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BLENDING SURFACES BY SMOOTHING PDE SPLINES

M. Pasadas and M.L. Rodríguez

Abstract. This work is concerned with how we can mix conditions of both interpolation and approximation in order to find a blending surface joining two or more surfaces when approximating a given data point set, and modelled from a certain partial differential equation. We establish a variational characterization for the solution of this problem and we establish some convergence result. Finally, we discretize this problem in a finite element space.

Keywords: surfaces, PDE, boundary value problem, approximation, fitting, finite elements

AMS classification: AMS 65D07, 65D10, 65D17

§1. Introduction

A *blending surface* is a surface that smoothly joins two or more given surfaces. In CAGD, solid objects are often represented by their boundary surfaces. These boundary surfaces can be classified into two types:

- *Primary surfaces*, that mainly define the shape of the objects, and
- *Blending surfaces*, that are important because they erase sharp edges and corners which are undesirable for functionality or aesthetic reasons.

The problem of constructing a system of surfaces that meet smoothly arises in a wide variety of applications in CAGD. For example, in designing a gate valve, sharp edges are avoided since they retard the fluid flow.

This problem appears particularly in the automotive and aerospace industries too, where the manufactured objects are designed from some interpolation and approximation data, and also when verifying some hydrodynamic properties that can be modelled by certain differential equations.

So, automatically constructing blending surfaces is important to facilitate the design process.

There are many possibilities to generate blending surfaces. In the 80's many authors like Warren[10] and Hopcroft and Hoffmann[6, 7, 8] studied blending surfaces from an algebraic viewpoint.

However, many problems in engineering, architecture, geology and other fields require smooth curves and surfaces whose shape cannot be described by an elementary representation. Because of this, new methods have been introduced.

Bloor and Wilson[1, 2, 3, 4] pioneered another modelling technique for the generation of blending surfaces, the PDE method, that generates a smooth surface by solving a partial differential equation with some given boundary conditions.

Greiner[5] presents an iterative procedure to obtain blending curves or surfaces.

This work is concerned with how we can mix conditions of both interpolation and approximation in order to find a surface that verifies some boundary conditions when approximating a given data point set and which is modelled from a certain partial differential equation. In [9], we studied a similar problem for explicit curves introducing the concept of ODE curve.

The paper is organized as follows. In Section 2, we briefly recall some preliminary notations. In Section 3, we define and characterize the notion of PDE surface. Section 4 is dedicated to the formulating of two convergence results. Section 5 is devoted to studying the associated discrete problem in a space of finite element. Finally, in Section 5 we present some graphical examples that state the efficiency of the method.

§2. Preliminaries

We shall use the following notations and assumptions:

- Ω is a bounded nonempty domain of $\mathbb{R}^2$;
- the Euclidean norm and inner product in $\mathbb{R}^3$ will be denoted by $\langle\cdot\rangle$ and $\langle\cdot,\cdot\rangle$ respectively.
- $L^2(\Omega;\mathbb{R}^3)$ stands for the linear space of real Lebesgue measurable functions such that $\int_\Omega \langle \mathbf{u}(\mathbf{x})\rangle^2 d\mathbf{x} < +\infty$;
- for each $n \in \mathbb{N}$, we designate by $H^n(\Omega;\mathbb{R}^3)$ the usual Sobolev space of (classes of) functions $\mathbf{u} \in L^2(\Omega;\mathbb{R}^3)$, together with all their partial derivatives $\partial^{\mathbf{i}}\mathbf{u}$, in the distribution sense, of order $|\mathbf{i}| \leq n$, where for all $\mathbf{i} = (i_1, i_2) \in \mathbb{N}^2$, $|\mathbf{i}| = i_1 + i_2$ and $\partial^{\mathbf{i}}\mathbf{u}(\mathbf{x}) = \dfrac{\partial^{|\mathbf{i}|}\mathbf{u}}{\partial x_1^{i_1}\partial x_2^{i_2}}$, for any $\mathbf{x} = (x_1, x_2) \in \Omega$;
- H_0^n will be denote the closure of $C_0^\infty(\Omega;\mathbb{R}^3)$ in $H^n(\Omega;\mathbb{R}^3)$.
  Obviously $H_0^0(\Omega;\mathbb{R}^3) = L^2(\Omega;\mathbb{R}^3)$.

