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A new ensemblist difference

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Abstract

We introduce a new operation for the difference of two sets A and C of $\mathbb{R}^n$ depending on a parameter α. This operation may yield as special cases the classical difference and the Minkowski difference. Continuity properties with respect to both the operands and the parameter of this operation are studied. Lipschitz properties of the Minkowski difference between two sets of a normed vector space are proved in the bounded case as well as in the unbounded case without condition on the dimension of the space, improving previous results.

Keywords: Minkowski difference, algebraic and geometric differences of sets.

AMS Classification: 52A30, 52A38

1 Introduction.

Recently, there has been a breakthrough in the literature concerning concrete applications of Mathematics to fields like biology, ecology, using the concept of shape evolution ([1]). If these phenomena may be represented by means of partial differential equations, a global approach is sometimes necessary; if one wants to describe a zone whose evolution depends on its shape, as a burning or epidemic zone, a differential calculus on the power set of $\mathbb{R}^n$, or, more generally, in a metric space, is required. The embedding methods (see [10], [15]) are in general not satisfying because the knowledge and the visualization of the embedded sets are not always available. Several authors have already introduced various notions of differentiability and integration of multi-valued mappings whose values are subsets of $\mathbb{R}^n$ that take into account the shape of the values of the map, to give a meaning to the differential equation $X'(t) = F(X(t))$, where X is a set-valued map from $\mathbb{R}_+$ to the power set of a normed vector space (see [6], [18], [11], [1]). Now, it would be interesting to apply these theories (see [5], for an example of a differential equation with semi-affine approximation). An evolution field (*i.e.* F) which may be interpreted in biological terms is needed to study concrete cases. In this paper, we are interested in the concept of differences. The classical difference $A - C = \{a - c : a \in A, c \in C\}$ of two subsets A and C of a normed vector space is large unless C is empty or a singleton. On the other hand, the Minkowski difference defined by $A \overset{*}{-} C = \{x : x + C \subset A\}$ (see [7]) is small thus more convenient and is already used in morphological analysis to describe erosion (as in [1]). However, it has the drawback of often being empty. Another difference is the Demyanov's difference, useful in optimization, non-smooth analysis and quasidifferential calculus (see [17]), constructed *via* the difference of supporting points. In the same direction, difference of support functions is also considered ([8]). Based on these, difference of directed sets is presented in [2]. There is no biological interpretation of theses differences. Consequently, they seem not to be adapted to certain biological problems.

The aim of this paper is, on the one hand, to overcome the above disadvantages and, on the other hand, to present an operation that can be interpreted in biological terms. To this end, a new operation scheme is proposed on the subsets of $\mathbb{R}^n$. Let us consider a simple example, the study of the evolution of a zone contaminated by a virus, satisfying a differential equation $X'(t) = F(X(t))$. $X(t)$ is the zone where $p\%$ of the population is infected by the virus at the time t. Which F can describe the evolution of $X(t)$? It is clear that more the contaminated zone near a point x is large more quick is the contamination. In this case, the set $X(t) \cap B(x, R)$ has to be sufficiently large to be infecting. We then introduce the set $X(t) \overset{\alpha}{-} B(x, R) = \{y \in \mathbb{R}^n : \mathcal{L}((y + B(x, R)) \cap X(t)) \geq \alpha\}$, where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^n$. R represents the maximum radius of influence of the virus. α measures the virulence of the virus: more α is small, more the virus is very infectious. We have extended this difference to a general case: $A \overset{\alpha}{-} C = \{x \in \mathbb{R}^n : \mathcal{L}((x + C) \cap A) \geq \alpha\}$. This difference coincides with the classical difference taking $\alpha = 0$ and with the Minkowski difference, taking $\alpha = \mathcal{L}(C)$ whenever A, C are closed, convex sets, $\mathrm{int}(C)$ is nonempty and A or C is bounded.

In the following section, we introduce an abstract scheme of difference on the subsets of $\mathbb{R}^n$ and we study its continuity properties. The last section is devoted to Lipschitz properties of the Minkowski difference, improving previous results (see [14], [18], [12]). These results, and their proofs, can be found in [13].

2 A new difference on the power set of $\mathbb{R}^n$

Let E be a normed vector space, whose closed unit ball is denoted by B. $C(E)$ represents the space of all the closed, convex, nonempty sets of E and $CB(E)$ the elements of $C(E)$ which are bounded. The convergence in the sense of Painlevé-Kuratowski is denoted P.K-convergence. The Hausdorff distance is represented by d_H. Continuity with respect to this distance is called H-continuity.

We introduce a new notion of difference between two sets in the particular case where $E = \mathbb{R}^n$. $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^n$ and by convention $\mathcal{L}(\emptyset) = -\infty$.

Definition 1 *Let A, C be two sets of $\mathbb{R}^n$. and $\alpha \geq 0$. We define $A \overset{\alpha}{-} C$ in the following way:*

$$A \overset{\alpha}{-} C = \{x \in \mathbb{R}^n : \mathcal{L}\left((x + C) \cap A\right) \geq \alpha\}.$$

It is clear that, if $\alpha > \mathcal{L}(C)$ or $\alpha > \mathcal{L}(A)$, then $A \overset{\alpha}{-} C = \emptyset$. In the sequel, we suppose that $0 \leq \alpha \leq min((A), (C))$.

