RICHARDSON EXTRAPOLATION ON GENERALIZED SHISHKIN MESHES FOR SINGULARLY PERTURBED PROBLEMS

J.L. Gracia and C. Clavero

Abstract. In this work we are interested in to apply the Richardson extrapolation technique on a type of finite difference schemes, which are used to solve 1D singularly perturbed problems of convection-diffusion type. The numerical method is constructed on generalized Shishkin meshes, which are defined by using a generating function; in all cases the mesh points are condensed in the boundary layer region, in order to obtain a good approximation in the maximum norm. We prove that, if the diffusion coefficient is sufficiently small, an appropriate Richardson extrapolation increase the order of uniform convergence associated to the basic finite difference scheme. Some numerical examples permit us to confirm in practice the theoretical results.

Keywords: Richardson extrapolation, Shishkin type meshes, generating function, uniform convergence

AMS classification: 65N12, 65N30, 65N06

§1. Introduction

We consider the Dirichlet boundary value problem

$$L_\varepsilon u \equiv -\varepsilon u'' + a(x)u' + b(x)u = f(x),\, 0 < x < 1, \quad u(0) = u_0, \;\; u(1) = u_1, \qquad (1)$$

where u_0 and u_1 are constants, a, b and f are sufficiently smooth functions satisfying $a(x) > 2\alpha > 0$, $b(x) \geq 0$, $\forall\, x \in [0, 1]$ and the diffusion parameter is sufficiently small, $0 < \varepsilon \ll 1$. This problem is the simple linear 1D model of convection-diffusion problems with dominating convection term and it appears in many areas of science (fluid mechanic, semiconductor devices and heat or mass transport by example). It is known (see [3]) that the exact solution has a boundary layer at $x = 1$.

In last years many works (see [3] and references therein) have been developed to solve efficiently this type of problems, using robust or ε-uniform convergent methods (finite differences or finite elements) defined on piecewise uniform Shishkin meshes, which condense the mesh points in the boundary layer region. In [8] new generalized Shishkin type meshes were introduced, proving also the uniform convergence of classical numerical schemes on this type of meshes. Nevertheless, only in few papers the numerical methods have order of convergence bigger than one; in [1] HODIE schemes on Shishkin meshes were analyzed, proving second and third order of uniform convergence; in [2] these results were extended for generalized Shihskin meshes; in [4] it was proved that the defect correction method, combining the upwind and the central finite difference schemes, is a second order ε-uniform convergent scheme; finally, in [7] it was proved that Richardson extrapolation increase the order of the upwind scheme defined on classical Shishkin meshes.

Richardson extrapolation is a well known procedure to improve the numerical solution using an appropriate combination of previously computed solutions (see [5, 6] for the application of this technique to non-singularly perturbed problems). In this paper we prove that the combination of a HODIE finite difference scheme, defined on generalized Shihskin meshes, and the Richardson extrapolation, is an efficient technique to increase the order of uniform convergence and also to reduce the errors associated to the numerical scheme.

Throughout the paper C denote any positive constant independent of ε and the discretization parameter N.

§2. The generalized Shishkin meshes

Before constructing the numerical method, we present the generalized Shishkin mesh, which was introduced in [8]. Let $N \geq 4$ be an even integer (the discretization parameter); we define the transition parameter by

$$\sigma = \min\{1/2, \sigma_0 \varepsilon \ln N\}, \tag{2}$$

where σ_0 is a positive constant to be fixed later. If $\sigma = 1/2$, we take a uniform mesh with $x_0 = 0,\ x_N = 1$; in this case the analysis could be made in a classical way; therefore, in the remainder of the paper we will suppose that $\sigma = \sigma_0 \varepsilon \ln N$. Now, on the interval $[0, 1-\sigma]$ we consider a uniform mesh such that $x_0 = 0,\ x_{N/2} = 1-\sigma$; however, on $[1-\sigma, 1]$ the mesh will be graded such that the step sizes, $h_j = x_j - x_{j-1}$, satisfy $h_j \geq h_{j+1},\ j = N/2+1, \cdots, N-1$. To define the mesh we use a continuous, monotone increasing and piecewise continuously differentiable function $\varphi(t),\ t \in [1/2, 1]$ such that $\varphi(1/2) = -\ln N$ and $\varphi(1) = 0$; the mesh points are given by

$$x_j = \begin{cases} (1-\sigma)2j/N, & j = 0, \cdots, N/2, \\ 1 + \sigma_0 \varepsilon \varphi(t_j),\ t_j = j/N, & j = N/2+1, \cdots, N. \end{cases} \tag{3}$$