The linear space $L^2(\Omega;\mathbb{R}^3)$ is equipped with the inner product $(\mathbf{u},\mathbf{v})_0 = \displaystyle\int_\Omega \langle \mathbf{u}(\mathbf{x}),\, \mathbf{v}(\mathbf{x})\rangle d\mathbf{x}$ and the corresponding norm $|\mathbf{u}|_0 = (\mathbf{u},\mathbf{u})_0^{\frac{1}{2}}$.

The Sobolev space $H^n(\Omega;\mathbb{R}^3)$ is equipped with the inner product

$$((\mathbf{u},\mathbf{v}))_n = \sum_{|\mathbf{i}|\leq n} \int_\Omega \langle \partial^{\mathbf{i}}\mathbf{u}(\mathbf{x}),\, \partial^{\mathbf{i}}\mathbf{v}(\mathbf{x})\rangle d\mathbf{x},$$

the corresponding norm $\|\mathbf{u}\|_n = ((\mathbf{u},\mathbf{u}))_n^{\frac{1}{2}}$, the semi–inner products

$$(\mathbf{u},\mathbf{v})_l = \sum_{|\mathbf{i}|=l} \int_\Omega \langle \partial^{\mathbf{i}}\mathbf{u}(\mathbf{x}),\, \partial^{\mathbf{i}}\mathbf{v}(\mathbf{x})\rangle\, d\mathbf{x},\ 0 \le l \le n,$$

and the corresponding semi–norms $|\mathbf{u}|_l = (\mathbf{u},\mathbf{u})_l^{\frac{1}{2}}$, for all $0 \le l \le n$.

We use the notation $\dfrac{\partial^j \mathbf{v}}{\partial \mathbf{n}^j}(\mathbf{x}) = D^j\mathbf{v}(\mathbf{x})\ (\mathbf{n}(\mathbf{x}), \overset{j)}{\ldots}, \mathbf{n}(\mathbf{x}))$ for all $j \in \mathbb{N}$, where $D^j\mathbf{v}(\mathbf{x})$ is the j–th Fréchet derivative of $\mathbf{v}$ and $\mathbf{x}$ is the unit outer normal vector at $\mathbf{x} \in \Omega$. For $j = 0$, $\dfrac{\partial^0 \mathbf{v}}{\partial \mathbf{n}^0}(\mathbf{x})$ indicates $\mathbf{v}(\mathbf{x})$.

Moreover, we denote by $(\mathbb{R}^k)^{k,l}$ the space of the real vectorial matrices with k rows and l columns, with the inner product and corresponding norm

$$\langle\langle A, B\rangle\rangle_{k,l} = \sum_{i=1}^{k}\sum_{j=1}^{l} \langle a_{ij}, b_{ij}\rangle, \qquad \langle\langle A\rangle\rangle_{k,l} = \langle\langle A, A\rangle\rangle_{k,3}^{\frac{1}{2}}$$

with $A = (\mathbf{a}_{ij})_{\substack{1\le i\le k\\ 1\le j\le 3}}$ and $B = (\mathbf{b}_{ij})_{\substack{1\le i\le k\\ 1\le j\le 3}}$. If $l = 1$ we write $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_k$ and $\langle\langle\,\cdot\,\rangle\rangle_k$ instead to $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle_{k,1}$ and $\langle\langle\,\cdot\,\rangle\rangle_{k,1}$, respectively.

Finally, for two real vectorial matrix $A = (\mathbf{a}_{ij}) \in (\mathbb{R}^3)^{k,l}$ and $B = (\mathbf{b}_{ij}) \in (\mathbb{R}^3)^{l,p}$ we denote by $AB \in \mathbb{R}^{k,p}$ the real matrix given by $AB = \left(\sum_{s=1}^{l} \langle \mathbf{a}_{is}, \mathbf{b}_{sj}\rangle\right)_{\substack{1\le i\le k\\ 1\le j\le p}}$.

