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RELATIVE EQUILIBRIA AND BIFURCATIONS IN A 2–D HAMILTONIAN SYSTEM IN RESONANCE 1:p.

V. Lanchares and A. I. Pascual

Abstract. In this work, we focus on a Hamiltonian system with two degrees of freedom whose normal form in a neighborhood of the equilibrium solution up to order two, corresponds to a subtraction of two harmonic oscillators in resonance 1:p, with p an odd number. We introduce appropriate coordinates in the reduced phase space in order to study the existence of relative equilibria and bifurcations in terms of the free parameters of the system. We do this for to the simplest case, the resonance 1:3, and then we comment how these results can be extended for a resonance 1:p with p an odd number.

Keywords: Equilibrium, stability, normal form, resonance, bifurcations.

AMS classification: 34D20, 37J25, 70H14, 93D05.

§1. Introduction

The dynamics around equilibrium positions is of special interest in nonlinear two degrees of freedom Hamiltonian systems. A close look to the orbits in a vicinity of them results in practical applications in real physical models. This is the case, for example, of the restricted three body problem, where stable periodic orbits arise around the well known lagrangian equilibrium points [5]. Besides, theoretical results can be given about the stability properties of the equilibrium point.

In this way, Arnold's theorem [1] establishes the conditions for stability when the eigenvalues of the linear approximation are rational independent. Otherwise, specialized theorems are needed. For instance, for concrete rational dependencies Markeev [8] gave the conditions of stability. Recently, Cabral & Meyer [2] gave a more general result including those of Arnold and Markeev as well as other cases of rational dependencies.

All the results mentioned above require to bring the Hamiltonian into its Birkhoff's normal form around the equilibrium position. It is worth to remark that the normal form is not the same for the resonant (rational dependence) than for the nonresonant cases. In the first situation the normal form provides a comprehensive description of the dynamics of the system around the equilibrium position. Thus, a convenient starting point to study the dynamics around an equilibrium position is its normal for up to a suitable order. In this sense,our goal will be to study the structure of the phase flow. This is done, usually, by means of the characterization of the critical points and the parametric bifurcations. We will focus on the case of a critical point in resonance 1:p, when the eigenvalues of the linear part satisfy the relation

$$\omega_1 \pm p\omega_2 = 0$$

The first step is to express the normal form of a two degrees of freedom Hamiltonian system in resonance 1:p in a convenient manner. To this purpose it is necessary to know the structure of those terms of the Hamiltonian that take part of the normal form. For instance, if we choose a set of canonical complex variables (u, v, U, V), the Hamiltonian function expresses as

$$\mathcal{H} = \sum_{s=2}^{\infty} \left[\sum_{2(\alpha_1+\alpha_2)+(p+1)(\alpha_3+\alpha_4)=s} C_{\alpha_1\alpha_2\alpha_3\alpha_4} I_1^{\alpha_1} I_2^{\alpha_2} I_3^{\alpha_3} I_4^{\alpha_4} \right],$$

where $I_1 = uU$, $I_2 = vV$, $I_3 = uv^p$ and $I_4 = UV^p$ (see [7] for details).

The variables I_1, I_2, I_3 and I_4 are named *invariants*, as they do not change by the process of normalization. They are also known as *generators*, because every term in the normal form is generated by appropriate combinations of them. Besides, they determine the structure of the phase space after normalization. Indeed, they are not all independent but satisfy the relation

$$I_1 I_2^p = I_3 I_4. \tag{1}$$

Moreover, the first term of the normal form, which in our case is supposed to be proportional to

$$pI_1 - I_2,$$

is an integral in the normalized system, so that the phase space is a collection of two dimensional manifolds.