From this definition we see that $h_j = H = 2(1-\sigma)/N,\ j = 1, \cdots, N/2$ and $N^{-1} \leq H \leq 2N^{-1}$.

Following [8], we consider a new function $\psi(t) = \exp(\varphi(t))$, which is also increasing and it satisfies $\psi(1/2) = N^{-1}$, $\psi(1) = 1$; some examples of mesh functions ψ (see [8]) are

$$\psi(t) = e^{-2(1-t)\ln N}, \quad \text{(S-mesh)} \tag{4}$$

$$\psi(t) = 1 - 2\left(1 - N^{-1}\right)(1-t), \quad \text{(S-B mesh)} \tag{5}$$

$$\psi(t) = e^{-(1-t)/(q-(1-t))},\ q = 1/2 + 1/(2\ln N). \quad \text{(S-B modified mesh)} \tag{6}$$

It is straightforward to prove that

$$\max_{t\in[1/2,1]} |\psi'(t)| \leq \begin{cases} C \ln N, & \text{(S-mesh)} \\ C, & \text{(S-B and S-B modified mesh)} \end{cases} \tag{7}$$

These meshes were used in [8] to prove that the simple upwind scheme is uniformly convergent with order at most 1; also, the same meshes were used in [4] to prove that the defect correction technique, based on central differences and the simple upwind scheme, gives a ε-uniform method having order $\mathcal{O}(N^{-1}\max|\psi'(t)|)^2)$.

Here we only are interested in meshes whose mesh-generating function φ satisfies

$$\int_{1/2}^{1} |\varphi'(t)|^2 dt \leq CN, \qquad \max_{t\in[1/2,1]} \varphi'(t) \leq CN. \tag{8}$$

Conditions (8) are sufficient to prove the ε-uniform convergence of the simple upwind scheme defined on these meshes (see [8]). Moreover, we will assume that there exists a fixed integer $1 \leq j_\psi < N/2$ independent of N such that

$$h_j a_j - 2\varepsilon < 0, \quad \forall\ j \geq j_\psi + N/2. \tag{9}$$

In [2] it was proved that (8) and (9) are satisfied by S, S-B and S-B modified meshes.

§3. HODIE finite difference scheme and the Richardson extrapolation

Before studying the Richardson extrapolation, we define the basic HODIE finite difference scheme, which is given as follows:

$$\begin{array}{l} L_\varepsilon^N U_j \equiv r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = q_j^1 f_{j-1} + q_j^2 f_j,\ j = 1, \cdots, N-1, \\ U_0 = u_0,\ \ U_N = u_1, \end{array} \tag{10}$$

where $f_j = f(x_j)$, $j = 0, \cdots, N$ (similarly for a_j and b_j). The coefficients $r_j^-,\ r_j^c,\ r_j^+,\ q_j^1,\ q_j^2$ associated to (10), are determined by imposing that the polynomials of degree less or equal 2, $P_2[x]$, belong to the kernel of the local error operator and also that the coefficients satisfy the normalization condition

$$q_j^1 + q_j^2 = 1, \quad q_j^i \geq 0, \quad i = 1, 2,\ j = 1, \cdots, N-1. \tag{11}$$

These coefficients were calculated in [2], where also it was proved the following result of uniform convergence.

