ORTHOGONALITY IN THE HYPERBOLIC PLANE

Domingo Gámez, Miguel Pasadas, Rafael Pérez and Ceferino Ruiz

Abstract. In this paper three essential construction are considered, construction which involve the orthogonality in the models of Poincaré for the hyperbolic plane: a) Determination of the line orthogonal to another one given through a point of it. b) Determination of the unique orthogonal line to a pair of ultraparallel lines. c) Determination of the line orthogonal to another one given through an outer point. This construction apply to the resolution of many problems of the plane hyperbolic geometry.

Keywords: Hyperbolic geometry, orthogonality, isometry

AMS classification: AMS 20H15, 51M15, 20H10, 51M10

§1. Introduction

Let $\mathbb{C}^+ = \{z \in \mathbb{C} \ : \ \mathrm{Im}\, z > 0\}$ the open upper half-plane endowed with the metric $ds = |dz| / \mathrm{Im}\, z$. We denote $\mathbb{C}^+$ with this metric by H^2. The geodesics (lines) of H^2 for this metric are obtained by minimizing the functional

$$s = \int_{t_0}^{t_1} \frac{\sqrt{(x'(t))^2 + (y'(t))^2}}{y(t)} dt.$$

Therefore the lines in H^2 are Euclidean half-circumferences centered at a point in the boundary of H^2 which correspond to the parametrization: $x(t) = r\cos t + k_1$, $y(t) = rsent$, $t \in (0, \pi)$; and Euclidean half-lines orthogonal to that boundary with the parametrization: $x(t) = k_2$, $y(t) = t$, $t \in \mathbb{R}^+$.

Let $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$, be the image of $\mathbb{C}^+$ by the Cayley transformation

$$f_c : \mathbb{C}^+ \longrightarrow D, \quad f_c(z) = \frac{z - i}{z + i}.$$

By mean of f_c the metric in H^2 it transforms in $ds = 2\dfrac{|dz|}{1 - |z|^2}$ for D and, with that metric, it will be denoted by D^2. That transformation induces an isomorfism of the group $Iso(H^2)$ of isometries of H^2 onto the group $Iso(D^2)$ of isometries of D^2. In the same way determines the line of D^2 as the arcs of Euclidean circumferences which are orthogonal to $fr(D^2) = \mathbb{S}^1$ or diameters of D^2.

Constructions which involve the orthogonality are necessary for the resolution of many problems of the plane hyperbolic geometry, such as the construction of the circumference, the horocycles, the hypercycles[4], translations according to a line, Saccheri[6] and Lambert[5] quadrilaterals and others, in the two models of Poincaré for the hyperbolic plane: H^2 and D^2[3]. If a Euclidean element appears in some construction or representation of both plane models, we shall note it in an explicit way.

We say that two lines are secant if they cut themselves at a point of H^2; they are parallel if they cut themselves at a point real or at a infinity point of the infinity line. Finally, they are ultraparallel in any other case.

Since H^2 does not have an intrinsic inner product (not even it has structure of linear space) makes no sense to define a concept of intrinsic angle in H^2. For that reason, we resorted to the extrinsic notation of simple angle: the angles will be Euclideanly understood.

At the same time, the angle determined by two geodesics at a common point of H^2 is equal to the angle determined by their normal vectors at this point, and this angle is considered oriented.

We denominated reflection the isometry that fixes the points of a geodesic ℓ and interchanges the two connected components of its complement and is denoted by σ_ℓ.

These constructions can be classified in three types[2]:

1.- Determination of the orthogonal line to another one given through a point of it.

2.- Determination of the unique orthogonal line to a pair of ultraparallel lines.

3.- Determination of the orthogonal line to another one given through an outer point.

§2. Orthogonal line to another one given through a point of it

Given a line ℓ, determine the orthogonal line to ℓ through a point A belonging to ℓ.

2.1. Determination in H^2

Let α be the angle determined by the tangent Euclidean line to ℓ at the point A with the Euclidean line We can consider two cases.