Due to the fact that the invariants introduced before are complex, a new set of real invariants is preferable. This new set is introduced by means of suitable action-angle variables specially useful to handle oscillators in resonance. These are the extended Lissajous variables [3]. In terms of the extended Lissajous variables the real invariants are defined as

$$M_1 = \tfrac{1}{2}\Phi_1, \qquad C_1 = 2^{-(m+n)/2}(\Phi_1 - \Phi_2)^{m/2}(\Phi_1 + \Phi_2)^{n/2}\cos 2nm\phi_1,$$

$$M_2 = \tfrac{1}{2}\Phi_2, \qquad S_1 = 2^{-(m+n)/2}(\Phi_1 - \Phi_2)^{m/2}(\Phi_1 + \Phi_2)^{n/2}\sin 2nm\phi_1.$$

Now, the normal form reads

$$\mathcal{H} = \sum_{s=2}^{\infty} \left[\sum_{2(\alpha_1+\alpha_2)+(p+1)(\alpha_3+\alpha_4)=s} C_{\alpha_1\alpha_2\alpha_3\alpha_4} M_1^{\alpha_1} M_2^{\alpha_2} S_1^{\alpha_3} C_1^{\alpha_4} \right], \tag{2}$$

where the first term is proportional to M_2, that becomes a new integral. Furthermore, the relation (1) becomes

$$C_1^2 + S_1^2 = (M_1 + M_2)(M_1 - M_2)^p, \tag{3}$$

together with the restriction

$$M_1 \geq |M_2|, \tag{4}$$

and M_2 a constant.

The equation (3), subjected to the restriction (4), defines the *reduced phase space*. This is a set of surfaces of revolution, one for each constant value of M_2. In figure 1 three of these surfaces are depicted for different values of M_2. It is worth to note the difference for $M_2 < 0$ and $M_2 \geq 0$. In the first case the vertex of the surface is a regular point whereas in the second case it is singular and it can be proved that then it is an equilibrium point of the reduced system.

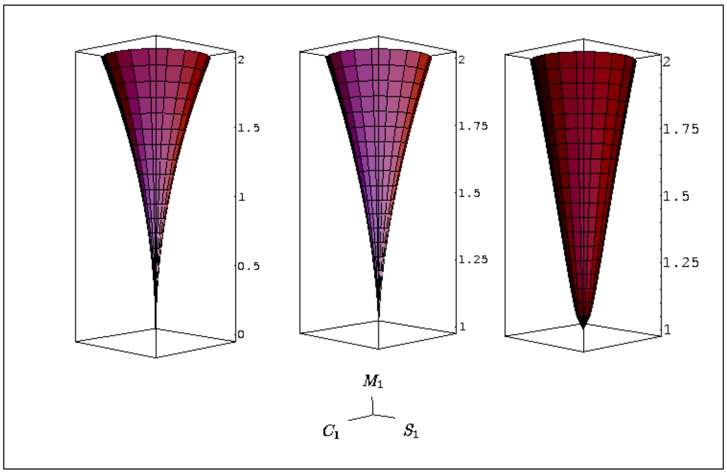

Figure 1: Resonance 1:3. Surfaces of revolution for different values of M_2; $M_2 = 0$, $M_2 = 1$ and $M_2 = -1$ respectively.

§2. The resonance $1{:}3$

We center on the phase flow around an equilibrium position in the case of a resonance $1{:}p$ with p an odd number. We begin with the simplest case, the resonance $1{:}3$, and later we will discuss how the results obtained for the $1{:}3$ resonance can be generalized.

Taking into account equation (2) for the resonance $1{:}3$, the Hamiltonian normal form up to order 4 can be written in terms of the invariants as

$$\mathcal{H} = \omega M_2 + a_1 M_1^2 + a_2 M_1 M_2 + a_3 M_2^2 + a_4 C_1 + a_5 S_1. \tag{5}$$

We will assume that $\mathcal{H}(M_2 = 0) \neq 0$ (non degenerate case), therefore $a_1^2 + a_4^2 + a_5^2 \neq 0$. At this point, some reductions allow to simplify the analysis of the Hamiltonian system. As M_2 is a constant, we can drop those terms which are integer powers of M_2. After that, Hamiltonian (5) reduces to

$$\mathcal{H} = a_1 M_1^2 + a_2 M_1 M_2 + a_4 C_1 + a_5 S_1. \tag{6}$$

If $a_4^2 + a_5^2 = 0$, then

$$\mathcal{H} = a_1 M_1^2 + a_2 M_1 M_2, \tag{7}$$

and the dynamics on the reduced phase space is trivial. The orbits are circumferences around the vertex of the reduced phase space provided (7) defines, for each appropriate $\mathcal{H}$, a plane parallel to $M_1 = 0$.