Theorem 1. *Let $\beta \leq \alpha$ be, $N \geq 4$ an even positive integer such that*

$$h_j \max\{||a'||_\infty, ||b||_\infty\} \leq \alpha, \qquad 1 \leq j \leq N,$$

u the solution of the continuous problem (1), $\{U_j\}_{j=0}^N$ the numerical solution of the finite difference scheme (10) defined on the mesh (3), where $\varphi(t)$ is such that (8) and (9) hold and we assume that $\varepsilon \leq CN^{-1}$. Then, the error satisfies

$$|u(x_j) - U_j| \leq C((N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^2 + N^{-\beta\sigma_0}), \quad 1 \leq j \leq N-1. \tag{12}$$

Remark 1. From Theorem 1 and bounds (7) we deduce that, if β satisfies $\beta\sigma_0 \geq 2$, the HODIE finite difference scheme has order two on S-B and S-B modified meshes and order near two (due to the logarithmic factor) on classical S-mesh.

In order to increase the order of convergence of this method, following the original idea of Natividad-Stynes [7], we consider now a new mesh $\bar{\Omega}_\sigma^{2N} = \{\tilde{x}_j\}$, which is obtained by bisecting each interval of the original mesh (3). On this mesh, we define the HODIE finite difference scheme

$$\begin{array}{l} \tilde{L}_\varepsilon^{2N}\tilde{U}_j^{2N} \equiv \tilde{r}_j^-\tilde{U}_{j-1}^{2N} + \tilde{r}_j^c\tilde{U}_j^{2N} + \tilde{r}_j^+\tilde{U}_{j+1}^{2N} = \tilde{q}_j^1 f_{j-1} + \tilde{q}_j^2 f_j, \quad 1 \le j \le 2N-1, \\ \tilde{U}_0^{2N} = u_0, \quad \tilde{U}_{2N}^{2N} = u_1, \end{array} \tag{13}$$

where now the coefficients are given by

$$\begin{array}{l} \tilde{q}_j^1 = a_j/(a_j + a_{j-1}), \quad \text{if } 1 \le j < N + 2j_\psi, \\ \tilde{q}_j^1 = (h_j - h_{j+1})/(3h_j), \quad \text{if } N + 2j_\psi \le j \le 2N-1, \\ \tilde{q}_j^2 = 1 - \tilde{q}_j^1, \quad \text{for } 1 \le j \le 2N-1, \\ \tilde{r}^\star(\varepsilon, \tilde{q}_j^1, \tilde{q}_j^2, h_j, a, b) = r^\star(\varepsilon, q_j^1, q_j^2, h_j/2, a, b), \quad \text{for } \star = -, c, +. \end{array} \tag{14}$$

We are interested in to prove that the extrapolated numerical solution defined on the mesh (3) by

$$\bar{U}_j^N = \left(4\tilde{U}_{2j}^{2N} - U_j^N\right)/3, \quad 1 \le j \le N-1, \tag{15}$$

improves the approximation obtained with the basic HODIE scheme and also that the order of uniform convergence is higher.

To study the ε-uniform convergence of the extrapolated solution, we need to know the asymptotic behaviour with respect to ε of the exact solution. In [3] it was proved that the exact solution can be written as $u = v + w$, where the regular component v and the singular component w satisfy $L_\varepsilon v = f$, $L_\varepsilon w = 0$, respectively, with appropriate boundary conditions such that for $0 \le k \le q$ (q is an integer depending on the data regularity) it holds

$$|v^{(k)}(x)| \le C, \quad |w^{(k)}(x)| \le C\varepsilon^{-k} e^{-2\alpha(1-x)/\varepsilon}. \tag{16}$$

Similarly to the continuous problem, we decompose the numerical solution as $\tilde{U}_j^{2N} = \tilde{V}_j^{2N} + \tilde{W}_j^{2N}$, where

$$\begin{array}{lll} \tilde{L}_\varepsilon^{2N}\tilde{V}_j = \tilde{q}_j^1 f_{j-1} + \tilde{q}_j^2 f_j, & 1 \le j \le 2N-1, & \tilde{V}_0 = v(0), \quad \tilde{V}_{2N} = v(1), \\ \tilde{L}_\varepsilon^{2N}\tilde{W}_j = 0, & 1 \le j \le 2N-1, & \tilde{W}_0 = w(0), \quad \tilde{W}_{2N} = w(1). \end{array}$$