1.- $\alpha \neq 0$

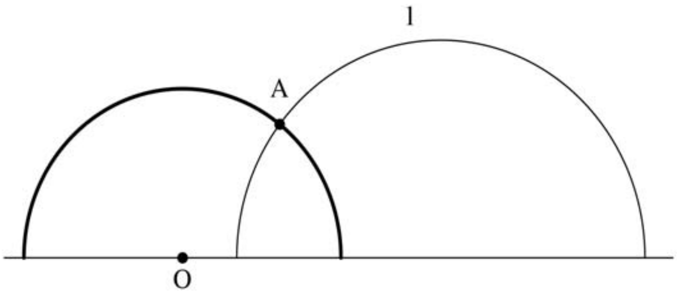
Then the tangent line of the orthogonal line to ℓ at the point A form an angle $\alpha + \frac{\pi}{2}$ with $y = 0$.

Imposing the conditions of orthogonality to the tangent Euclidean lines, we obtain that the center O of the sought orthogonal Euclidean half-circumference is

$$\left(b \tan\left(\alpha + \frac{\pi}{2}\right) + a, 0\right)$$

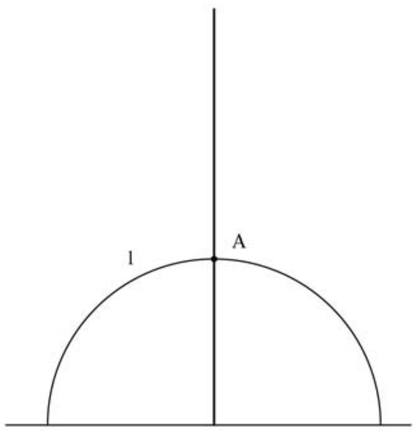
and the radius is

$$b\sqrt{1+\tan^2\left(\alpha+\frac{\pi}{2}\right)}.$$



2.- $\alpha = 0$

In this case, the equation of the sought orthogonal line is $x = a$.



2.2. Determination in D^2

Let A be as point in ℓ. By the Cayley transformation f_c, we construct the line $\widetilde{\ell} = f_c^{-1}(\ell)$ and the point $\widetilde{A} = f_c^{-1}(A)$ in H^2.

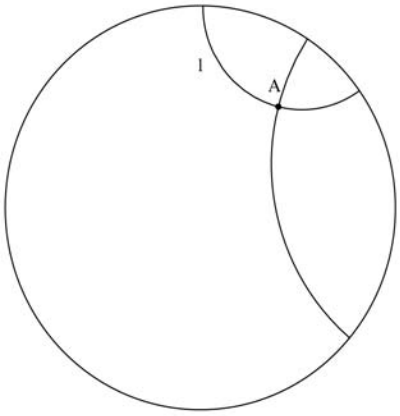
Now, in H^2, we construct the orthogonal line to $\widetilde{\ell}$ through $\widetilde{A}$. Applying f_c we obtain the corresponding orthogonal line in D^2.

Let $\widetilde{\alpha}$ be the angle detemined by the euclidean tangent line to $\widetilde{\ell}$ at the point $\widetilde{A}$ with the euclidean line $y = 0$.

We can consider two cases:

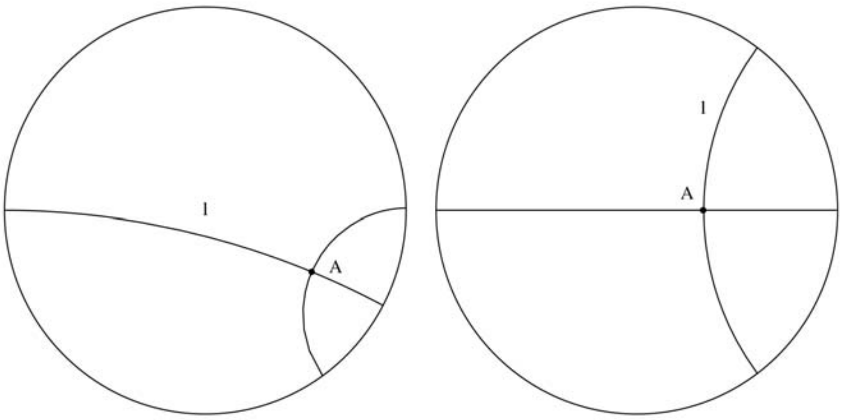
1.- $\widetilde{\alpha} \neq 0$

If $(k_1, 0)$ and r_1 is the center and the radius, respectively, of the Euclidean half-circumference in $\mathbb{C}^+$ corresponding to the orthogonal line to $\widetilde{\ell}$ through $\widetilde{A}$, the orthogonal line to ℓ through A in D^2 is the arc of Euclidean circumference orthogonal to $\mathbb{S}^1$ at the points $f_c(k_1 - r_1, 0)$ and $f_c(k_1 + r_1, 0)$.