If $a_4^2 + a_5^2 \neq 0$, the dynamic is more intricate. First of all, we notice that it is possible to reduce the number of parameters in (6) by means of a suitable rotation around the axis M_1. This allows to eliminate the term in S_1 or C_1. So as a_4 or a_5 are not zero, we can suppose without loss of generality that $a_5 \neq 0$, and then, $\mathcal{H}$ can be written as

$$\mathcal{H} = \alpha M_1^2 + \beta M_1 M_2 + \gamma S_1, \tag{8}$$

with $\gamma \neq 0$. Furthermore another reduction can be performed. It consists on dividing (8) by γ and then, the final form of the Hamiltonian is

$$\mathcal{H} = a M_1^2 + b M_1 M_2 + S_1, \tag{9}$$

where a and b are the essential parameters.

2.1. Equilibria and stability

The equations of the motion corresponding to equation (9) are derived from the Poisson brackets between the variables S_1, C_1, M_1 (see [3] for details). They are

$$\begin{aligned} \dot{M}_1 &= -3\,C_1, \\ \dot{S}_1 &= 3\,C_1\,(2aM_1 + bM_2), \\ \dot{C}_1 &= -3\,S_1\,(2aM_1 + bM_2) - 3\,(M_1 - M_2)^2\,(M_2 + 2M_1). \end{aligned}$$

Fixed a value of M_2, a point (C_1, S_1, M_1) is an equilibrium if it verifies one of the two following conditions

- It is the vertex and $M_2 \geq 0$.

- $C_1 = 0$ and M_1 is a root of the polynomial

$$\mathcal{P}(M_1) = 4(1-a^2)M_1^3 - 4a(a+b)M_2 M_1^2 - (3+b^2+4ab)M_2^2 M_1 - (1+b^2)M_2^3, \tag{10}$$

  subjected to the restriction (4).

To begin with, we consider the simplest case $M_2 = 0$. When $M_2 = 0$, three cases arise:

- if $a^2 < 1$, there are two asymptotic orbits to the vertex, which is unstable. Indeed, it is the only critical point and the rest of the orbits are unbounded (see figure 2).

- if $a^2 > 1$, the vertex is again the only critical point and all the orbits are bounded surrounding the vertex (see figure 2).

- if $a^2 = 1$, the vertex is not an isolated equilibrium. In fact, all the points on the curve $S_1^2 = M_1^4$ with $M_1 > 0$ are equilibria. The rest of the orbits are unbounded as in the case $a^2 < 1$. This case can be considered as a limit case which separate the stable situation for $a^2 > 1$ from the unstable one for $a^2 < 1$. Indeed, if $a^2 = 1$, the stability of the vertex does not follow from the Hamiltonian function (9) and higher order terms must be considered.

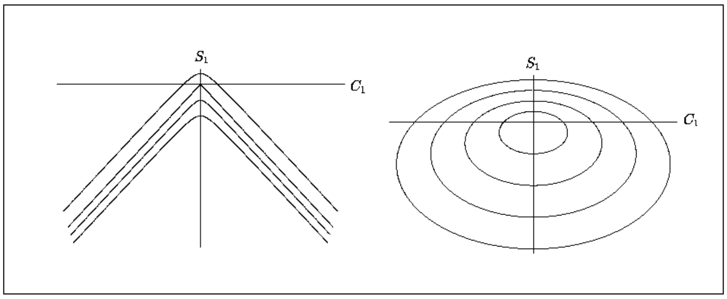

Figure 2: Orbits for $M_2 = 0$ when $a^2 \neq 1$, unstable ($a^2 < 1$) and stable ($a^2 > 1$) cases (orbits are projected onto the plane $M_1 = 0$).