Lemma 1. The local error associated to the regular component satisfies

$$L_\varepsilon^N(V^N - v)(x_j) = \zeta(x_j)h_j^2 + \mathcal{O}(\varepsilon N^{-1} + N^{-3}), \quad 1 \le j \le N-1, \tag{17}$$

where $\zeta(x)$ is a sufficiently smooth function such that in the mesh points it satisfies

$$\zeta(x_j) = \left(\frac{q_j^1 a_{j-1}}{2} + \frac{h_{j+1}^2(a_j - q_j^1(a_{j-1} + a_j))}{3!h_j(h_j + h_{j+1})} + \frac{q_j^1(2h_j + h_{j+1})a_{j-1} + q_j^2 h_{j+1} a_j}{3!(h_j + h_{j+1})}\right) v_j''',$$

and its derivatives are bounded with respect to ε.

Proof. The proof is straightforward from Taylor expansion and the bounds (16). □

Let E be the solution of the following boundary value problem

$$L_\varepsilon E(x) = \zeta(x), \quad x \in (0,1), \quad E(0) = E(1) = 0. \tag{18}$$

Similarly to the original problem we can write $E = \eta + \vartheta$, where

$$L_\varepsilon \eta(x) = \zeta(x),\ L_\varepsilon \vartheta(x) = 0,\ \ x \in (0,1),\ \eta(0) = -\vartheta(0),\ \eta(1) = -\vartheta(1),$$

and the functions η, ϑ satisfy

$$|\eta^{(k)}| \leq C, \quad |\vartheta^{(k)}| \leq C\varepsilon^{-k} e^{-2\alpha(1-x)/\varepsilon}, \tag{19}$$

for $0 \leq k \leq q$ with q a positive integer, again depending on the data regularity.

Lemma 2. Let $h_0 = h_1$ be. Then, it holds

$$V_j^N - v(x_j) = \eta(x_j)h_j^2 + \mathcal{O}(\varepsilon N^{-1} + N^{-3}), \quad 1 \leq j \leq N-1. \tag{20}$$

Proof. We easily prove that

$$L_\varepsilon^N \eta(x_j) = L_\varepsilon \eta(x_j) + L_\varepsilon^N \eta(x_j) - L_\varepsilon \eta(x_j) = L_\varepsilon \eta(x_j) + \mathcal{O}(N^{-2}) = \zeta(x_j) + \mathcal{O}(N^{-2}),$$

or equivalently

$$h_j^2 L_\varepsilon^N \eta(x_j) = h_j^2 \zeta(x_j) + \mathcal{O}(N^{-4}).$$

From (17) it follows that

$$\begin{aligned} &L_\varepsilon^N (V^N - v - h_j^2\eta)(x_j) = L^N(V^N - v)(x_j) - h_j^2 L_\varepsilon^N \eta(x_j) = \\ &= \zeta(x_j)h_j^2 + \mathcal{O}(\varepsilon N^{-1} + N^{-3}) - h_j^2\zeta(x_j) + \mathcal{O}(N^{-4}) = \mathcal{O}(\varepsilon N^{-1} + N^{-3}). \end{aligned}$$

We consider now the barrier function

$$Z_j = C(1 + x_j)(\varepsilon + N^{-2})N^{-1}, \quad j = 0, \ldots, N,$$

with C a positive constant large enough, which satisfies

$$L_\varepsilon^N Z_j \geq |L_\varepsilon^N (V^N - v - h_j^2\eta)(x_j)|, \quad 1 \leq j \leq N-1,$$

and

$$\begin{aligned} Z_N &= C(\varepsilon + N^{-2})N^{-1} \geq C\varepsilon^2 \geq Ch_N^2|\eta(1)|, \\ Z_0 &= C(\varepsilon + N^{-2})N^{-1} \geq C\varepsilon^2 \geq C\alpha^2 e^{-\alpha/\varepsilon} \geq Ch_1^2|\eta(0)|. \end{aligned}$$