Let $n \ge 1$ and let $\mathbf{L} : H^{2n}(\Omega;\mathbb{R}^3) \to L^2(\Omega;\mathbb{R}^3)$ be a differential operator given by

$$\mathbf{Lu}(\mathbf{x}) = \sum_{|\mathbf{i}|,|\mathbf{j}|\le n} (-1)^{|\mathbf{j}|}\partial^{\mathbf{j}}(\mathbf{p_{ij}}(\mathbf{x})\, \partial^{\mathbf{i}}\mathbf{u}(\mathbf{x})),\ \mathbf{x} \in \Omega, \tag{1}$$

where $\mathbf{p_{ij}} \in C^{|\mathbf{j}|}(\Omega;\mathbb{R}^3)$ and $\mathbf{p_{ij}} = \mathbf{p_{ji}}$, for all $|\mathbf{i}|, |\mathbf{j}| \le n$.

We note that we can write

$$\mathbf{Lu} = (Lu_i)_{1\le i\le 3},\ \text{where } Lu_i = \sum_{|\mathbf{i}|,|\mathbf{j}|\le n} (-1)^{|\mathbf{j}|}\partial^{\mathbf{j}}((p_i \circ \mathbf{p_{ij}})\,(p_i \circ \partial^{\mathbf{i}}\mathbf{u})),$$

with p_i, $i = 1, 2, 3$, the orthogonal projections of $\mathbb{R}^3$ into $\mathbb{R}$.

Now, we consider the symmetric bilinear form associated with $\mathbf{L}$ defined on $H^n(\Omega;\mathbb{R}^3) \times H^n(\Omega;\mathbb{R}^3)$ by

$$(\mathbf{u},\mathbf{v})_L = \sum_{|\mathbf{i}|,|\mathbf{j}|\le n} (\mathbf{p_{ij}}\partial^{\mathbf{i}}\mathbf{u}, \partial^{\mathbf{j}}\mathbf{v})_0$$

and we assume that

$$\sum_{|\mathbf{i}|,|\mathbf{j}|\le n-1} \boldsymbol{\xi}^{\mathbf{i}}(p_k \circ \mathbf{p_{ij}})(\mathbf{x})\boldsymbol{\xi}^{\mathbf{j}} \ge 0,\ \forall\, \mathbf{x} \in \Omega,\ k = 1, 2, 3, \tag{2}$$

and that there exists $\nu > 0$ such that

$$\sum_{|\mathbf{i}|,|\mathbf{j}|= n} \boldsymbol{\xi}^{\mathbf{i}}(p_k \circ \mathbf{p_{ij}})(\mathbf{x})\boldsymbol{\xi}^{\mathbf{j}} \ge \nu\langle\boldsymbol{\xi}\rangle_2^{2n}, \forall\, \mathbf{x} \in \Omega,\ k = 1, 2, 3, \tag{3}$$

for all $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$, where $\boldsymbol{\xi}^{\mathbf{i}} = \xi_1^{i_1}\xi_2^{i_2}$, for any $\mathbf{i} = (i_1, i_2) \in \mathbb{N}^2$, and $\langle \cdot \rangle_2$ indicates the Euclidean norm in $\mathbb{R}^2$.

Due to (3), the differential operator $\mathbf{L}$ is said to be strongly elliptic on Ω.

It can be easily shown that under the hypotheses (2)–(3) the bilinear form $(\cdot, \cdot)_L$ defines a semi–inner product on $H^n(\Omega; \mathbb{R}^3)$ whose associated semi–norm is denoted by $|\mathbf{u}|_L = (\mathbf{u}, \mathbf{u})_L^{\frac{1}{2}}$.

In addition, we suppose that $(\cdot, \cdot)_L$ is coercive on $H_0^n(\Omega; \mathbb{R}^3)$, that is, there exists $C > 0$ such that

$$(\mathbf{u}, \mathbf{u})_L \geq C\|\mathbf{u}\|_n^2, \ \forall\, \mathbf{u} \in H_0^n(\Omega; \mathbb{R}^3).$$

§3. Formulation of the problem

We introduce the following assumptions. Suppose we are given:

- the functions $\mathbf{f} \in L^2(\Omega; \mathbb{R}^3)$ and $\mathbf{h}_j \in C(\overline{\Omega}; \mathbb{R}^3)$, for $j = 0, \ldots, n-1$;
- a set $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ of $m = m(r)$ distinct points of Ω, with $r \in \mathbb{N}$;
- a set $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ of N distinct points of $\partial\Omega$;
- a data vector $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m) \in \mathbb{R}^{3m}$.