2.- $\widetilde{\alpha} = 0$

The equation of the orthogonal line to $\widetilde{\ell}$ at $\widetilde{A}$ is $x = a$, thus the orthogonal line to ℓ at A is an arc of Euclidean circumference through $(1, 0)$ and $f_c(a, 0) \in \mathbb{S}^1$, if $a \neq 0$, or the Euclidean line $y = 0$, if $a = 0$.



§3. Orthogonal line to a pair of ultraparallel lines

Given two ultraparallel lines ℓ and m, determine the unique ([1]) orthogonal line to both.

3.1. Determination en H^2

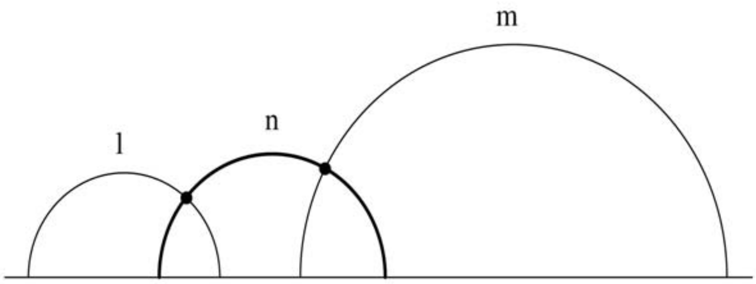
From the euclidean point of view exist four possibilities of relative position for these two ultraparallel lines. From this, we can obtain the following situations:

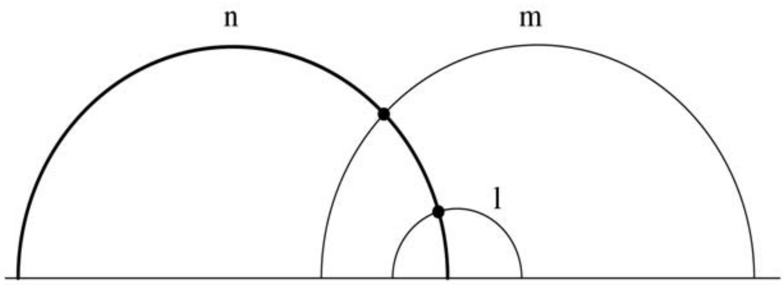
Let $O_1(k_1, 0)$, r_1 and $O_2(k_2, 0)$, r_2 be the center and the radius of the Euclidean half-circumferences corresponding to ℓ and m, respectively, with $k_2 \neq k_1$. The center $O(k, 0)$ and radius r of the Euclidean half-circumference corresponding to the orthogonal line n to both

$$\left(\frac{k_2^2 - r_2^2 - k_1^2 + r_1^2}{2(k_2 - k_1)}, 0\right) \quad \text{and} \quad \sqrt{(k - k_1)^2 - r_1^2},$$

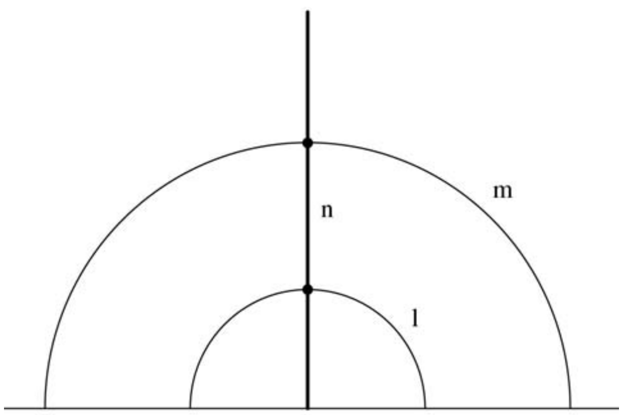
respectively, since one verifies that

$$\begin{cases} r_1^2 + r^2 = (k - k_1)^2 \\ r_2^2 + r^2 = (k_2 - k)^2 \end{cases}$$



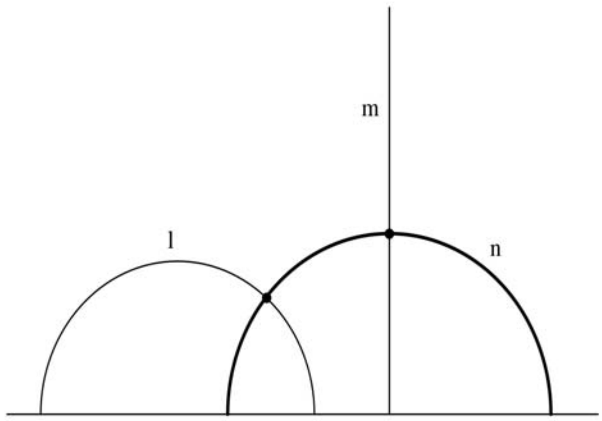


If $k_2 = k_1$ then the orthogonal line to ℓ and m is the Euclidean line of equation $x = k_1$.



