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The approximated variance of the output of nonlinear stochastic systems

Ruiz-Fuentes, N.[1], Ruiz-Molina, J.C.[1] & Valderrama, M.J.[2]

[1] Departamento de Estadística e I.O. Universidad de Jaén

[2] Departamento de Estadística e I.O. Facultad de Farmacia. Universidad de Granada

Abstract

The equivalent linearization technique is applied to approximate and analyze the input-output relationships of nonlinear stochastic systems in the frequency domain. When it is applied, the original coefficients of the system are approximated to give the structure of the equivalent system. These new coefficients depend on the variance of the output, our aim is to approximate it. The spectral representation of the involved processes allows us to express the variance of the output as the solution of an integral equation depending on itself. We apply the technique to combined Duffing-van der Pol systems and we deal with the use of numerical methods to accelerate the convergence of the iterative procedure used to approximate the variance. The procedure depends on the nature of the input, therefore the results are presented for several inputs.

Keywords: Equivalent Linearization Technique, Duffig-van der Pol, spectral representation

AMS Classification: 93E03, 93B18, 62M15

1 Introduction

The equivalent linearization technique is applied to nonlinear systems whose nonlinear term can be neglected to yield an underlying linear system whose dynamic equations are suitable to be treated by the technique. Our main objective is to carry out a fast algorithm to approximate the variance of the output of the system. This goal allow us to approximate some features of the real output and it raised when the approximation procedures were applied because the coefficients of the linearized system depend on the unknown variance of the output and we need to implement an algorithm to approximate it. The procedure also depends on the input, thus we present the results under several inputs and for various parameters. Our aim is also to compare the results for different degrees of non-linearity.

We consider a combined system so that it is necessary to make additional suppositions about the output, however the method is applicable to both combined systems widely used in several fields. The combined systems are the Duffing system and the van der Pol system, both systems are represented by the same linear dynamics and a term that represents their nonlinear behaviour. These systems have a nonlinear component that can be neglected leading to a linear system.

Taking into account that the technique provides a linear system, it is possible to apply the input-output relationships available for such systems to characterize the original one. These relationships are the basic tool in this paper because the spectral density of the approximated output is expressed through the spectral density of the input and the amplitude response of the linearized system. The variance of the output can be obtained as the total area under its auto-spectral density, so that we find that the variance of the output is the solution of an integral equation and the integrand is a function depending on it. To solve the problem we implement an iterative algorithm to approximate the variance and finally we can approximate the spectral density of the output.

2 The equivalent Linearization Technique

The following equation represents a broad class of nonlinear dynamical systems

$$\ddot{\mathbf{y}}_t + \alpha\dot{\mathbf{y}}_t + \omega_0^2\mathbf{y}_t + f_0\left(\mathbf{y}_t, \dot{\mathbf{y}}_t\right) = \mathbf{x}_t,$$

where $\mathbf{x}_t$ is a stochastic process representing the input, $\mathbf{y}_t$ is a stochastic process representing the associated output and $f_0\left(\mathbf{y}_t, \dot{\mathbf{y}}_t\right)$ is the nonlinear contribution which is neglected by the technique leading to a linear system, α and ω_0^2 are the coefficients of the system. The system is called combined Duffing-van der Pol system when

$$f_0\left(\mathbf{y}_t, \dot{\mathbf{y}}_t\right) = \frac{k}{m}\mathbf{y}_t^3 + \frac{c}{m}\mathbf{y}_t^2\dot{\mathbf{y}}_t$$

The technique requires that the input or applied perturbation $\mathbf{x}_t$ is a zero mean, continuous in quadratic mean wide-sense stationary gaussian process. The linearization technique also considers that the output of the system is approximated by the output obtained after filtering the original input distorted by the stochastic process ε_t^e, *i.e.* $\mathbf{x}_t + \varepsilon_t^e$, through the original system neglecting the nonlinear component, that is

$$\ddot{\mathbf{y}}_t + \alpha\dot{\mathbf{y}}_t + \omega^2\mathbf{y}_t = \mathbf{x}_t + \varepsilon_t^e.$$

The perturbation ε_t^e is considered as the linearization error. However, what we really consider as an approximation of the output is the stochastic process $\mathbf{y}_t^e$ obtained by filtering the input $\mathbf{x}_t$ through a linear equivalent system of a family of systems depending on the coefficients α_e, ω_e^2, this fact is represented by the following expression

$$\ddot{\mathbf{y}}_t^e + \alpha_e \dot{\mathbf{y}}_t^e + \omega_e^2 \mathbf{y}_t^e = \mathbf{x}_t.$$

