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FREDHOLM INTEGRAL EQUATIONS AND SCHAUDER BASES

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Abstract. In this work we present a new numerical method to solve the linear Fredholm integral equation of the second kind which is based on the use of Schauder bases and the geometric series theorem.

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§1. Preliminaries

The aim of this work is to show a new numerical method to approximate the solution of the linear Fredholm integral equation of the second kind

$$\lambda u(x) - \int_a^b k(x,y)u(y)\,dy = f(x), \quad (a \le x \le b),$$

where $k : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ and $f : [a,b] \longrightarrow \mathbb{R}$ are continuous functions and $\lambda \in \mathbb{R} \setminus \{0\}$. We propose a method which makes use of the classical Schauder basis for a suitable Banach space and the geometric series theorem. It provides us a sequence of continuous functions which approximates the solution of the equation.

Let us denote $C([a,b])$ (respectively $C([a,b]^2)$) for the Banach space of all continuous and real–valued functions defined on $[a,b]$ (respectively $[a,b] \times [a,b]$), endowed with its usual sup norm $\|\cdot\|_\infty$ (respectively its usual sup norm $\|\cdot\|_\infty$). We shall also write $C^1([a,b])$ (respectively $C^1([a,b]^2)$) for the space of all functions of C^1 class on $[a,b]$ (respectively $[a,b] \times [a,b]$).

Let $(X, \|\cdot\|)$ be a Banach space. We will use the notation $\mathcal{L}(X)$ for the Banach space of all the continuous linear operators from X to X with the usual operator norm, i.e., given $T \in \mathcal{L}(X)$,

$$\|T\| = \sup_{x \in X, \|x\| \le 1} \|Tx\|.$$

Let us consider the linear integral operator

$$K : C([a,b]) \longrightarrow C([a,b])$$

defined by

$$Ku(x) := \int_a^b k(x,y)u(y)\,dy, \quad (u \in C([a,b]),\ a \le x \le b)\,.$$

Then we can write the linear Fredholm integral equation of the second kind as

$$(\lambda I - K)u = f.$$

To assure that there exists a unique solution of this integral equation, we make use of the geometric series theorem (see [1]): let X be a Banach space, $L \in \mathcal{L}(X)$ and assume that $\|L\| < 1$. Then $I - L$ is a bijection on X, its inverse is a bounded linear operator and

$$(I - L)^{-1} = \sum_{n=0}^{\infty} L^n.$$

Consequently, let us write the integral equation in the following equivalent way:

$$(I - L)u = g,$$

where

$$L = \frac{K}{\lambda}\ ,\ g = \frac{f}{\lambda}\ .$$

So if we assume (for the rest of this work) that $\|L\| < 1$, i.e.,

$$\|K\| = \max_{a \le x \le b} \int_a^b |k(x,y)|\,dy < |\lambda|,$$

then the integral equation has a unique solution which is given by

$$u = (I - L)^{-1} g = \sum_{n=0}^{\infty} L^n g.$$

Therefore, if we consider the sequence $\{u_n\}_{n\ge1}$ of partial sums of this series, whose general term is $u_n = \sum_{k=0}^{n} L^k g$, then the solution u is the limit of that sequence. However, for $n \ge 1$, the expression of $L^n g$ is

$$L^n g(t_1) = \frac{1}{\lambda^n} \int_a^b \overset{n)}{\cdots} \int_a^b k(t_1,t_2)k(t_2,t_3)\cdots k(t_n,t_{n+1})g(t_{n+1})\,dt_{n+1}\cdots dt_2, \quad (t_1 \in [a,b]),$$

which is quite difficult to obtain explicitly. To solve this problem we proceed as follows: we equivalently write the sequence $\{u_n\}$ as

$$\begin{cases} u_0 = g \\ u_n = g + Lu_{n-1} = g(\cdot) + \dfrac{1}{\lambda}\displaystyle\int_a^b k(\cdot,y)u_{n-1}(y)\,dy, \quad n \ge 1 \end{cases} \tag{1}$$

The integrand in the expression of u_n is a function in $C([a,b]^2)$. This function can be written as an infinite series using an appropriate Schauder basis for that Banach space. After that, we consider an approximation of u_n truncating the previous series.

