A TAX-INVESTMENT DYNAMIC REACTION MODEL.

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Abstract. This paper introduces a two-agent dynamic model for studying optimal behaviour of Government and a representative private firm regarding tax and investment policies.

Each agent tries to maximize an objective function over different relevant variables. Government will focus on stating the appropriate tax rate in order to achieve long run budget equilibrium.

On the other side firms will try to determine the optimal investment policy according prices for the single good produced and the cost of use of capital equipment.

We will derive the conditions for optimal long run equilibrium and the reaction functions for each agent in the model.

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§1. Introduction.

We consider a dynamic game with two players: the government (which plays the role of a dominant player) and a single firm in the economy.

The firm uses a unique input, its capital, to get a single output. At each time t its demand function is given by:

$$p_t = A_0 - \frac{1}{2}A_1 Y_t, \qquad \text{with} \qquad A_0,\, A_1 > 0 \tag{1}$$

where, p_t is the price of output at time t, Y_t is the output of the firm which is given by the expression $f_0\, k_t$, where, k_t is the stock of capital at time t and f_0 is a positive constant, so we have:

$$Y_t = f_0 k_t. \tag{2}$$

At each time moment t Government determines the tax rate $\{\tau_t\}_{t=0}^T$ and the firm tries to maximize its profit by choosing the appropriate $\{k_t\}$ over time as a reaction to Government policy. We will focus only in the non-stochastic case where $\{J_t\}$, and $\{p_t\}$ are known values by the firm at each time, and both of exponential order lower than $\frac{1}{\sqrt{b}}$ and J_t the cost of use of capital at time t.

§2. The firm problem.

According to previous definitions, from the firm side we state the following optimization program **to maximize the present discounted value profits**:

$$(P)\left\{\begin{array}{lll} \underset{\{k_t\}}{\text{Max.}} & \sum_{t=0}^{T} b^t\{p_t f_0 k_t - J_t(k_t-k_{t-1}) - \frac{d}{2}(k_t-k_{t-1})^2 - \tau_t p_t f_0 k_t\} \\ \text{where} & k_{-1} \quad \text{is known.} \end{array}\right.$$

Where b is the discount factor, $0 < b < 1$, $p_t f_0 k_t$ are the revenues the firm has got derived from its production and the term $-\frac{d}{2}(k_t - k_{t-1})^2$ where $d > 0$, reflects the profit reductions derived from depreciation of capital equipment and $\tau_t p_t f_0 k_t$ are the taxes on production to be paid by the firm.

Substitution of equation (1), into problem (P) yields:

$$(P^*)\left\{\begin{array}{ll} \underset{\{k_t\}}{\text{Max.}} & \sum_{t=0}^{T} b^t\{(A_0 - \frac{1}{2}A_1 f_0 k_t) f_0 k_t - J_t(k_t-k_{t-1}) - \\ & \quad \frac{d}{2}(k_t-k_{t-1})^2 - \tau_t(A_0 - \frac{1}{2}A_1 f_0 k_t) f_0 k_t\} \\ \text{where} & k_{-1} \quad \text{is known.} \end{array}\right.$$

To solve this problem the state the first order conditions (FOC), and computing derivative with respect to k_t, we get:

$$\begin{aligned} & b^t\{A_0 f_0 - A_1 f_0^2 k_t - J_t - d(k_t - k_{t-1}) - \tau_t A_0 f_0 + \tau_t A_1 f_0^2 k_t\} + b^{t+1}\{J_{t+1} + d(k_{t+1} - k_t)\} \\ = \; & b^t\{A_0 f_0(1-\tau_t) - J_t + bJ_{t+1} + bdk_{t+1} - [A_1 f_0^2(1-\tau_t) + d(1+b)]k_t + dk_{t-1}\} = 0. \end{aligned}$$

Multiplying by $\frac{1}{b^t d}$ and rearranging:

$$bk_{t+1} - [\frac{A_1 f_0^2}{d}(1-\tau_t) + (1+b)]k_t + k_{t-1} = \frac{1}{d}\{J_t - bJ_{t+1} - A_0 f_0(1-\tau_t)\},$$

we now make use of the *lag operator* L, and the previous expression results in:

$$bk_{t+1} - [\frac{A_1 f_0^2}{d}(1-\tau_t) + (1+b)]Lk_{t+1} + L^2 k_{t+1} = \frac{1}{d}\{J_t - bJ_{t+1} - A_0 f_0(1-\tau_t)\}$$

so that:

$$b\{k_{t+1} - \frac{1}{b}[\frac{A_1 f_0^2}{d}(1-\tau_t) + (1+b)]Lk_{t+1} + \frac{1}{b}L^2 k_{t+1}\} \quad = \quad \frac{1}{d}\{J_t - bJ_{t+1} - A_0 f_0(1-\tau_t)\}.$$