Then, the discrete maximum principle proves

$$|(V^N - v - h_j^2\eta)(x_j)| \leq Z_j \leq C(\varepsilon + N^{-2})N^{-1},$$

which permit us to deduce (20). □

Lemma 3. The error associated to the regular component satisfies

$$|v(x_j) - \bar{V}_j| \leq C(\varepsilon + N^{-2})N^{-1}, \quad 1 \leq j \leq N-1. \tag{21}$$

Proof. Taking into account the definition of numerical extrapolated solution giving by (15), the proof is straightforward from (20). □

Lemma 4. The error associated to the singular component satisfies

$$|w(x_j) - \bar{W}_j^N| \le CN^{-\beta\sigma_0}, \quad 1 \le j < N/2 + j_\psi.$$

Proof. In [2] it was proved that

$$|w(x_j) - W_j^N| \le CN^{-\beta\sigma_0}, \quad 1 \le j < N/2 + j_\psi.$$

In a similar way we can also obtain

$$|w(x_j) - \tilde{W}_j^{2N}| \le CN^{-\beta\sigma_0}, \quad 1 \le j < N + 2j_\psi.$$

Therefore, for $1 \le j < N/2 + j_\psi$, the result follows. □

For $N/2 + j_\psi \le j \le N - 1$ the study is not so easy. Firstly, we will prove that the error can be written in the form

$$W_j^N - w(x_j) = (h_j/\varepsilon)^2 F(x_j) + \mathcal{O}(N^{-\beta\sigma_0} + (N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^4),$$

where the function F is the solution of the boundary value problem

$$\begin{array}{l} L_\varepsilon F(x) = \theta(x), \quad x \in (x_{N/2+j_\psi-1}, 1), \\ F(x_{N/2+j_\psi-1}) = W_{N/2+j_\psi-1}^N - w(x_{N/2+j_\psi-1}), \quad F(1) = 0, \end{array} \tag{22}$$

whit θ a sufficiently smooth function satisfying

$$\theta(x_j) = \varepsilon^2 \left(\frac{h_{j+1}^3}{3!h_j^2} r_j^+ - \frac{h_j}{3!} r_j^- - \frac{q_j^1\varepsilon}{h_j} + \frac{q_j^1 a_{j-1}}{2} \right) w_j''', \quad N/2 + j_\psi \le j \le N - 1,$$

and also we assume that

$$|\theta^{(i)}(x)| \le C\varepsilon^{-(1+i)} e^{-2\alpha(1-x)/\varepsilon}, \quad x \in [x_{N/2+j_\psi-1}, 1].$$

Lemma 5. Let σ_0 be such that $\beta\sigma_0 \ge 4$. Then, it holds

$$W_j^N - w(x_j) = (h_j/\varepsilon)^2 F(x_j) + \mathcal{O}(N^{-\beta\sigma_0} + (N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^4), \; N/2 + j_\psi \le j \le N - 1.$$

Proof. Clearly, from the hypotheses on θ, it follows that F satisfies

$$|F^{(k)}(x)| \le C(1 + \varepsilon^{-k} e^{-2\alpha(1-x)/\varepsilon}). \tag{23}$$

Taylor expansions of $L_\varepsilon^N F(x_j)$ around x_j and (23) permit us to prove

$$L_\varepsilon^N F(x_j) = L_\varepsilon F(x_j) + \mathcal{O}(\varepsilon^{-1}(h_j/\varepsilon)^2 e^{-2\alpha(1-x_j)/\varepsilon}), \quad N/2 + j_\psi \le j \le N - 1. \tag{24}$$

Using the Shishkin decomposition of the continuous problem (1) (see [3]), the bounds (16) for the derivatives of w and appropriate Taylor expansion, we deduce