To this end, let us recall (see [3]) that a Schauder basis in a Banach space X is a sequence $\{s_n\}_{n\geq 1}$ in X satisfying that for all $x \in X$ there exists a unique sequence $\{a_n\}_{n\geq 1}$ of scalars such that

$$x = \sum_{n\geq 1} a_n s_n.$$

For such a basis and for each positive integer k, the k^{th} *biorthogonal functional* s_k^* associated to $\{s_n\}_{n\geq 1}$ is the continuous linear functional from X to $\mathbb{R}$ that provides us the k^{th} coefficient of the series, i.e.,

$$s_k^* \left(\sum_{n\geq 1} a_n s_n \right) = a_k$$

and the k^{th} *natural projection* P_k associated to $\{s_n\}_{n\geq 1}$ is the continuous linear operator from X to X that gives us the k^{th} partial sum of the series, i.e.,

$$P_k \left(\sum_{n\geq 1} a_n s_n \right) = \sum_{n=1}^{k} a_n s_n.$$

Now let $\{t_i : i \geq 1\}$ be a dense subset of distinct points in $[a,b]$, with $t_1 = a$ and $t_2 = b$. Then the classical Schauder basis $\{b_n\}_{n\geq 1}$ for $C([a,b])$ associated with such points is given in the following way:

$$b_1(t) = 1, \quad \forall t \in [a,b]$$

and for $j \geq 2$, b_j is the function from $[a,b]$ to $\mathbb{R}$ whose graph is the polygonal passing through the points $(t_1, 0), \ldots (t_{j-1}, 0), (t_j, 1)$.

Finally, the classical Schauder basis $\{B_n\}_{n\geq 1}$ for the Banach space $C([a,b]^2)$ is given by the following expression (see [2] and [4]):

$$B_n(x,y) = b_{\tau_1(n)}(x) b_{\tau_2(n)}(y), \quad (x, y \in [a,b]),$$

where $\tau = (\tau_1, \tau_2) : \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ is the bijective mapping defined by

$$\tau(n) := \begin{cases} (\sqrt{n}, \sqrt{n}), & \text{if} \quad [\sqrt{n}] = \sqrt{n} \\ (n - [\sqrt{n}]^2, [\sqrt{n}] + 1), & \text{if} \quad 0 < n - [\sqrt{n}]^2 \leq [\sqrt{n}] \\ ([\sqrt{n}] + 1, n - [\sqrt{n}]^2 - [\sqrt{n}]), & \text{if} \quad [\sqrt{n}] < n - [\sqrt{n}]^2 \end{cases}$$

being $[x] = \max\{k \in \mathbb{Z} : k \leq x\}, \ (x \in \mathbb{R})$.

If we define

$$\Phi(x,y) := k(x,y) v(y), \quad (x, y \in [a,b], v \in C([a,b])),$$

then

$$\Phi(x,y) = \sum_{n=1}^{\infty} B_n^*(\Phi) B_n(x,y). \tag{2}$$

Therefore, the image of a continuous function v under the integral operator L is easily obtained (using (2), the uniform convergence of the previous series and the linearity of the integral) as follows:

$$\begin{aligned}(Lv)(x) &= \frac{1}{\lambda}\int_a^b \Phi(x,y)\,dy = \\ &= \frac{1}{\lambda}\int_a^b \left(\sum_{n=1}^{\infty} B_n^*(\Phi)B_n(x,y)\right) dy = \\ &= \frac{1}{\lambda}\left(\sum_{n=1}^{\infty} B_n^*(\Phi)\int_a^b B_n(x,y)\,dy\right) = \\ &= \frac{1}{\lambda}\sum_{n=1}^{\infty}\left(B_n^*(\Phi)b_{\tau_1(n)}(x)\int_a^b b_{\tau_2(n)}(y)\,dy\right).\end{aligned}$$

In the following, $\{P_n\}_{n\geq 1}$ will denote the sequence of natural projections associated to the basis $\{B_n\}_{n\geq 1}$.

The next result, consequence of some elementary properties of the basis $\{B_n\}_{n\geq 1}$ and the mean value theorem, estimates the difference of a function and its n^{th} natural projection.

Proposition 1. *Let* $\varphi \in C^1([a,b]^2)$ *and*

$$M := \max\left\{\left\|\frac{\partial\varphi}{\partial x}\right\|_\infty, \left\|\frac{\partial\varphi}{\partial y}\right\|_\infty\right\}.$$

Suppose that $M \neq 0$ *(otherwise the statement is trivially satisfied). Given* $\varepsilon > 0$, $\{t_i\}_{i\geq 1}$ *a dense subset of distinct points in* $[a,b]$*, for all* $n \geq 2$*, we note* $\Delta_n := \{a = x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ *the points* $\{t_1,\ldots,t_n\}$ *ordered in an increasing way and assume that*

$$\max_{i=2,\cdots,n}(x_i - x_{i-1}) < \frac{\varepsilon}{4M},$$

then

$$\|\varphi - P_{n^2}(\varphi)\|_\infty \leq \varepsilon.$$

§2. Numerical method

The geometric series theorem provides us the sequence $\{u_n\}_{n\geq 1}$ which converges uniformly on $[a,b]$ to the solution u of the linear Fredholm integral equation of the second kind. In the next theorem we present an approximation of the function u_n, using for that purpose the sequence of natural projections $\{P_n\}_{n\geq 1}$.