Let $O_1(k_1, 0)$ and r_1 be the center and the radius, respectively, of the Euclidean half-circumference corresponding to ℓ, and $x = a$ the Euclidean line corresponding to m. The center and the radius of the Euclidean circumference corresponding to the orthogonal line n to both, are respectively,

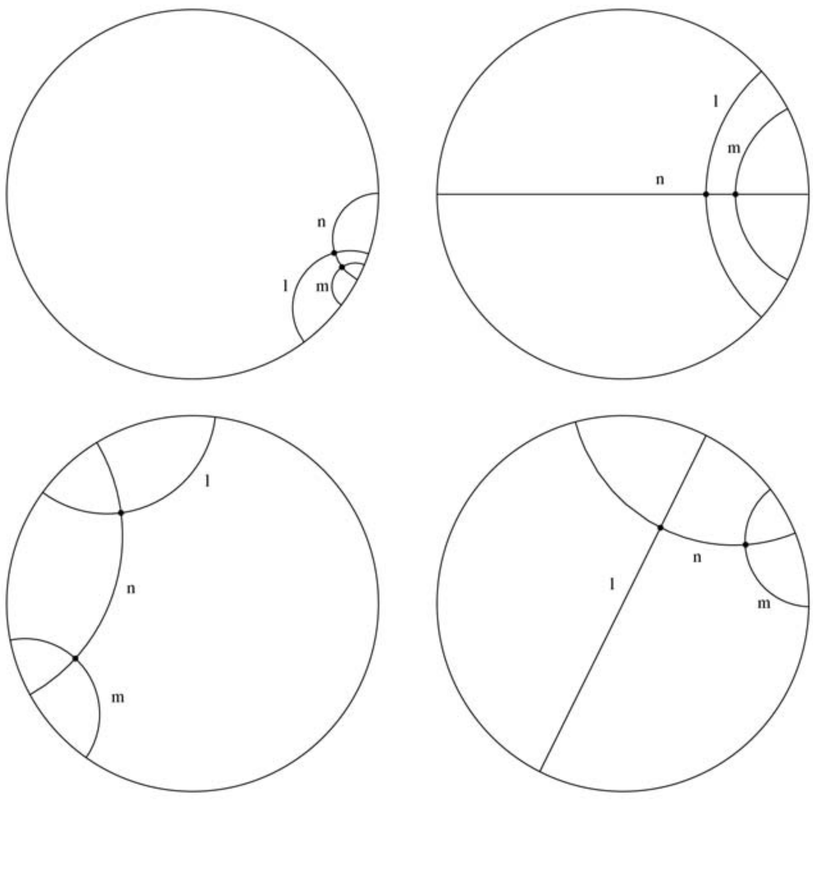
$$(a, 0) \quad \text{and} \quad \sqrt{(a-k_1)^2 - r_1^2}.$$



3.2. Determination in D^2

Let ℓ and m two ultraparallel lines in D^2. By the Cayley transformation f_c, we construct the lines $\widetilde{\ell} = f_c^{-1}(\ell)$ and $\widetilde{m} = f_c^{-1}(m)$ in H^2.

Now, in H^2, we construct the unique orthogonal line to $\widetilde{\ell}$ and $\widetilde{m}$. Applying f_c we obtain the corresponding orthogonal line in D^2:



§4. Orthogonal line to another one given through an outer point

Given a line ℓ and an outer point of it A, determine the orthogonal line to ℓ through A.

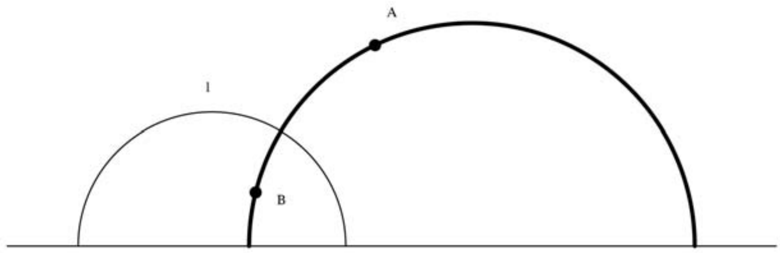
4.1. Determination in H^2

Let B be the refection of the point A respect to the line ℓ. The sought orthogonal line is the line through A and B.