$$\begin{aligned} L_\varepsilon^N[W_j^N - w(x_j)] = w_j''' \left(\frac{h_{j+1}^3}{3!} r_j^+ - \frac{h_j^3}{3!} r_j^- - q_j^1 h_j \varepsilon + q_j^1 \frac{h_j^2}{2} a_{j-1} \right) + \\ + \frac{h_{j+1}^4}{4!} r_j^+ w^{(4)}(\delta) + \frac{h_j^4}{4!} r_j^- w^{(4)}(\alpha_1) + q_j^1 \frac{h_j^2}{2} \varepsilon w^{(4)}(\alpha_2) - q_j^1 \frac{h_j^3}{3!} a_{j-1} w^{(4)}(\alpha_3) - \\ - q_j^1 b_{j-1} \frac{h_j^3}{3!} w^{(3)}(\alpha_4) = (h_j/\varepsilon)^2 \varepsilon^2 w_j''' \left(\frac{h_{j+1}^3}{3! h_j^2} r_j^+ - \frac{h_j}{3!} r_j^- - q_j^1 \frac{\varepsilon}{h_j} + q_j^1 \frac{a_{j-1}}{2} \right) + \\ + \mathcal{O}(\varepsilon^{-1} (h_j/\varepsilon)^2 e^{-2\alpha(1-x_j)/\varepsilon}), \end{aligned} \tag{25}$$

where $\alpha_i \in (x_{j-1}, x_j)$, $i = 1, 2, 3, 4$, and $\delta \in (x_j, x_{j+1})$. From (22), (24) and (25), it follows

$$\begin{aligned} (h_j/\varepsilon)^2 L_\varepsilon^N F(x_j) &= (h_j/\varepsilon)^2 L_\varepsilon F(x_j) + \mathcal{O}(\varepsilon^{-1}(h_j/\varepsilon)^4 e^{-2\alpha(1-x_j)/\varepsilon}) = \\ &= L_\varepsilon^N [W_j^N - w(x_j)] + \mathcal{O}(\varepsilon^{-1}(h_j/\varepsilon)^4 e^{-2\alpha(1-x_j)/\varepsilon}). \end{aligned}$$

Therefore,

$$|L_\varepsilon^N [W_j^N - w(x_j) - (h_j/\varepsilon)^2 F(x_j)]| = \mathcal{O}(\varepsilon^{-1}(h_j/\varepsilon)^4 e^{-2\alpha(1-x_j)/\varepsilon}). \tag{26}$$

Taking the barrier function

$$\xi_j(\beta) = C\left(N^{-\beta\sigma_0}(1 + x_j) + (h_j/\varepsilon)^4 \phi_j(\beta)\right),$$

where C is a positive constant large enough, it holds that

$$\begin{aligned} &L_\varepsilon^N \xi_j(\beta) \geq |L^N[W_j^N - w(x_j) - (h_j/\varepsilon)^2 F(x_j)]|, \qquad N/2 + j_\psi \leq j \leq N-1, \\ &\xi_{N/2+j_\psi-1}(\beta) \geq |W_{N/2+j_\psi-1}^N - w(x_{N/2+j_\psi-1}) - (h_j/\varepsilon)^2 F(x_{N/2+j_\psi-1})|, \\ &\xi_N(\beta) \geq |W_N^N - w(1) - (h_j/\varepsilon)^2 F(1)| = 0. \end{aligned}$$

Finally, using the discrete maximum principle on $[x_{N/2+j_\psi-1}, 1]$, we obtain

$$\begin{aligned} |W_j^N - w(x_j) - (h_j/\varepsilon)^2 F(x_j)| &\leq \xi_j \leq C(N^{-\beta\sigma_0} + (h_j/\varepsilon)^4 \phi_j(\beta)) \leq \\ &\leq \mathcal{O}(N^{-\beta\sigma_0} + (N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^4), \end{aligned}$$

taking β such that $\beta\sigma_0 \geq 4$, which is the required result. □

Lemma 6. Let σ_0 be such that $\beta\sigma_0 \geq 4$. Then, the error associated to the singular component satisfies

$$|w(x_j) - \bar{W}_j^N| \leq C(N^{-\beta\sigma_0} + (N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^4), \quad N/2 + j_\psi \leq j \leq N-1.$$

Proof. From Lemma 5 we have

$$\begin{aligned} w(x_j) - W_j^N &= (h_j/\varepsilon)^2 F(x_j) + \mathcal{O}(N^{-\beta\sigma_0} + (N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^4), \\ w(x_j) - \tilde{W}_j^{2N} &= ((h_j/\varepsilon)^2/4) F(x_j) + \mathcal{O}(N^{-\beta\sigma_0} + (N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^4). \end{aligned}$$

Therefore, it immediately follows

$$w(x_j) - \bar{W}_j^N = \mathcal{O}(N^{-\beta\sigma_0} + (N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^4), \quad N/2 + j_\psi \leq j \leq N-1.$$

□

From Lemmas 3, 4 and 6 we obtain the main result of this work.