Theorem 2. *Let $k \in C([a,b]^2)$, $g \in C([a,b])$, $\lambda \in \mathbb{R}\setminus\{0\}$ and the corresponding Fredholm integral operator L. Let us consider $m \in \mathbb{N}$, $n_1, \ldots, n_m \in \mathbb{N}$ and for $i = 1, \ldots, m$ we inductively define the functions*

$$\tilde{u}_i(x) := g(x) + \frac{1}{\lambda}\int_a^b P_{n_i}(k(x,y)\tilde{u}_{i-1}(y))\,dy, \qquad (x \in [a,b]),$$

where $\tilde{u}_0 = g$. For all $i = 1, \ldots, m$, let $\varepsilon_i > 0$ and assume that

$$\|(g + L\tilde{u}_{i-1}) - \tilde{u}_i\|_\infty < \varepsilon_i.$$

Then

$$\|u - \tilde{u}_m\|_\infty \leq \|g\|_\infty \frac{\|L\|^{m+1}}{1-\|L\|} + \sum_{i=1}^m \varepsilon_i,$$

where u is the solution of the Fredholm integral equation.

Proof. Using the triangular inequality, we obtain

$$\|u - \tilde{u}_m\|_\infty \leq \|u - u_m\|_\infty + \|u_m - \tilde{u}_m\|_\infty. \tag{3}$$

On the one hand,

$$\begin{aligned} \|u - u_m\|_\infty &= \left\| \sum_{j\geq 0} L^j g - \sum_{j=0}^m L^j g \right\|_\infty = \left\| \sum_{j\geq m+1} L^j g \right\|_\infty \leq \sum_{j\geq m+1} \left\|L^j g\right\|_\infty \leq \\ &\leq \sum_{j\geq m+1} \|L\|^j \|g\|_\infty = \|g\|_\infty \left(\sum_{j\geq m+1} \|L\|^j \right) = \|g\|_\infty \frac{\|L\|^{m+1}}{1-\|L\|}. \end{aligned} \tag{4}$$

And on the other hand,

$$\begin{aligned} \|u_m - \tilde{u}_m\|_\infty &\leq \|u_m - (g + L\tilde{u}_{m-1})\|_\infty + \|(g + L\tilde{u}_{m-1}) - \tilde{u}_m\|_\infty \leq \\ &\leq \|g + Lu_{m-1} - g - L\tilde{u}_{m-1}\|_\infty + \varepsilon_m = \\ &= \|L(u_{m-1} - \tilde{u}_{m-1})\|_\infty + \varepsilon_m < \\ &< \|u_{m-1} - \tilde{u}_{m-1}\|_\infty + \varepsilon_m. \end{aligned}$$

If we recurrently repeat the previous process, we have that

$$\|u_m - \tilde{u}_m\|_\infty \leq \sum_{i=1}^m \varepsilon_i. \tag{5}$$

Now, substituting the upper bounds (4) and (5) in (3), we conclude the proof. □

Finally, with the next proposition we pretend to complete the previous theorem in order to determine which natural numbers $n_1, \ldots, n_m$ must be taken.

Proposition 3. *Let us consider* $k \in C^1([a,b]^2)$, $g \in C^1([a,b])$, $\lambda \in \mathbb{R} \setminus \{0\}$ *and the functions* $\{\tilde{u}_n\}_{n\geq 1}$ *defined in Theorem 2. We assume that for* $p \in \mathbb{N}$, $M_p \neq 0$, *where*

$$M_p := \max\left\{\left\|\frac{\partial k}{\partial x}\right\|_\infty \|\tilde{u}_{p-1}\|_\infty, \left\|\frac{\partial k}{\partial y}\right\|_\infty \|\tilde{u}_{p-1}\|_\infty + \|k\|_\infty \|\tilde{u}'_{p-1}\|_\infty\right\}.$$

Given $\varepsilon_p > 0$, *fix* $n_p \geq 2$ *and suppose that* $\Delta_{n_p} = \{a = x_1 < x_2 < \ldots < x_{n_p-1} < x_{n_p} = b\}$ *satisfies that*

$$\max_{i=2,\cdots,n_p}(x_i - x_{i-1}) < \frac{\varepsilon_p|\lambda|}{4M_p(b-a)}.$$

