intricate nature. In particular, their poisedness with respect to Π_k is all but obvious and depends on the way how the points are multiindexed. Nevertheless, we have the following result from which Theorem 1 follows from immediately by choosing $\alpha = 0$.

Theorem 2. *If $\mathcal{X}$ is poised with respect to Π_n then it can be multiindexed in such a way that all the subsets*

$$\mathcal{X}_{\alpha+\Gamma_k} := \{x_\beta \ : \ \beta \in \alpha + \Gamma_k\} \tag{2}$$

are poised with respect to Π_k whenever $|\alpha| + \Gamma_k \le n$.

Before we prove the result in section 4, let us briefly illustrate its meaning by looking at the simplest possible and most naturally multiindexed configuration, namely $x_\alpha = \alpha$. This

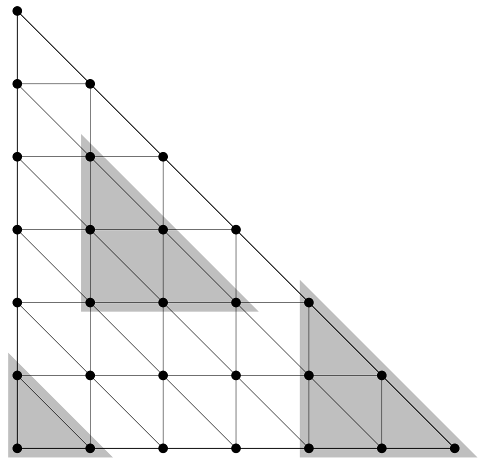

Figure 1: The points $x_\alpha = \alpha$, $\alpha \in \Gamma_6$, and some natural subsets thereof.

triangular grid configuration, depicted in Fig. 1 has already been considered in the earliest references on multivariate (or, to be precise, bivariate) polynomial interpolation [2, 18] as it can still be treated by tensor product methods. Since the index sets of the form $\alpha + \Gamma_k$ are clearly isomorphic to Γ_k, the subsets $\mathcal{X}_{\alpha+\Gamma_k}$ are in this case the natural subsets of lower degree among $\mathcal{X}$ with the same structure as the original index set.

§4. Construction and proof

Suppose that a set $\mathcal{X}$ of interpolation points of cardinality $N = \binom{n+d}{d}$ is given. We first describe the algorithm to multiindex the points. To that end, we let $\prec$ denote any degree compatible *term order*. Recall that a term order is a well–ordering on (the free monoid) Γ such that $\alpha \prec \beta$ implies $\alpha + \gamma \prec \beta + \gamma$ for any $\gamma \in \Gamma$, and that degree compatibility requires that $|\alpha| < |\beta|$ implies $\alpha \prec \beta$ for any $\alpha, \beta \in \Gamma$. To be specific, we can choose the *graded lexicographical* term order that defines $\alpha \prec \beta$ iff either $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and there exists an index $1 \le j < d$ such that $\alpha_k = \beta_k$, $k = 1, \dots, j-1$, and $\alpha_j < \beta_j$. Let, in addition, $\Lambda(f)$ denote the leading term of a polynomial $f \in \Pi$ with respect to the term order $\prec$, i.e.,

$$\Lambda(f) = f_\delta \, x^\delta, \qquad \delta := \max_{\prec} \{\alpha \in \Gamma \ : \ f_\alpha \neq 0\}.$$

The multiindex $\delta = \delta(f)$ is often called the *multidegree* of f, cf. [8]. A fundamental property of multidegree and leading term is the fact that for any two polynomials $f, g \in \Pi$ one has

$$\delta(fg) = \delta(f) + \delta(g) \qquad \text{and} \qquad \Lambda(fg) = \Lambda(f) + \Lambda(g). \tag{3}$$

Lemma 3. *If the point set $\mathcal{X}$ is poised for Π_n then it can be indexed such that there exist polynomials $p_\alpha, q_\alpha \in \Pi_{|\alpha|}$, $\alpha \in \Gamma_n$, that have the properties*

$$p_\alpha \left(x_\beta\right) = \delta_{\alpha,\beta}, \qquad |\beta| \le |\alpha|, \tag{4}$$

and

$$q_\alpha \left(x_\beta\right) = \delta_{\alpha,\beta}, \qquad \beta \prec \alpha, \tag{5}$$

respectively.

The *Newton fundamental polynomials* p_α from (4) were constructed by a Gram–Schmidt orthogonalization method in [16], but the process can also be interpreted as a Gaussian elimination on the Vandermonde matrix with pivoting by points. Indeed, the difference between (4) and (5) consists in choosing the upper triangular matrix U in the LU–factorization block upper triangular with identity matrices on the diagonal or simply upper triangular, see also [4] where this is called *Gauß elimination by blocks*.