Theorem 2. *Let* σ_0 *be such that* $\beta\sigma_0 \geq 4$, u *the solution of the continuous problem (1),* $\{\bar{U}_j\}_{j=0}^N$ *the extrapolated numerical solution defined by (15) and we assume that* $\varepsilon \leq CN^{-1}$*. Then, the error satisfies*

$$|u(x_j) - \bar{U}_j^N| \leq \begin{cases} C((\varepsilon + N^{-2})N^{-1} + N^{-\beta\sigma_0}), & \text{if } 1 \leq j < N/2 + j_\psi, \\ C((\varepsilon + N^{-2})N^{-1} + N^{-\beta\sigma_0} + \\ \quad +(N^{-1}\sigma_0 \max_{t\in[1/2,1]} |\psi'(t)|)^4), & \text{if } N/2 + j_\psi \leq j \leq N-1. \end{cases} \tag{27}$$

Remark 2. From Theorem 2 we deduce that, if the diffusion parameter is sufficiently small with respect to the discretization parameter ($\varepsilon \leq CN^{-2}$) and β satisfies $\beta\sigma_0 \geq 3$, the extrapolated solution converges with third order on S-B and S-B modified mesh and with order near three on S-mesh.

§4. Numerical examples

We consider the test problem:

$$-\varepsilon u'' + u' + (e^x + 1 - x^3)u = f,$$

where the source term f and the boundary conditions are such that the exact solution is

$$u(x, \varepsilon) = xe^{-(1-x)/\varepsilon} + \sin(x).$$

For any values of ε and N we calculate exactly the pointwise errors $e_j^{\varepsilon,N} = |u(x_j) - U_j^N|$, $0 \leq j \leq N$, where $\{U_j^N\}$ is the numerical solution obtained with the HODIE finite difference scheme. From these values, the maximum errors are $E^{\varepsilon,N} = \max_{0\leq j\leq N} e_j^{\varepsilon,N}$ and the numerical order of convergence are $p_{\varepsilon,N} = \log(E^{\varepsilon,N}/E^{\varepsilon,2N})/\log 2$.

Table 1 shows the maximum errors and the numerical order of convergence of the HODIE scheme on some particular generalized Shishkin meshes, taking $\varepsilon = 10^{-8}$ and $\sigma_0 = 4$.

Table 1: Errors and order of convergence without extrapolation

$\varepsilon = 10^{-8}$	N=64	N=128	N=256	N=512	N=1024	N=2048
S-mesh	8.541E-3 1.589	2.840E-3 1.620	9.237E-4 1.663	2.917E-4 1.697	8.995E-5 1.725	2.721E-5
S-B mesh	1.268E-3 1.969	3.239E-4 1.987	8.170E-5 1.994	2.051E-5 1.997	5.138E-6 1.999	1.286E-6
S-B m. mesh	1.548E-3 1.919	4.095E-4 1.939	1.068E-5 1.951	2.763E-5 1.960	7.099E-6 1.967	1.816E-6

For the same values of ε and σ_0, table 2 shows the results obtained using Richardson extrapolation. From both tables we see that the results are in agreement with the theoretical results.

Table 2: Errors and order of convergence with extrapolation

$\varepsilon = 10^{-8}$	N=64	N=128	N=256	N=512	N=1024	N=2048
S-mesh	6.388E-5 3.109	7.406E-6 3.222	7.936E-7 3.325	7.919E-8 3.392	7.541E-9 3.451	6.898E-10
S-B mesh	3.597E-6 3.820	2.547E-7 3.850	1.767E-8 3.869	1.209E-9 3.872	8.254E-11 3.803	5.912E-12
S-B m. mesh	2.735E-6 5.109	7.924E-8 3.844	5.178E-9 3.880	3.748E-10 3.867	2.566E-11 3.642	2.054E-12

Moreover, table 2 gives better results than Theorem 2 proves; the reason is that, for this example, the maximum errors are associated to the singular component of the exact solution and therefore the order is given by Lemma 6.