The selection of the equivalent system from the previous family, that means, the selection of the coefficients to characterize the equivalent linear system, are obtained by minimizing the quadratic mean error due to the linearization procedure. The best choice for the Duffing-van der Pol system (Ruiz-Fuentes, 1999) is the following

$$\alpha_e \simeq \alpha + \frac{c}{m}\sigma^2_{\mathbf{y}_t^e},$$

$$\omega_e^2 \simeq \omega_0^2 + 3\frac{k}{m}\sigma^2_{\mathbf{y}_t^e},$$

where α, ω_0^2 and k are the parameters of the original system. However α_e and ω_e^2 depend on $\sigma^2_{\mathbf{y}_t^e}$ which is unknown and our main objective is to approximate it. These parameters characterize the equivalent system and they determine its transfer function given by

$$\eta_e(\omega) = \frac{1}{2\pi\omega\alpha_e i + (\omega_e^2 - 4\pi^2\omega^2)}, \quad \forall \omega \in (-\infty, \infty).$$

If ε_t^e represents the error process, the quadratic mean error which measures the accuracy of the technique is given by $E\left[(\varepsilon_t^e)^2\right] = 2\left(\frac{c}{m}\right)^2 \sigma^4_{\mathbf{y}_t^e}\sigma^2_{\dot{\mathbf{y}}_t^e} + 6\left(\frac{k}{m}\right)^2 \sigma^6_{\mathbf{y}_t^e}$ also depending on $\sigma^2_{\mathbf{y}_t^e}$.

In the case of the combined system we have to make additional suppositions about the output because the nonlinear term $\mathrm{f}_0(\mathbf{y}_t, \dot{\mathbf{y}}_t)$ cannot be decomposed in two components depending on y_t and $\dot{y}_t$, respectively. We suppose that the output is a narrow band process.

3 Approximation algorithm

We use the spectral representation of stochastic processes and the properties of linear systems to approximate the variance of the approximated output through an iterative algorithm. If we consider that the equivalent system is linear, the spectral densities of the input and of the output are related by

$$S_{\mathbf{y}_t^e}(\omega) = |\eta_e(\omega)|^2 S_x(\omega)$$

and

$$\sigma^2_{\mathbf{y}_t^e} = \int_{-\infty}^{\infty} |\eta_e(\omega)|^2 S_x(\omega)\ d\omega.$$

Hence, the variance of the output is the solution of an integral equation whose integrand is also a function depending on it through the amplitude response operator $|\eta_e(\omega)|^2$ that depends on the parameters α_e and ω_e^2. We write this fact in a reduced form as

$$\theta = \sigma^2_{\mathbf{y}_t^e} = \Psi(\theta)$$

We propose and initial value $\sigma_0^2 = \theta_0$, for the parameter θ and we fix the desired tolerance. In the following steps, for $i > 1$, we apply the Steffensen and Aitken methods, that is, we calculate

$$\theta_i = \Psi\left(\theta_{i-1}\right) \quad \text{and} \quad \theta_{i+1} = \Psi\left(\theta_i\right)$$

and then evaluate

$$\sigma_i^2 = \theta_{i-1} - \frac{\left(\theta_i - \theta_{i-1}\right)^2}{\theta_{i+1} - 2\theta_i + \theta_{i-1}}$$

Afterwards we compare the difference $\left|\sigma_i^2 - \sigma_{i-1}^2\right|$ with the desired tolerance and if it is exceeded the procedure begins again with $\theta_0 = \sigma_i^2$. When the procedure finishes we have estimated the value of the variance of the approximated output. As we have mentioned before the applied input determine the procedure of estimation because the evaluation of the integral at each step is different depending on the spectral density of the input.

4 Numerical results

To shorten we outline the results only for two cases: white noise and wide-band processes.