Now, we suppose that

$$A \cup B \quad \text{contains a } \mathbb{P}_{n-1}(\Omega; \mathbb{R}^3)\text{–unisolvent subset,}$$

where $\mathbb{P}_{n-1}(\Omega; \mathbb{R}^3)$ is the linear space of polynomial functions defined from Ω into $\mathbb{R}^3$ of total degree less than $n-1$.

Moreover, we define the operators

$$\boldsymbol{\rho} : H^n(\Omega; \mathbb{R}^3) \to (\mathbb{R}^3)^m, \qquad \boldsymbol{\tau} : H^n(\Omega; \mathbb{R}^3) \cap C^{n-1}(\overline{\Omega}; \mathbb{R}^3) \to (\mathbb{R}^3)^{N,n}$$

given by

$$\boldsymbol{\rho}\mathbf{v} = (\mathbf{v}(\mathbf{a}_i))_{1 \leq i \leq m}, \qquad \boldsymbol{\tau}\mathbf{v} = \left(\frac{\partial^j \mathbf{v}}{\partial \mathbf{n}^j}(\mathbf{b}_i)\right)_{\substack{1 \leq i \leq N \\ 0 \leq j \leq n-1}},$$

the convex set

$$H = \{\mathbf{u} \in H^n(\Omega; \mathbb{R}^3) \cap C^{n-1}(\overline{\Omega}; \mathbb{R}^3) : \ \boldsymbol{\tau}\mathbf{u} = \mathbf{y}\}$$

where $\mathbf{y} = (\mathbf{h}_j(\mathbf{b}_i))_{\substack{1 \leq i \leq N \\ 0 \leq j \leq n-1}}$, and the linear space

$$H_0 = \{\mathbf{u} \in H^n(\Omega; \mathbb{R}^3) \cap C^{n-1}(\overline{\Omega}; \mathbb{R}^3) : \ \boldsymbol{\tau}\mathbf{u} = \mathbf{0}\}.$$

Finally, we consider the boundary value problem

$$\left| \begin{array}{l} \mathbf{L}\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \ \mathbf{x} \in \Omega, \\ \dfrac{\partial^j \mathbf{u}}{\partial \mathbf{n}^j}(\mathbf{x}) = \mathbf{h}_j(\mathbf{x}), \ \mathbf{x} \in \partial\Omega, \ 0 \leq j \leq n-1, \end{array} \right. \tag{4}$$

where $\mathbf{L}$ is the differential operator defined in (1).

Definition 1. Given $\varepsilon > 0$, we say that $\boldsymbol{\sigma} \in H$ is a *PDE spline* associated to (4), A, B, $\boldsymbol{\beta}$, $\mathbf{y}$ and ε if $\boldsymbol{\sigma}$ is a solution of the problem

$$\forall\, \mathbf{v} \in H,\ J(\boldsymbol{\sigma}) \le J(\mathbf{v}), \tag{5}$$

where J is the functional defined on $H^n(\Omega;\mathbb{R}^3)$ by

$$J(\mathbf{v}) = \langle\langle \boldsymbol{\rho}\mathbf{v} - \boldsymbol{\beta}\rangle\rangle_m^2 + \varepsilon(|\mathbf{v}|_L^2 - 2(\mathbf{f},\mathbf{v})_0). \tag{6}$$

Now, we present a variational characterization of the PDE surfaces.

Theorem 1. *Problem (5) has a unique solution which is characterized as the unique solution of the following variational problem: Find $\boldsymbol{\sigma} \in H$ such that*

$$\forall\, \mathbf{v} \in H_0,\ \langle\langle \boldsymbol{\rho\sigma}, \boldsymbol{\rho}\mathbf{v}\rangle\rangle_m + \varepsilon(\boldsymbol{\sigma},\mathbf{v})_L = \langle\langle \boldsymbol{\beta}, \boldsymbol{\rho}\mathbf{v}\rangle\rangle_m + \varepsilon(\mathbf{f},\mathbf{v})_0.$$

Analogously, we present a method of Lagrangian multipliers to solve Problem (5).