When $M_2 \neq 0$, the vertex is an equilibrium point if and only if $M_2 \geq 0$. The rest of equilibria satisfy $C_1 = 0$ and M_1 a root of polynomial (10) under restriction (4). Despite of $\mathcal{P}(M_1)$ is a polynomial of degree 3 and there are exact formulae for the roots, they are not practical because the coefficients of $\mathcal{P}$ depend on the parameters a and b. So it is not easy to decide whether if they are real or complex and if they satisfy the condition (4). It is for this reason that we focus on the number of equilibria, rather on the explicit coordinates for them. This is a useful technique to find *bifurcations* because they take place when the number of equilibria changes. In this way, the number of equilibria changes when

- some of the roots of $\mathcal{P}(M_1)$ reach the extremes of the interval $[|M_2|, +\infty)$. This occurs if
    * $\mathcal{P}(|M_2|) = 0$,
    * the degree of $\mathcal{P}(M_1)$ is less than 3, that is, $a^2 = 1$.
- some of the roots of $\mathcal{P}(M_1)$ have a multiplicity bigger than 1.

In fact, the curves defined by $\mathcal{P}(|M_2|) = 0$ and $a^2 = 1$ are bifurcation lines in the parameter plane (a, b). Nevertheless, a careful study is necessary. First of all, we note that

$$\mathcal{P}(|M_2|) = -M_2^2 \left[M_2 - |M_2| + (2a+b)^2(M_2 + |M_2|)\right]. \tag{11}$$

Then, when $M_2 < 0$, $\mathcal{P}(|M_2|)$ is proportional to M_2^3 and, thus, it can not be zero and no bifurcation line appears. On the other hand, if $M_2 > 0$ we have

$$\mathcal{P}(M_2) = -2(2a+b)^2 M_2^3$$

which is zero if and only if $2a + b = 0$. However, if $2a + b \neq 0$, $\mathcal{P}(M_2) < 0$ and then the line $2a + b = 0$ does not give rise to any bifurcation line in the sense that if it is crossed the number of equilibrium points changes. Eventually, on this line two equilibrium points can collide but as soon as the line is crossed the two equilibria appear again.

On the contrary, $a^2 = 1$ is a bifurcation line in the sense that as it is crossed the number of equilibria changes. Nevertheless, the number of critical points along the two straight lines $a = \pm 1$ changes with the values of b. This is not difficult to see if we take into account the discriminant of the polynomial $\mathcal{P}$ which becomes

$$\mathcal{P}(M_1) = -4a(a+b)M_2M_1^2 - (3 + b^2 + 4ab)M_2^2M_1 - (1 + b^2)M_2^3.$$

We note that if $a+b=0$, $\mathcal{P}$ is a constant and none equilibrium is derived. On the contrary, if $a+b\neq 0$, $\mathcal{P}$ is a second degree polynomial whose discriminant is

$$\Delta = M_2^2(8ab(1-b^2)+(b+3)^2-16a^2).$$

In the special case $a=\pm 1$ Δ can be factorized as

$$\Delta = -M_2^2(a-b)^2(7a-b)(a+b).$$

and it can be concluded that

- if $M_2>0$,
  * if $(a=1, b<-1, b\neq -2)$ or $(a=-1, b>1, b\neq 2)$, 2 equilibria (one of them, the vertex).
  * another case, 1 equilibrium (the vertex).
- if $M_2<0$,
  * if $b=7a$ or $(a=1, b<-1)$ or $(a=-1, b>1)$, 1 equilibrium.
  * if $(a=1, b>7)$ or if $(a=-1, b<-7)$, 2 equilibria.
  * another case, 0 equilibria.

Once we have analyzed the number of roots occurring when a root of $\mathcal{P}(M_1)$ reaches the extremes of the interval $[|M_2|, +\infty)$, we proceed to establish the conditions under a multiple root of $\mathcal{P}(M_1)$ appears. Before that, we recall a result related on the concept of *discriminant* [4].

Theorem 1. *Let be* $p(x)=x^3+ax^2+bx+c$ *a monic polynomial of degree 3. Then*

- $\mathcal{D}(p)>0 \Longleftrightarrow p(x)$ *has 3 real and different roots.*
- $\mathcal{D}(p)<0 \Longleftrightarrow p(x)$ *has 1 real root and 2 conjugate complex roots.*
- $\mathcal{D}(p)=0 \Longleftrightarrow p(x)$ *has 3 real roots, some of them multiple.*