White noise process

Let us consider that the input is a white noise process with constant spectral density over all the frequencies, $S\left(\omega\right) = S_0,\ \forall\omega \in \mathbb{R}$. Thus, $S_{\mathbf{y}_t^e}\left(\omega\right) = S_0 \left|\eta_e\left(\omega\right)\right|^2,\ \forall\omega \in \mathbb{R}$ and the iterative algorithm has to solve the following integral equation

$$\sigma_{\mathbf{y}_t^e}^2 = \int_{-\infty}^{\infty} \frac{S_0}{4\pi^2\omega^2\alpha_e^2 + \left(\omega_e^2 - 4\pi^2\,\omega^2\right)^2}\, d\omega$$

The results of the iterative algorithm are summarized in Table 1 for some values of S_0 and in Table 2 for several values of k. Obviously as the spectrum increases so do the variance (Table 1). Columns *It 1* and *It 2* show the number of iterations needed to estimate the variance, the first column is the number of iterations without considering the Aitken and Steffensen procedures and *It 2* shows the results within these procedures. Although it is not necessary to justify that these procedures are adequate, we can check by comparing the values given in these columns that the number of iterations has been significantly reduced. The variance and the error values were the same for both procedures, there were only insignificant differences.

Wide-band process

If the input is a wide-band process it has constant spectrum in a bounded interval, $S\left(\omega\right) = S_0,\ \forall\omega \in [0, \omega_0]$ and it is zero valued elsewhere. Hence, $S_{\mathbf{y}_t^e}\left(\omega\right) = S_0 \left|\eta_e\left(\omega\right)\right|^2, \forall\omega \in [0, \omega_0]$, and

$$\sigma_{\mathbf{y}_t^e}^2 = \int_0^{\omega_0} \frac{S_0}{4\pi^2\omega^2\alpha_e^2 + \left(\omega_e^2 - 4\pi^2\,\omega^2\right)^2}\, d\omega$$

	Input: white noise				
Parameters	Spectrum	Variance	Error	It 1	It 2
$k = 1$	1	0.236529	0.080534	22	3
$c = 1$	1.5	0.302127	0.168147	33	4
$m = 1$	3	0.444773	0.538292	90	4
$\alpha = 1$ $\omega_0^2 = 1$	5	0.578737	1.18991	123	5

Table 1: Same parameters-different spectrum S_0

	Input: white noise				
Parameters	k	Variance	Error	It 1	It 2
$S_0 = 3$	0.1	0.718235	0.045044	24	3
$c = 1$	0.5	0.538966	0.249173	52	4
$m = 1$	1.5	0.390881	0.814656	133	4
$\alpha = 1$ $\omega_0^2 = 1$	3	0.305971	1.5524	333	5

Table 2: Same spectrum and parameters-different values of k

Table 3 summarizes some of the results in this case and Figures 1 and 2 show the behaviour of the error at each step for the last values of the parameters shown in Table 3.

	Input: wide band process				
Parameters	Spectrum	Variance	Error	It 1	It 2
$k = 1$	1	0.149947	0.0204666	13	3
$c = 1$	1.5	0.196935	0.0464139	18	3
$m = 1$	3	0.302113	0.168007	33	4
$\alpha = 1$ $\omega_0^2 = 1$	5	0.40316	0.400134	64	4

Table 3: Same parameters-different spectrum S_0

5 Conclusions

After the troubles raised with the application of the equivalent linearization technique have been solved, the solution of the iterative algorithm leads to acceptable results after few iterations for a small tolerance, the procedure allows to approximate the spectral

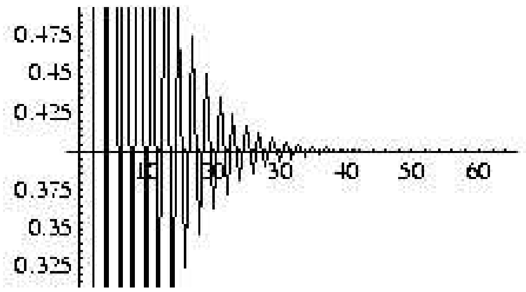

Figure 1: Plot of error at each step. Unitary parameters, $S_0 = 5$

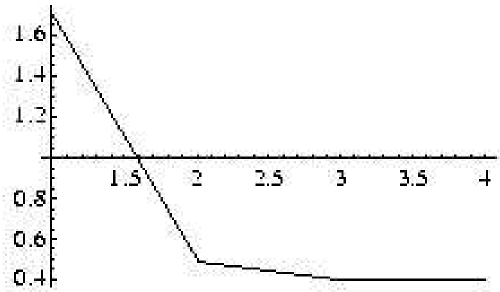

Figure 2: Plot of error at each step. Fast method. Unitary parameters, $S_0 = 5$

density of the output of the nonlinear system.

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