Theorem 2. *There exists a unique $(\boldsymbol{\sigma},\boldsymbol{\lambda}) \in H \times (\mathbb{R}^3)^{N,n}$ such that*

$$\langle\langle \boldsymbol{\rho}\sigma, \boldsymbol{\rho}\mathbf{v}\rangle\rangle_m + \varepsilon(\boldsymbol{\sigma},\mathbf{v})_L + \langle\langle \boldsymbol{\lambda}, \boldsymbol{\tau}\mathbf{v}\rangle\rangle_{N,n} = \langle\langle \boldsymbol{\beta}, \boldsymbol{\rho}\mathbf{v}\rangle\rangle_m + \varepsilon(\mathbf{f},\mathbf{v})_0,$$

for all $\mathbf{v} \in H^n(\Omega;\mathbb{R}^3) \cap C^{n-1}(\overline{\Omega};\mathbb{R}^3)$, where $\boldsymbol{\sigma}$ is the unique solution of Problem (5).

§4. Convergence

We keep all the former notations and hypotheses. We can see that $\boldsymbol{\rho}$ depends on r and $\boldsymbol{\tau}$ depends on N.

First, we suppose that N is fixed and that $\varepsilon = \varepsilon(m)$. Let $\mathbf{g} \in H$ and we denote by $\boldsymbol{\sigma}_\varepsilon^m$ the PDE spline associated with (4), A, B, $\boldsymbol{\rho}\mathbf{g}$, $\boldsymbol{\tau}\mathbf{g}$ and ε.

Theorem 3. *Suppose that the following hypotheses hold:*

$$\sup_{\mathbf{x}\in\Omega} \min_{\mathbf{a}\in A} \langle \mathbf{x} - \mathbf{a}\rangle_2 = o\left(\frac{1}{r}\right),\ r \to +\infty,$$

and

$$\varepsilon = o(r),\ r \to +\infty.$$

Then, one has

$$\lim_{r\to+\infty} \|\boldsymbol{\sigma}_\varepsilon^m - \mathbf{g}\|_n = 0.$$

Now, we suppose that the boundary value problem (4) has a solution $\mathbf{u} \in H^{2n}(\Omega;\mathbb{R}^3)$. We denote by $\boldsymbol{\sigma}_\varepsilon^N$ the PDE spline associated with (4), A, B, $\boldsymbol{\rho}\mathbf{u}$, $\boldsymbol{\tau}\mathbf{u}$ and ε. Then we can prove the following result:

Theorem 4. *Suppose that*

$$\sup_{\mathbf{x}\in\partial\Omega} \min_{\mathbf{b}\in B} \langle \mathbf{x} - \mathbf{b}\rangle_2 = o(1),\ N \to +\infty.$$

Then, one has

$$\lim_{N\to+\infty} \|\boldsymbol{\sigma}_\varepsilon^N - \mathbf{u}\|_n = 0.$$

§5. Discretization

The solution of the above described problem is usually not easy to find explicitly, so we are going to discretize it in order to apply a numerical method for its resolution.

Therefore, we have to consider a space of finite dimension, where we will formulate and solve the discrete problem.

In this case, the functional finite-dimensional space that we have chosen is a finite element space but we could consider solving other spaces: for instance, a spline space of tensor product type, or a box-spline space.

Thus, let $\Omega \in \mathbb{R}^2$ be a polygonal bounded open set and suppose we are given:

- a subset $\mathcal{H}$ of $\mathbb{R}_+^*$ where 0 is an accumulation point;
- for all $h \in \mathcal{H}$, a partition $\mathcal{T}_h$ of $\overline{\Omega}$ made of rectangles or triangles of diameter less than h;
- for any $h \in \mathcal{H}$, a parametric finite element space X_h constructed on $\mathcal{T}_h$ such that

$$X_h \subset H^n(\Omega;\mathbb{R}^3) \cap C^{n-1}(\overline{\Omega};\mathbb{R}^3).$$

Now, we suppose that

$$\forall\, \mathbf{b} \in B,\ \mathbf{b} \text{ is a knot of } \mathcal{T}_h,$$

and we consider the convex set

$$H_h = \{\mathbf{u} \in X_h :\ \boldsymbol{\tau}\mathbf{u} = \mathbf{y}\}$$

and the linear space

$$H_h^0 = \{\mathbf{u} \in X_h :\ \boldsymbol{\tau}\mathbf{u} = \mathbf{0}\}.$$

Definition 2. Given $\varepsilon > 0$, we say that $\boldsymbol{\sigma}_h \in H_h$ is a *discrete PDE spline* associated with (4), A, B, $\boldsymbol{\beta}$, $\mathbf{y}$ and ε if $\boldsymbol{\sigma}_h$ is a solution of the problem

$$\forall\, \mathbf{v} \in H_h,\ J(\boldsymbol{\sigma}_h) \le J(\mathbf{v}), \tag{7}$$

where J is the functional defined in (6).