In second place, we take $\sigma_0 = 4$ and $\varepsilon = 10^{-4}$, for which the restriction $\varepsilon < N^{-2}$ is violated. Table 3 shows the results obtained without extrapolation, in agreement with the theoretical results.

Table 3: Errors and order of convergence without extrapolation

$\varepsilon = 10^{-4}$	N=64	N=128	N=256	N=512	N=1024	N=2048
S-mesh	8.538E-3 1.589	2.839E-3 1.620	9.234E-4 1.663	2.916E-4 1.697	8.993E-5 1.725	2.721E-5
S-B mesh	1.267E-3 1.969	3.238E-4 1.987	8.171E-5 1.993	2.053E-5 1.994	5.153E-6 1.993	1.294E-6
S-B m. mesh	1.547E-3 1.919	4.093E-4 1.938	1.068E-4 1.950	2.764E-5 1.958	7.113E-6 1.963	1.824E-6

For these same values, table 4 gives the results associated to the extrapolation; from it we see that for all meshes, principally for S-B and S-B modified meshes, the numerical order of convergence is reduced. Nevertheless, the errors are smaller than in table 3, which is important in practice applications of the method.

Table 4: Errors and order of convergence with extrapolation

$\varepsilon = 10^{-4}$	N=64	N=128	N=256	N=512	N=1024	N=2048
S-mesh	6.380E-5 3.113	7.373E-6 3.247	7.763E-7 3.458	7.065E-8 2.897	9.485E-9 1.000	4.741E-9
S-B mesh	3.722E-6 3.535	3.212E-7 2.646	5.131E-8 1.435	1.897E-8 1.000	9.485E-9 1.000	4.741E-9
S-B m. mesh	2.859E-6 4.285	1.467E-7 1.902	3.925E-8 1.049	1.897E-8 1.000	9.485E-9 1.000	4.741E-9

Acknowledgements

This research has been partially supported by MCYT/FEDER Project BFM2001–2521

References

[1] C. Clavero, J.L. Gracia, F. Lisbona High order methods on Shishkin meshes for singular perturbation problems of convection-diffusion type. *Numerical Algorithms*, 22 (1999), 73–97.

[2] C. Clavero, J.L. Gracia HODIE finite difference schemes on generalized Shishkin meshes. *To appear in J. Comp. Appl. Math.* (2004).

[3] P. Farrell, A. Hegarty, J.J.H. Miller, E. O'Riordan, G.I. Shishkin *Robust computational techniques for boundary layers*. CRC Press, 2000.

[4] A. Fröhner, T. Linss, H.G. Roos Defect correction on Shishkin–type meshes. *Numerical Algorithms*, 26 (2001), 281–299.

[5] H.B. Keller Accurate difference methods for ordinary differential systems subject to linear constraints. *SIAM J. Numer. Anal.*, 6 (1969), 8–30.

[6] H.B. Keller *Numerical methods for two-point boundary value problems*. Dover Publications Inc, New York, 1992.

[7] M.C. Natividad, M. Stynes Richardson extrapolation for a convection–diffusion problem using a Shishkin mesh. *Appl. Numer. Math.*, 45 (2003), 315-329.

[8] H.G. Roos, T. Linss Sufficient conditions for uniform convergence on layer-adapted grids. *Computing*, 63 (1999), 27–45.

J.L. Gracia
Departamento de Matemática Aplicada
Universidad de Zaragoza
Escuela Universitaria Politécnica, Ciudad Escolar, s/n, 44003, Teruel (Spain)
jlgracia@unizar.es

C. Clavero
Departamento de Matemática Aplicada
Universidad de Zaragoza
Centro Politécnico Superior, María de Luna 3, 50018, Zaragoza (Spain)
clavero@unizar.es