Proof. Let α^k, $k = 1, \dots, N$, denote the multiindices in Γ_n, ordered according to "$\prec$" and set $f_k(x) = x^{\alpha^k}$. Then the matrix

$$\left[f_k \left(x_j\right) \ : \ j, k = 1, \dots, N\right]$$

is nonsingular and Gauß elimination with piont (column) interchanging (if needed) computes a decomposition

$$\left[f_k \left(x_j'\right) \ : \ j, k = 1, \dots, N\right] = LU, \quad \text{where} \quad L_{kk} \neq 0,\, U_{kk} = 1, \quad k = 1, \dots, N,$$

and $\mathcal{X} = \{x_1', \dots, x_N'\}$. Then the polynomials q_α are defined as

$$q_{\alpha^k} = \left(L^{-1} \left[f_j \ : \ j = 1, \dots, N\right]\right)_k, \qquad k = 1, \dots, N,$$

and satisfy

$$\Lambda \left(q_{\alpha^k}\right)(x) = L_{kk}^{-1}\, x^{\alpha^k}, \qquad k = 1, \dots, N, \tag{6}$$

as well as (5). In particular, for $k = 0, \dots, n$ the matrix

$$\left[q_\alpha \left(x_\beta\right) \ : \ |\alpha| = |\beta| = k\right]$$

ordered with respect to $\prec$ is an upper triangular matrix U_k with all diagonal entries being equal to one and so we define

$$p_\alpha = \left(U_{|\alpha|}^{-1} \left[q_\beta \ : \ |\beta| = |\alpha|\right]\right)_\alpha, \qquad \alpha \in \Gamma_n,$$

to obtain the polynomials p_α that satisfy (4). $\square$

A direct consequence of equation (6) in the proof above is the following observation.

Corollary 4. *For $\alpha \in \Gamma_n$ the polynomials q_α satisfy $\Lambda\left(q_\alpha\right)(x) = c_\alpha\, x^\alpha$ where $q_\alpha \neq 0$.*

Another interesting observation in the proof is that a rearrangement of points in the Gauß elimination process is necessary if an only if a pivot element during the elimination becomes zero, otherwise *any* arbitrary arrangement of the interpolation points would do. However, if the situation of zero pivot occurs, then an arbitrarily small translation of the point in question results in a nonzero, though still very small and numerically inacceptable, pivot element. In other words: in the "generic case" the actual way how the points are arranged is completely irrelevant, at least from a theoretical point of view. Observe, however, that the "nicely structured" point configuration as the one in Fig. 1 is the opposite of general position as there are many points lying on lines and intersections thereof. The same holds true for the Aitken–Neville configurations from [17] which inherit the intersection structure of the multiindex grid.

Now we can state and prove the following result that immediately verifies Theorem 2.

Proposition 5. *The multiindexing of points given in the proof of Lemma 3 has the property that $\mathcal{X}_{\alpha+\Gamma_k}$ is poised with respect to Π_k for any $\alpha \in \Gamma_{n-k}$ and $k = 0, \dots, n$.*

Proof. The proof will be done by contradiction. Assuming that there exist $0 \le k \le n$ and $\alpha \in \Gamma_{n-k}$ such that $\mathcal{X}_{\alpha+\Gamma_k}$ is not poised for Π_k, we will show that $\mathcal{X}_{|\alpha|+k} = \mathcal{X}_{0+\Gamma_{|\alpha|+k}}$ is not poised for $\Pi_{|\alpha|+k}$. Since (4) implies that $\mathcal{X}_j$ is poised for Π_j, $j = 0, \dots, n$, whenever $\mathcal{X}$ is poised for Π_n, see also Theorem 1, this then yields a contradiction.

So, suppose that $\mathcal{X}_{\alpha+\Gamma_k}$ is not poised for Π_k and that α is minimal with respect to $\prec$ among all such possibilities. In addition, let k be chosen minimally as well and set $m := |\alpha| + k$. Consequently, there exists a polynomial $q \in \Pi_k \setminus \Pi_{k-1}$ such that

$$q\left(\mathcal{X}_{\alpha+\Gamma_k}\right) = 0, \qquad \text{and} \qquad \Lambda(q) \in \Pi_k.$$

Define $f := q_\alpha \, q$, then $f \in \Pi_m$ satisfies

$$\Lambda(f) = \Lambda\left(q_\alpha\right)\, \Lambda(q) \in \Pi_m \qquad \text{and} \qquad f\left(\mathcal{X}_{|\alpha|} \cup \mathcal{X}_{\alpha+\Gamma_k}\right) = 0.$$