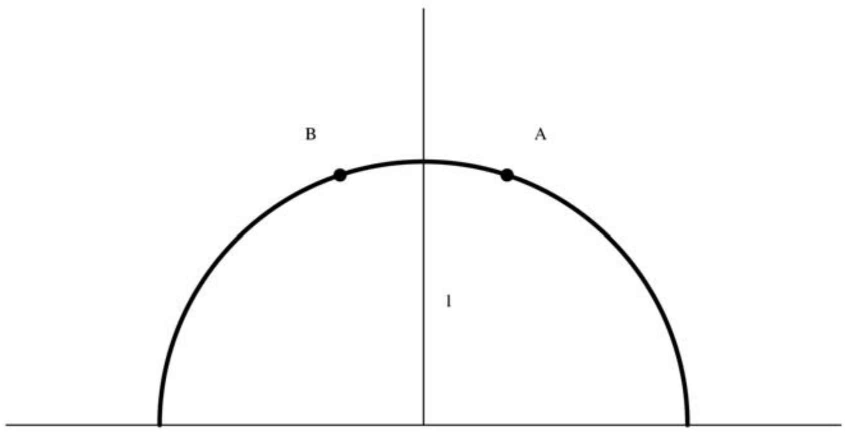
If line ℓ is a Euclidean half-circumference, then the transformed point B of A (p, q) has coordinates

$$\left(\frac{r^2(p-k)}{(p-k)^2+q^2}+k, \frac{r^2 q}{(p-k)^2+q^2}\right),$$

being $(k, 0)$ and r the center and the radius of the Euclidean circumference.



If the equation of the line is $x = a$, the coordinates of B are $(2a - p, q)$.

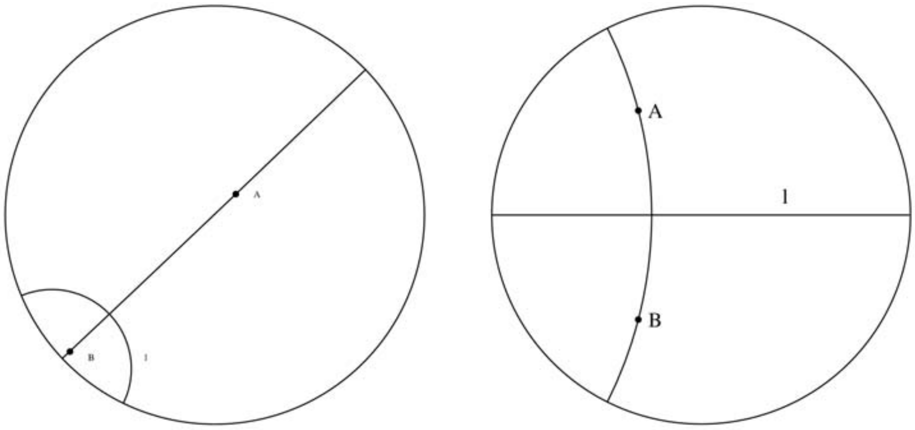


In both cases, uniting the points A and B, the orthogonal line to ℓ through the point A outside to her is obtained.

4.2. Determination in D^2

Let ℓ and A a line and an outer point in D^2, respectively. By the Cayley transformation f_c, we construct the line $\widetilde{\ell} = f_c^{-1}(\ell)$ and $\widetilde{A} = f_c^{-1}(A)$ in H^2.

Now, in H^2, we construct the orthogonal line to $\widetilde{\ell}$ through $\widetilde{A}$. Applying f_c we obtain the corresponding orthogonal line in D^2:



§5. Remarks

We refer the reader to [3] which presents and electronic tool named `Hyperbol` ([1]) whose computational support is *Mathematica* software. this tool consists of modules that allow us to draw different hyperbolic constructions in Poincare's models for the hyperbolic plane, usually denoted by H^2 and D^2. Such constructions include reflections, rotations, translations, glide reflections and the orbits of a point. These isometries and geometric loci act on the hyperbolic plane.

[1] Software available at http://www.ugr.es/local/ruiz/software.htm

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