Now, we present two results which are the discrete versions of Theorems 1 and 2.

Theorem 5. *Problem (7) has a unique solution which is characterized as the unique solution of the following variational problem: Find $\boldsymbol{\sigma}_h \in H_h$ such that*

$$\forall\, \mathbf{v} \in H_h^0,\ \langle\langle \boldsymbol{\rho}\boldsymbol{\sigma}_h, \boldsymbol{\rho}\mathbf{v}\rangle\rangle_m + \varepsilon(\boldsymbol{\sigma}_h, \mathbf{v})_L = \langle\langle \boldsymbol{\beta}, \boldsymbol{\rho}\mathbf{v}\rangle\rangle_m + \varepsilon(\mathbf{f}, \mathbf{v})_0.$$

Theorem 6. *There exists a unique $(\boldsymbol{\sigma}_h, \boldsymbol{\lambda}) \in H_h \times \mathbb{R}^{N,n}$ such that*

$$\langle\langle \boldsymbol{\rho}\boldsymbol{\sigma}_h, \boldsymbol{\rho}\mathbf{v}\rangle\rangle_m + \varepsilon(\boldsymbol{\sigma}_h, \mathbf{v})_L + \langle\langle \boldsymbol{\lambda}, \boldsymbol{\tau}\mathbf{v}\rangle\rangle_{N,n} = \langle\langle \boldsymbol{\beta}, \boldsymbol{\rho}\mathbf{v}\rangle\rangle_m + \varepsilon(\mathbf{f}, \mathbf{v})_0, \quad \forall\, \mathbf{v} \in X_h, \tag{8}$$

where $\boldsymbol{\sigma}_h$ is the unique solution of Problem (7).

We consider a basis $\mathcal{B} = \{\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_Z\}$ of the finite element space X_h. Let us compute the (unique) solution of Problem (7).

Obviously $\boldsymbol{\sigma}_h = \sum_{i=1}^{Z} \alpha_i \boldsymbol{\omega}_i$ and, from (8), it follows that there exists $\boldsymbol{\lambda} \in \mathbb{R}^{N,n}$ such that

$$\sum_{i=1}^{Z} \alpha_i \left(\langle\langle \boldsymbol{\rho}\boldsymbol{\omega}_i, \boldsymbol{\rho}\mathbf{v} \rangle\rangle_m + \varepsilon(\boldsymbol{\omega}_i, \mathbf{v})_L \right) + \langle\langle \boldsymbol{\lambda}, \boldsymbol{\tau}\mathbf{v} \rangle\rangle_{N,n} = \langle\langle \boldsymbol{\beta}, \boldsymbol{\rho} v \rangle\rangle_m + \varepsilon(\mathbf{f}, \mathbf{v})_0, \ \forall\, \mathbf{v} \in X_h,$$

and verifying

$$\boldsymbol{\tau} \left(\sum_{i=1}^{Z} \alpha_i \boldsymbol{\omega}_i \right) = \mathbf{y}.$$

Taking $\mathbf{v} = \boldsymbol{\omega}_i$, for $i = 1, \ldots, Z$, we obtain a linear system of order $Z + Nn$ with the unknown

$$\alpha_1, \ldots, \alpha_Z, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{Nn}.$$