Set $\Gamma' := \Gamma_m \setminus \left(\Gamma_{|\alpha|} \cup \alpha + \Gamma_k\right)$. Because of (5) there exist coefficients c_β, $\beta \in \Gamma'$, such that

$$g := f - \sum_{\beta \in \Gamma'} c_\beta \, q_\beta$$

vanishes at all of $\mathcal{X}_m$, i.e., $g\left(\mathcal{X}_m\right) = 0$. In particular, it follows that $\Lambda(g) \preceq \Lambda(f)$ with equality if and only if $f\left(x_\beta\right) = 0$ for all $\beta \in \Gamma_m$ such that $\beta \prec \alpha + \Gamma_k$. Since $\Lambda(f) \in \Pi_m$ and $g \in \Pi_m$, this implies that g cannot be the zero polynomial and therefore $\mathcal{X}_m$ cannot be poised with respect to Π_m which yields the desired contradiction. $\square$

Indeed, Theorem 2 can be used as a characterization of poised configurations in terms of poised subproblems.

Corollary 6. *A set $\mathcal{X} \subset \mathbb{R}^d$ of cardinality $\binom{n+d}{d}$ is poised with respect to Π_n if and only if it can be multiindexed in such a way that all the subsets $\mathcal{X}_{\alpha+\Gamma_k}$, $\alpha \in \Gamma_{n-k}$, $k = 0, \dots, n$, are poised with respect to Π_k.*

§5. Minimal degree interpolation

The above result can be extended to a more general situation, namely to minimal degree interpolation, cf. [13], where the idea is to choose the interpolation space according to the point set. This approach has been pursued first by de Boor and Ron in [3] by a scheme the named *least interpolation*, see also [5] for further details. Algebraically, all the minimal degree degree interpolation approaches can be viewed as interpolation by normal forms with respect to Gröbner or H-bases, respectively, for the ideal

$$I_{\mathcal{X}} = \{ f \in \Pi \ : \ f(\mathcal{X}) = 0 \},$$

cf. [14, 15]. In this presentation here, we will only consider the case of what has been called "minimal interpolation with *minimal monomials*" relative to a term order "$\prec$" in the terminology of [13]. For that end, let $\Theta \subset \Gamma$ be a finite set of multiindices and denote by

$$\Pi_\Theta := \left\{ f(x) = \sum_{\alpha \in \Theta} f_\alpha \, x^\alpha \ : \ f_\alpha \in \mathbb{R} \right\}$$

the polynomial subspace spanned by the set of monomials x^Θ. It has been shown in [13] that for any set $\mathcal{X}$ of distinct interpolation points of *any* cardinality N there exists an index set $\Theta \subset \Gamma$, $\#\Theta = N$, such that $\mathcal{X}$ is poised with respect to Θ and Θ is $\prec$–minimal in the following sense: for any $\alpha \in \Gamma \setminus \Theta$ satisfying $\alpha \prec \max_\prec \Theta$ there exists a polynomial $f \in \Pi_{\Theta \cup \{\alpha\}}$ such that

$$\delta(f) = \alpha, \qquad \text{and} \qquad f(\mathcal{X}) = 0.$$

The construction of Θ is most conveniently done by a Gram–Schmidt orthogonalization process in the following way.

Algorithm 1. For given $\mathcal{X} \subset \mathbb{R}^d$

1. initialize $\Theta \leftarrow \emptyset$, $\mathcal{X}' \leftarrow \mathcal{X}$, $q_\beta(x) = x^\beta$, $\beta \in \Gamma_{|\mathcal{X}|}$, and $\alpha = 0$.
2. While $\mathcal{X}' \neq \emptyset$
   (a) If $q_\alpha(\mathcal{X}') \neq 0$ then
      i. choose $x_\alpha \in \mathcal{X}'$ such that $q_\alpha(x_\alpha) \neq 0$.
      ii. Set $q_\alpha \leftarrow \frac{q_\alpha}{q_\alpha(x_\alpha)}$, $\mathcal{X}' \leftarrow \mathcal{X}' \setminus \{x_\alpha\}$ and $\Theta \leftarrow \Theta \cup \{\alpha\}$.
      iii. For $\beta \in \Gamma_{|\mathcal{X}|}$, $\beta \succ \alpha$, set $q_\beta \leftarrow q_\beta - q_\beta(x_\alpha)\, q_\alpha$.
   (b) If $q_\alpha(\mathcal{X}') \neq 0$ then q_α is an element of a Gröbner basis for $I_{\mathcal{X}}$.

Result: index set Θ and points indexed as $\mathcal{X} = \{x_\alpha \ : \ \alpha \in \Theta\}$.