Then

$$\|(g + L\tilde{u}_{p-1}) - \tilde{u}_p\|_\infty \leq \varepsilon_p.$$

Proof. Since

$$\frac{\partial(k(x,y)\tilde{u}_{p-1}(y))}{\partial x}(x,y) = \frac{\partial k}{\partial x}(x,y)\tilde{u}_{p-1}(y)$$

and

$$\frac{\partial(k(x,y)\tilde{u}_{p-1}(y))}{\partial y}(x,y) = \frac{\partial k}{\partial y}(x,y)\tilde{u}_{p-1}(y) + k(x,y)\tilde{u}'_{p-1}(y),$$

then

$$\max\left\{\|\frac{\partial(k(x,y)\tilde{u}_{p-1}(y))}{\partial x}\|_\infty, \|\frac{\partial(k(x,y)\tilde{u}_{p-1}(y))}{\partial y}\|_\infty\right\} \leq M_p\,.$$

Now, applying Proposition 1 for $x \in [a,b]$, we obtain

$$|(g + L\tilde{u}_{p-1})(x) - \tilde{u}_p(x)| = \left|\frac{1}{\lambda}\int_a^b \big(k(x,y)\tilde{u}_{p-1}(y) - P_{n_p}(k(x,y)\tilde{u}_{p-1}(y))\big)\ dy\right| \leq$$

$$\leq \frac{b-a}{|\lambda|}\ \|k(x,\cdot)\tilde{u}_{p-1}(\cdot) - P_{n_p}(k(x,\cdot)\tilde{u}_{p-1}(\cdot))\|_\infty \ \ \leq \frac{b-a}{|\lambda|}\ \frac{|\lambda|}{b-a}\ \varepsilon_p = \varepsilon_p.$$

Finally, since x is arbitrary in $[a,b]$,

$$\|(g + L\tilde{u}_{p-1}) - \tilde{u}_p\|_\infty \leq \varepsilon_p,$$

as required. □

§3. Numerical example

In this section we approximate the solution of the integral equation

$$5u(x) - \int_0^1 e^{xy}u(y)\ dy = f(x), \quad (0 \leq x \leq 1) \tag{6}$$

using our method and compare our results with those one given by the classical collocation method (see [1]).
First of all, we note that this integral equation has a unique solution because of

$$\|K\| = e - 1 < 5 = |\lambda|.$$

Let us consider the functions

$$u^{(1)}(x) = e^{-x}\cos x, \quad u^{(2)}(x) = \sqrt{x}, \quad (0 \leq x \leq 1)$$

as exact solutions of the equation (6) and obtain $f(x)$ accordingly. For $i = 1, 2,$ we denote

$$E_n^{(i)} = \max_{1\leq j\leq n+1} \left| u^{(i)}(x_j) - u_n^{(i)}(x_j) \right|,$$

where $u_n^{(i)}(x)$ is the approximation of the exact solution $u^{(i)}(x)$ given by the collocation method and $\{x_j\}_{j=1}^{n+1}$ are the nodes of this method.
To compare these results with those one obtained by our method, we proceed as follows: in the definition of the classical Schauder basis for $C([0,1])$ we consider the dense subset

$$\{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots, \frac{1}{2^k}, \frac{3}{2^k}, \ldots, \frac{2^k-1}{2^k}, \ldots\}.$$

Fixed k, the set

$$\{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots, \frac{1}{2^k}, \frac{3}{2^k}, \ldots, \frac{2^k-1}{2^k}\}$$

coincides with the nodes of the collocation method for $n = 2^k$, and the cardinal of this set is $n+1$. We take the values $n_1, \ldots, n_m$ in Theorem 2 as $n_1 = \ldots = n_m = n+1$. We also denote $\tilde{u}_{n,p}^{(i)}(x)$ for the approximation, obtained by our numerical method, of the exact solution $u^{(i)}(x)$. The natural number p specifies the number of iterations for each fixed n. For each iteration we denote

$$F_{n,p}^{(i)} = \max_{1\leq j\leq n+1} \left| u^{(i)}(x_j) - \tilde{u}_{n,p}^{(i)}(x_j) \right|.$$

To determine this number p, we have established the criterion of choosing p such that

$$\frac{F_{n,p}^{(i)}}{F_{n,p+1}^{(i)}} < 1 + 10^{-2}.$$

The results we have obtained when programming both methods are presented in the following tables:

n	p	$E_n^{(1)}$	$F_{n,p}^{(1)}$
$n=8$	$p=9$	3.27×10^{-4}	2.55×10^{-4}
$n=16$	$p=10$	8.18×10^{-5}	6.36×10^{-5}
$n=32$	$p=11$	2.04×10^{-5}	1.58×10^{-5}

n	p	$E_n^{(2)}$	$F_{n,p}^{(2)}$
$n=8$	$p=7$	2.75×10^{-3}	2.09×10^{-3}
$n=16$	$p=8$	9.65×10^{-4}	7.62×10^{-4}
$n=32$	$p=9$	3.40×10^{-4}	2.76×10^{-4}

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