The matrix form of this system is

$$\begin{pmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{D}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{y} \end{pmatrix},$$

where

$$\mathbf{C} = \mathbf{A}^T\mathbf{A} + \varepsilon\mathbf{R}, \quad \mathbf{D} = (\boldsymbol{\tau}\boldsymbol{\omega}_i)_{1 \le i \le Z}, \quad \mathbf{f} = \mathbf{A}^T\boldsymbol{\beta} + ((\boldsymbol{\omega}_i, f)_0)_{1 \le i \le Z},$$

with

$$\mathbf{A} = (\boldsymbol{\omega}_j(\mathbf{a}_i))_{\substack{1 \le i \le m \\ 1 \le j \le Z}}, \quad \mathbf{R} = ((\boldsymbol{\omega}_i, \boldsymbol{\omega}_j)_L)_{1 \le i,j \le Z}.$$

§6. Numerical Examples

Let $\Omega = (0,1) \times (0,1)$ and we consider the cylindric surfaces parameterized respectively by

$$\mathbf{f}(x,y) = (2y - 1, 2\cos(2\pi x), 2\sin(2\pi x)), \ (x,y) \in \Omega,$$

and

$$\mathbf{g}(x,y) = (2y + 1, \cos(2\pi x), \sin(2\pi x)), \ (x,y) \in \Omega.$$

We apply our construction method of blending surfaces using discrete PDE splines from a partition in 8×4 equal rectangles whose sides are parallel to the coordinate axes, the BFS's rectangle of class C^1 and the boundary–value problem

$$\begin{cases} \dfrac{\partial^4 \mathbf{u}}{\partial x^4}(x,y) + 2a\dfrac{\partial^4 \mathbf{u}}{\partial x^2 \partial y^2}(x,y) + a^2\dfrac{\partial^4 \mathbf{u}}{\partial y^4}(x,y) = 0, \quad (x,y) \in \Omega, \\ \mathbf{u}(x,0) = \mathbf{f}(x,y), \ \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}}(x,0) = \dfrac{\partial \mathbf{f}}{\partial \mathbf{n}}(x,1), \quad 0 \le x \le 1, \\ \mathbf{u}(x,0) = \mathbf{f}(x,y), \ \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}}(x,0) = \dfrac{\partial \mathbf{f}}{\partial \mathbf{n}}(x,1), \quad 0 \le x \le 1, \end{cases}$$

with $a > 0$.

Consider the vectorial function $\mathbf{h}(x, y) = (2y - 1, 0.5\cos(2\pi x), 0.5\sin(2\pi x))$.

Figure 1 shows the two cylindrical surfaces and the graph of curve $\mathbf{h}(x, 0.5)$, with $0 \leq x \leq 1$.

From top to bottom and from left to right, respectively, Figure 2 shows the blending surface that smoothly join the two sylindrical surfaces by the discrete PDE spline obtained from:

- $A = \emptyset$, $\varepsilon = 1$, $a = 1$;
- $A = \emptyset$, $\varepsilon = 1$, $a = 10$;
- $A = \{(i/9, 0.5),\ i = 0, \ldots, 8\}$, $\boldsymbol{\beta} = \boldsymbol{\rho}\mathbf{h}$, $\varepsilon = 10^{-3}$, $a = 10$;
- $A = \{(i/9, 0.5),\ i = 0, \ldots, 8\}$, $\boldsymbol{\beta} = \boldsymbol{\rho}\mathbf{h}$, $\varepsilon = 10^{-5}$, $a = 10$.

§7. Conclusions

We have developed a construction method of blending surfaces using PDE splines. These splines are constructed minimizing a quadratic functional which includes two measures of how well the spline approximates a data point set and the solution of a given boundary–value problem of a partial differential equation. Boundary conditions are given by the surfaces that we want to blend.

The problem has been discretized in a finite element space and the constructed discrete PDE surface is obtained by a linear system whose order depends only on the dimension of the discretization space.

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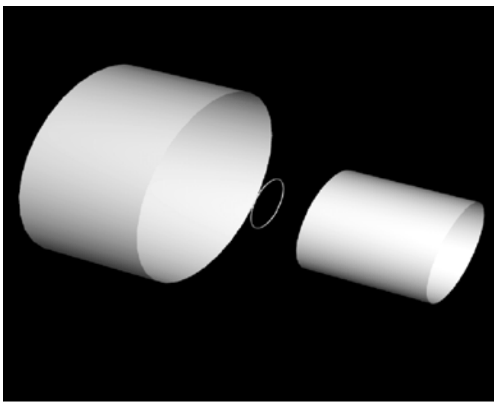

Figure 1: Two cylindrical surfaces and a circumference that contains the approximation points.