We now make the following substitution:

$$\phi_t = \frac{A_1 f_0^2(1-\tau_t)}{d} + (1+b)$$

and previous expression can be written as:

$$b\{1 - \frac{\phi_t}{b}L + \frac{1}{b}L^2\}k_{t+1} \quad = \quad \frac{1}{d}\{J_t - bJ_{t+1} - A_0 f_0(1-\tau_t)\}$$

if in the left hand side we make the following substitution:

$$\{1 - \frac{\phi_t}{b}L + \frac{1}{b}L^2\}k_{t+1} \quad = \quad \{(1-\lambda_1 L)(1-\lambda_2 L)\}k_{t+1}$$

where:

$$\lambda_1 + \lambda_2 \quad = \quad \frac{\phi_t}{b}; \quad \lambda_1\lambda_2 \quad = \quad \frac{1}{b},$$

then:

$$b\{(1-\lambda_1 L)(1-\lambda_2 L)\}k_{t+1} = \frac{1}{d}\{J_t - bJ_{t+1} - A_0 f_0(1-\tau_t)\}. \tag{3}$$

And having in mind that for any real numbers λ and b and for any function X_t, is easy to prove that:

$$\frac{1}{b(1-\lambda L)}X_t = \frac{-\lambda^{-1}L^{-1}}{b(1-\lambda^{-1}L^{-1})}X_t, \tag{4}$$

with $L^{-1}X_t = X_{t+1}$, and appliying (4) to (3), we get:

$$(1-\lambda_1 L)k_{t+1} \quad = \quad \frac{-\lambda_2^{-1}L^{-1}}{1-\lambda_2^{-1}L^{-1}}\frac{1}{bd}\{J_t - bJ_{t+1} - A_0 f_0(1-\tau_t)\}$$

using the previous substitution:

$$\lambda_2^{-1} = b\lambda_1$$

using the inverse of the lag operator, L^{-1}, and simplifying we get:

$$(1-\lambda_1 L)k_{t+1} \quad = \quad \frac{-\lambda_1}{1-b\lambda_1 L^{-1}}\frac{1}{d}\{(1-bL^{-1})J_{t+1} - A_0 f_0(1-\tau_{t+1}),\} \tag{5}$$

at time t, we get the corresponding value

$$(1-\lambda_1 L)k_t \quad = \quad \frac{-\lambda_1}{1-b\lambda_1 L^{-1}}\frac{1}{d}\{(1-bL^{-1})J_t - A_0 f_0(1-\tau_t)\}. \tag{6}$$

In this way, we have got the critical value k_t, (6). To check we have got a maximum we check for the second order conditions (SOC) through the second order derivative of the objective function (P^*), thus:

$$b^t\{-A_1 f_0^2 - d + \tau_t A_1 f_0^2 - bd\} = b^t\{-A_1 f_0^2(1-\tau_t) - (1+b)d\}$$

since $0 < b < 1$ and $A_1,\ f_0,\ d > 0$, for k_t to be a maximum, we need $\tau_t < 1$, and this is always satisfied given that τ_t is the tax rate.

Expression (6) states that *the amount of investment at time* t is a function which depends on the output price, the cost of use of capital and the tax rate imposed by Government, therefore solution to problem (P^*) is the sequence $\{k_t^*\}$ satisfying expression (6) which can be stated as follows:

$$k_t = \lambda k_{t-1} - \frac{\lambda}{d}\frac{1}{1-b\lambda L^{-1}}[(1-bL^{-1})J_t - A_0 f_0(1-\tau_t)]. \tag{7}$$

Therefore, the firm tries to find an equilibrium according to sequences $\{k_t^*\}$ and $\{p_t^*\}$ satisfying (8) and (9) respectively

$$\{k_t^*\} \quad \text{maximize} \quad (P^*) \tag{8}$$

$$p_t^* = A_0 - \frac{1}{2}A_1 k_t^* f_0. \tag{9}$$

§3. The problem of the Government.

We now describe the Government problem trying to get an amount of tax enough to finance a given public expenditure policy g_t, with running from 0 to the final horizon T. The government has to set sequence of taxes $\{\tau_t\}_{t=0}^T$ on firm production Y_t over time which is the only source to finance Government expenditure. In the long run, the discounted value of expenses and revenues must balance, hence the Government will try to minimize the objective function:

$$\sum_{t=0}^{T} b^t\Big(\tau_t p_t Y_t - g_t\Big),$$

where g_t represents the exogenous government expenditure at time t according to its economic policy, b is again the discount factor, and the rest of the expressions have been previously defined for the firm problem. We can now state the problem the Government tries to solve:

$$(P_g)\left\{\begin{array}{ll} \text{Min.} & \sum_{t=0}^{T} b^t\Big(\tau_t p_t Y_t - g_t\Big) \\ \text{s. t.} & k_t = \lambda k_{t-1} - \frac{\lambda}{d}\frac{1}{1-b\lambda L^{-1}}[(1-bL^{-1})J_t - A_0 f_0(1-\tau_t)], \end{array}\right.$$

where the restriction is computed according the optimal values obtained for the firm problem in (7).