Remark 1. This algorithm from [13] serves a double purpose: it computes fundamental interpolation polynomials q_α, $\alpha \in \Theta$, of multidegree α such that

$$q_\alpha(x_\beta) = \delta_{\alpha,\beta}, \qquad \alpha, \beta \in \Theta, \quad \beta \prec \alpha, \tag{7}$$

from which the Newton fundamental polynomials p_α, $\alpha \in \Theta$, with

$$p_\alpha(x_\beta) = \delta_{\alpha,\beta}, \qquad \alpha, \beta \in \Theta, \quad |\beta| \leq |\alpha|, \tag{8}$$

can be obtained like in Lemma 3, but at the same time the algorithm also computes a (restricted) Gröbner basis for the radical ideal $I_{\mathcal{X}}$. For the latter purpose, an almost identical algorithm has already been used in [6].

Since the polynomials q_α again have leading term $c_\alpha x^\alpha$, $c_\alpha \neq 0$, we can use a refined version of the argumentation as in Proposition 5 to obtain the following result.

Theorem 7. *With the point arrangement $\mathcal{X} = \{x_\alpha \;:\; \alpha \in \Theta\}$ from Algorithm 1 we have that for any $\alpha \in \Theta$ and $k \in \mathbb{N}_0$ the point sets*

$$\mathcal{X}^\Theta_{\alpha+\Gamma_k} := \{x_\beta \;:\; \beta \in (\alpha + \Gamma_k) \cap \Theta\}$$

are poised for

$$\Pi^\Theta_{\alpha+\Gamma^k} = \{p \in \Pi_k \;:\; x^\alpha\, p(x) \cap \Pi_\Theta \neq \emptyset\}\,. \tag{9}$$

Here, (9) has to be understood in the sense that the polynomial $x^\alpha p(x)$ has at least one nonzero coefficient with an index in Θ.

Proof. Observe that the construction in Algorithm 1 ensures that for $\alpha \in \Theta$ we have $q_\alpha \in \Pi_\Theta$ and that $\delta\,(q_\alpha) = \alpha$. Therefore, the polynomial space spanned by q_β, $\beta \preceq \alpha$, is poised for the points x_β, $\beta \preceq \alpha$. Assume again that there exist $\alpha \in \Theta$, a minimal $k \in \mathbb{N}_0$ and a nontrivial polynomial $q \in \Pi_k$ such that $q\left(\mathcal{X}^\Theta_{\alpha+\Gamma_k}\right) = 0$ and $x^\alpha q(x) \cap \Pi_\Theta \neq \emptyset$. Let $\alpha^* \in \Theta$ denote the maximal one of them and note that $|\alpha^*| = |\alpha| + k$, $\alpha^* \succ \alpha$ as well as $\alpha^* = \delta\,(q')$, where q' denotes the *reduction* of $x^\alpha q(x)$ modulo $I_{\mathcal{X}}$ or, more precisely, with respect to a Gröbner basis of $I_{\mathcal{X}}$ (for example the one computed as a by–product of Algorithm 1) according to "$\prec$", cf. [8]. Set $f = q_\alpha\, q$ and observe that usually f' will *not* belong to Π_Θ any more as this vector space is not closed under multiplication.

Let f' be the reduction of f modulo $I_{\mathcal{X}}$ then $f' \in \Pi_\Theta$ and $\delta\,(f') = \delta\,(q') = \alpha^*$. By subtracting suitable multiples of q_β, $\beta \prec \alpha^*$, we can modify f' in such a way that

$$f'\,(x_\beta) = 0, \qquad \beta \prec \alpha^*,$$

while still $f \in \Pi_\Theta$ and $\delta\,(f') = \alpha^*$. But this contradicts (7) which ensures for any $\alpha \in \Theta$ that the polynomial spaces spanned by the monomials x^β, $\beta \in \Theta$, $\beta \preceq \alpha$ are poised for the associated point sets x_β, $\beta \in \Theta$, $\beta \preceq \alpha$. □

Combining Theorem 7 with the results from [13, 14], we find that Π_Θ is a "good" interpolation space for $\mathcal{X}$.

Corollary 8. *The space Π_Θ is a degree reducing interpolation space for the point set $\mathcal{X}$. It is even the vector space of all normal forms modulo the unique reduced Gröbner basis for the ideal $I_{\mathcal{X}}$ with respect to the degree compatible term order "$\prec$".*

Acknowledgment

I want to thank Mariano Gasca, Jesus Carnicer and the conference organizers for making it possible for me to participate at the meeting in Jaca as well as Christophe Rabut for very interesting and stimulating discussions.

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