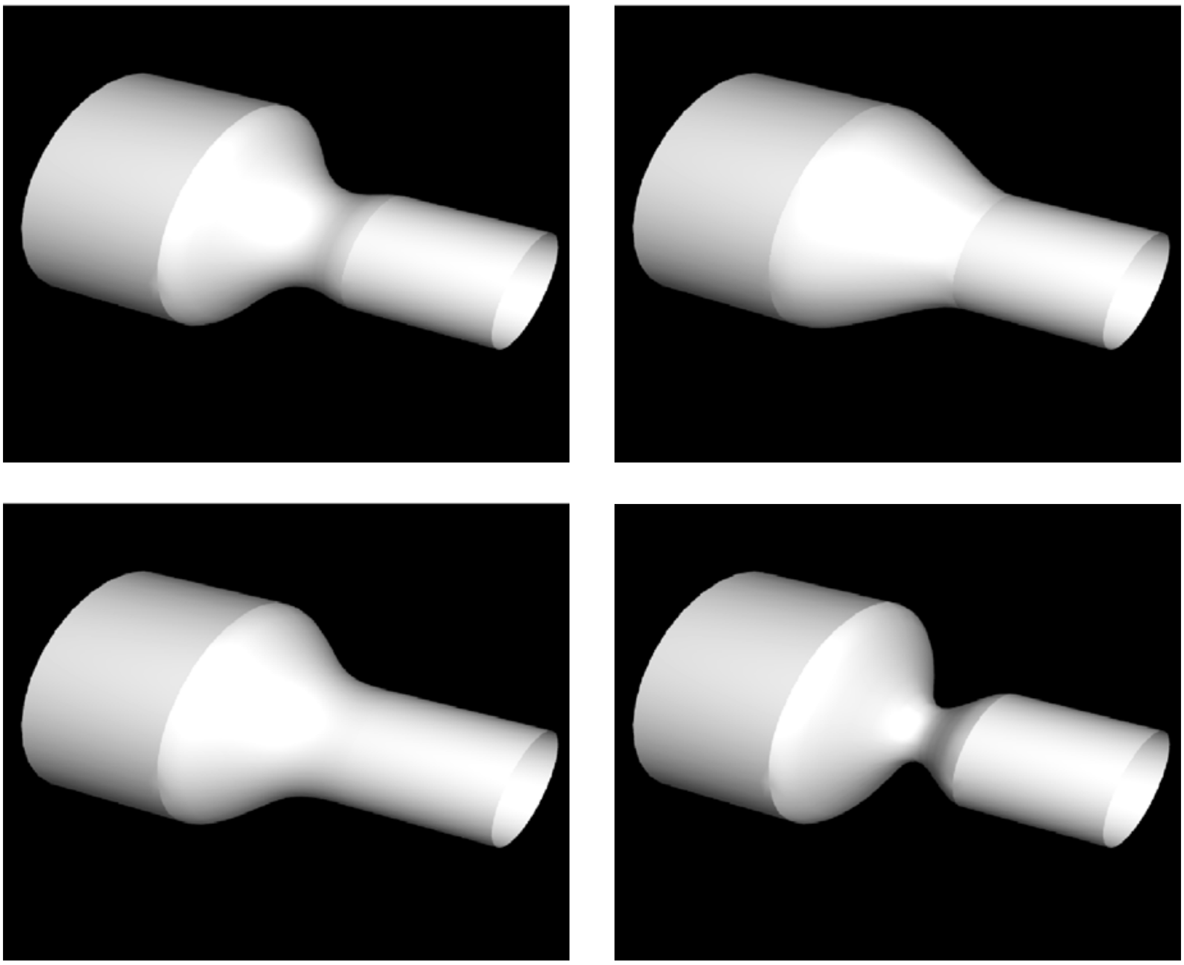

Figure 2: Some blending surfaces that smoothly join the two cylindrical surfaces.

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Miguel Pasadas and Miguel L. Rodríguez
Dept. de Matemática Aplicada
E.T.S.I. Caminos, Canales y Puertos
Campus de Fuentenueva,s/n 18071 Granada, España.
mpasadas@ugr.es and zamorano@ugr.es