Having in mind (1) and (2) the objective function results in:

$$\sum_{t=0}^{T} b^t\{\tau_t(A_0 f_0 k_t - \frac{1}{2}A_1 f_0^2 k_t^2) - g_t\}. \tag{10}$$

The problem restriction can be expressed as:

$$(1-\lambda L)k_t = \frac{-\lambda}{d}\frac{1}{1-b\lambda L^{-1}}\left[(1-bL^{-1})J_t - A_0 f_0(1-\tau_t)\right]$$

or equivalently:

$$k_t = \frac{1}{1-\lambda L}\left\{\frac{-\lambda}{d}\frac{1}{1-b\lambda L^{-1}}\left[(1-bL^{-1})J_t - A_0 f_0(1-\tau_t)\right]\right\}, \tag{11}$$

we recall that J_t is the known cost of use of capital at time t.

To simplify, define:

$$P = \frac{-\lambda}{d(1-\lambda L)(1-b\lambda L^{-1})};\ H = (1-bL^{-1})J_t - A_0 f_0. \tag{12}$$

And substitute (12) into (11) to get

$$k_t = P(H + A_0 f_0 \tau_t) \tag{13}$$

by substituting (13) into the objective function, the problem (P_g) is converted into the expression:

$$\boxed{\text{Min.}\quad h(\tau_t) = \sum_{t=0}^{T} b^t\left\{\tau_t\left[A_0 f_0(PH + PA_0 f_0\tau_t) - \frac{1}{2}A_1 f_0^2(PH + PA_0 f_0 \tau_t)^2\right] - g_t\right\},}$$

hence, the problem to solve by the Government results in choosing the sequence $\tau_t,\ t = 1,\dots T$ for τ_0 given minimizing the objective function.
If we have in mind the (FOC) $h'(\tau_t) = 0$, and after rearranging, we get the following second grade equation in τ_t :

$$\frac{-1}{2}A_0^2 A_1 f_0^4 P^2 \tau_t^2 + \left\{A_0^2 f_0^2 P(H+1) - A_0 A_1 f_0^3 P^2(H + A_0 f_0)\right\}\tau_t +$$

$$f_0 PH\left\{A_0 - \frac{1}{2}A_1 f_0 PH - A_0 A_1 f_0^2 P\right\} = 0$$

and its solution is the critical value τ_t^*, which can be obtained after some tedious calculus.
The second order derivative of the objective function is:

$$h''(\tau_t) = b^t\left\{PA_0^2 f_0^2(H+1) - A_0 A_1 P^2 f_0^4(\tau_t + 1) - P^2 H A_0 A_1 f_0^3\right\},$$

if we substitute for the critical value τ_t^* we get:

$$h''(\tau_t^*) = b^t P A_0 f_0^2\sqrt{(H+1)^2 - 2HPA_1 f_0(H + A_0 f_0 + 1 - A_0) + PA_1 f_0^2 A_0(Pf_0^2 A_0 - 2)} > 0$$

If the following inequality is satisfied

$$PA_0^2 f_0^3 < 2(A_0 f_0 + H)(H + 1) - 2A_0 H,$$

Therefore, according to (13) the previous expression results in:

$$2(1 + bL^{-1})^2 J_t^2 + 2(1 + bL^{-1})(A_0 f_0 - A_0 + 1)J_t + A_0^2 f_0 (2 + \frac{\lambda f_0^2}{d(1 - \lambda L)(1 - b\lambda L^{-1})}) > 0$$

§4. Conclusions.

Both equations corresponding to optimal k_t and τ_t obtained by solving the firm and the Government problems respectively allow to find the sequences for both variables in order to make both firm and Government targets compatible. To achieve this goal initial values for constants $b,\ A_0,\ A_1,\ f_0 \ldots$ and so on should be provided in order to get positive values for k_t and τ_t at each time t.

We can use the software program *Mathematica* to simulate by providing different values for constant parameters, $b,\ A_0,\ A_1,\ f_0 \ldots$, the solutions obtained for both problems give us the optimal sequences for capital $\{k_t\}$ and tax values $\{\tau_t\}$ which are compatible with firm and Government optimal behaviour.

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