We can suppose that $a^2\neq 1$. Then, roots of polynomial $\mathcal{P}$ are the same as the roots of the monic polynomial

$$\hat{\mathcal{P}} = M_1^3 - \frac{4a(a+b)}{4(1-a^2)}M_2M_1^2 - \frac{(3+b^2+4ab)}{4(1-a^2)}M_2^2M_1 - \frac{(1+b^2)}{4(1-a^2)}M_2^3,$$

whose discriminant is given by

$$\mathcal{D}(\hat{\mathcal{P}}) = -\frac{M_2^6(a-b)^2 f(a,b)}{16(a-1)^4(a+1)^4},$$

where

$$f(a,b) = 27 - 18(2a^2-2ab-b^2) + (2a-b)^2(4a^2-b^2).$$

From theorem 1, it follows that

Theorem 2. *There exists functions* $f_1(a)$, $f_2(a)$ *such that*

- $\mathcal{P}$ *has one real root* $\iff f_2(a) < b < f_1(a)$ *with* $b \neq a$.
- $\mathcal{P}$ *has 3 different real roots* $\iff b > f_1(a)$ *or* $b < f_2(a)$.
- $\mathcal{P}$ *has 3 real roots, some of them multiple* $\iff b = f_1(a)$ *or* $b = f_2(a)$ *or* $b = a$.

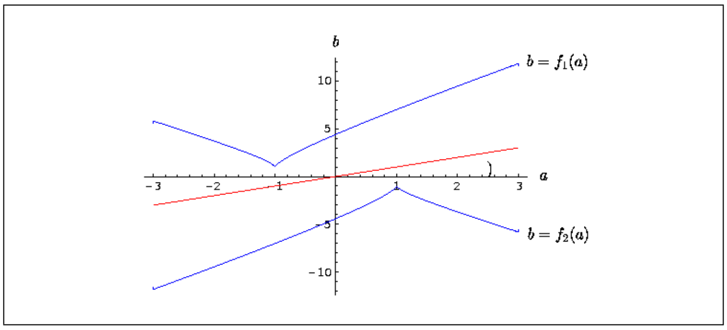

Figure 3: The straight line $b = a$ and the curves $f_1(a)$ and $f_2(a)$.

The expression of these two functions are too much involved so we do not write them. They are depicted in figure 3. We note that they are symmetric respect to the origin and that they present a singularity for $a = \pm 1$ respectively. In fact, they can be considered as two branches of an algebraic curve with a cusp for $a = \pm 1$. A careful application of the theorem 2 proves that each branch of the algebraic curve determines a bifurcation line depending on the sign of M_2.

Collecting all the considerations made previously it is possible to derive all the bifurcation lines taking into account whether the number of equilibrium points changes when they are crossed. In figure 4, a partition of the parameter plane is depicted where the bifurcation lines divide the plane into several regions where the number of equilibria changes. It is worth to note that different partitions appear when $M_2 > 0$ or $M_2 < 0$.

- For $M_2 > 0$, there are 5 regions where the number of equilibria is 1, 2 or 3.
- For $M_2 < 0$, there are 7 regions where the number of equilibria ranges from 0 to 3.

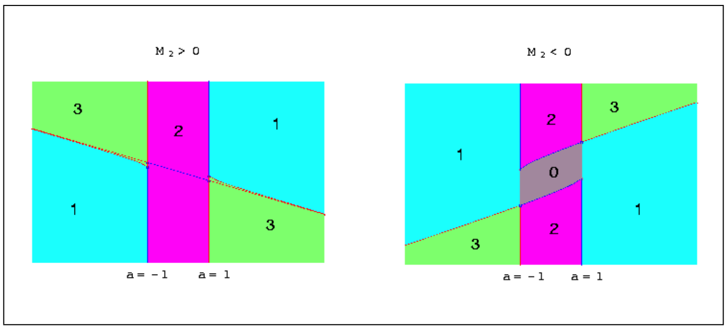

Figure 4: Division of the parameter plane (a, b) for $M_2 > 0$ and $M_2 < 0$. The number in the regions indicates the number of equilibria.

We also note that if $a^2 < 1$ (the condition for instability), the number of equilibria is even (0 or 2). On the other hand, if $a^2 > 1$ (the condition for stability), the number of equilibria is odd (1 or 3). Finally, we note that there are four different types of possible flows as a function of the number of equilibria and they are depicted in figure 5.

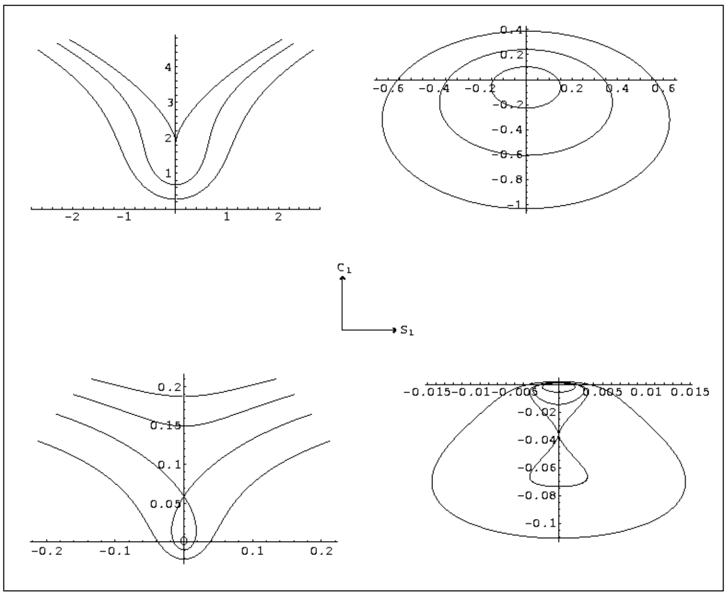

Figure 5: The four different types of flows depending on the number of equilibria projected onto the plane $M_1 = 0$. Note that for an odd number of equilibria all the orbits are bounded.

§3. The resonance $1{:}p$ for p an odd number

Now, we consider a resonance $1{:}p$ with p an odd number. The Hamiltonian normal form up to order $p+1$ can be written in terms of the invariants, as

$$\mathcal{H} = \mathcal{P}(M_1, M_2) + \beta_1 C_1 + \beta_2 S_1,$$

where $\mathcal{P}(M_1, M_2)$ is a polynomial of degree less or equal than $\frac{p+1}{2}$. We can perform similar reductions as those made in resonance 1:3. As M_2 is a constant, we can drop the terms depending only on M_2. If $\beta_1^2 + \beta_2^2 = 0$ the orbits are circumferences around the vertex of the surface (3). In addition, the vertex is the unique equilibrium point and it is stable.

If $\beta_1^2 + \beta_2^2 \neq 0$, we can make a suitable rotation about the axis M_1 that allows to drop the term in C_1 (or in S_1). We suppose that, after reduction, $\mathcal{H}$ is written as

$$\mathcal{H} = \mathcal{P}(M_1, M_2) + \gamma S_1,$$

with $\gamma \neq 0$. Finally, we can divide $\mathcal{H}$ by γ, obtaining

$$\mathcal{H} = \mathcal{P}(M_1, M_2) + S_1,$$

with $\mathcal{P}(M_1, M_2)$ a polynomial with director coefficient equal to a.

As in the case of the resonance 1:3, we are interested on the number of equilibrium points rather than in the explicit expressions of them. In this way, we look for bifurcation hypersurfaces which are mainly determined for the changes in the number of equilibria. Now, apart from the vertex, the rest of equilibria are related with the roots of a p degree polynomial in M_1 such that $M_1^* > |M_2|$, being M_1^* one of those roots.

It is worth to note that $a = \pm 1$ is a hypersurface of bifurcation and, as in the resonance 1:3, if $a^2 > 1$ we have a condition of stability in the sense that all the orbits are bounded. Moreover, the number of equilibrium points must be odd. On the other hand, if $a^2 < 1$ we have a condition of instability and there always exists a set of unbounded orbits. In addition, the number of equilibria must be even. From this considerations it is not difficult to figure the different types of phase flow, although we are not able to determine the bifurcation hypersurfaces in terms of the free parameters of the system.

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Lanchares, V. and Pascual, A. I.
Dpto. Matemáticas y Computación.
Universidad de La Rioja.
26004. Logroño. La Rioja. Spain.
vlancha@dmc.unirioja.es and aipasc@dmc.unirioja.es