Proposition 1 *Let A and C be two sets of $\mathbb{R}^n$ and $\alpha \geq 0$. If $A \overset{\alpha}{-} C$ is nonemty,the subset $A \overset{\alpha}{-} C$ has the following properties:*

(i) If A and C are closed, convex, nonempty sets and if C or A is bounded then $A \overset{\alpha}{-} C$ is a closed set.

(ii) If A and C are convex then $A \overset{\alpha}{-} C$ is convex.

(iii) If A and C are bounded sets then $A \overset{\alpha}{-} C$ is bounded and

$$\sup_{z \in A \overset{\alpha}{-} C} \|z\| \leq \sup_{a \in A} \|a\| + \sup_{c \in C} \|c\|.$$

The assertion (i) comes from the continuity of the Lebesgue measure on the compact, convex, nonempty sets of $\mathbb{R}^n$. The proof of (ii) is based on the Brunn-Minkowski inequality (cf [4]).

The two following propositions link the classical difference $A - C$, the Minkowski difference $A \overset{*}{-} C$ and the new difference $A \overset{\alpha}{-} C$:

Proposition 2 *Let A and C be two nonempty subsets of $\mathbb{R}^n$. We have the following properties:*

(i) $A - C = A \overset{0}{-} C$.

(ii) If $0 \leq \alpha \leq \mathcal{L}(C)$, then $A \overset{*}{-} C \subset A \overset{\mathcal{L}(C)}{-} C \subset A \overset{\alpha}{-} C \subset A - C$.

(iii) If A and C are two closed, convex sets, $\operatorname{int}(C)$ *being nonempty and A or C being bounded, then $A \overset{*}{-} C = A \overset{\mathcal{L}(C)}{-} C$.*

We also have:

Proposition 3 *Let A and C be two closed, convex, nonempty sets,* $\operatorname{int}(C)$ *being nonempty.*

(i) Assume that A or C is bounded. If $A \overset{*}{-} C \neq \emptyset$ and if $(\alpha_m)_{m \in \mathbb{N}}$ is a sequence of $[0, \mathcal{L}(C)]$, converging to $\mathcal{L}(C)$ then $\left(A \overset{\alpha_m}{-} C\right)_{m \in \mathbb{N}}$ P.K-converges to $A \overset{*}{-} C$.

(ii) Assume that A and C are bounded. If $A \overset{*}{-} C = \emptyset$ and if $(\alpha_m)_{m \in \mathbb{N}}$ is a sequence of $[0, \mathcal{L}(C)]$, converging to $\mathcal{L}(C)$ then, for m sufficiently large, $A \overset{\alpha_m}{-} C = \emptyset$.

(iii) Assume that C is bounded and A has a nonempty interior. If $(\alpha_m)_{m\in\mathbb{N}}$ is a sequence of nonnegative reals converging to 0 then $\left(A \overset{\alpha_m}{-} C\right)_{m\in\mathbb{N}}$ P.K-converges to $A - C$.

Let us give another property of $A \overset{\alpha}{-} C$:

Theorem 1 *Let A and C be two closed convex subsets of $\mathbb{R}^n$, C bounded, and let $\alpha \geq 0$. We suppose that*

$$\{x \in \mathbb{R}^n : \mathcal{L}\left((x + C) \cap A\right) > \alpha\}$$

is nonempty. Then,

$$A \overset{\alpha}{-} C = cl(\{x \in \mathbb{R}^n : \mathcal{L}\left((x + C) \cap A\right) > \alpha\}).$$

Proposition 4 *Let A and C be two closed, convex sets of $\mathbb{R}^n$ such that C is bounded and $\alpha > 0$. If x satisfies $\mathcal{L}\left((x + C) \cap A\right) > \alpha$, then x belongs to $\operatorname{int}\left(A \overset{\alpha}{-} C\right)$.*

We now study continuity properties of this difference with respect to both the parameter α and the operands.

Theorem 2 *Let $(A_m)_{m\in N}$ and $(C_m)_{m\in\mathbb{N}}$ be two sequences of nonempty, closed, convex subsets of $\mathbb{R}^n$ P.K-converging respectively to A and to a bounded set C. Let $(\alpha_m)_{m\in\mathbb{N}}$ be a sequence of $\mathbb{R}_+$, converging to $\alpha \geq 0$. We further suppose that*

$$\{x \in \mathbb{R}^n : \mathcal{L}\left((x + C) \cap A\right) > \alpha\} \tag{1}$$

is nonempty.
Then, $\left(A_m \overset{\alpha_m}{-} C_m\right)_{m\in\mathbb{N}}$ P.K-converges to $A \overset{\alpha}{-} C$.

Corollary 1 *Let A and C be two nonempty, closed, convex, bounded subsets of $\mathbb{R}^n$ and let $\alpha > 0$ such that*

$$\{x \in \mathbb{R}^n : \mathcal{L}\left((x + C) \cap A\right) > \alpha\} \neq \emptyset.$$

Then, the function

$$F : CB\left(\mathbb{R}^n\right) \times CB\left(\mathbb{R}^n\right) \times \ R_+ \to CB\left(\mathbb{R}^n\right) \cup \{\emptyset\}$$

defined by $F\left(A', C', \alpha'\right) = A' \overset{\alpha'}{-} C'$ is uniformly H-continuous on a neighbourhood of (A, C, α) .

In Theorem 2, the hypothesis $\{x \in \mathbb{R}^n : \mathcal{L}\left((x + C) \cap A\right) > \alpha\} \neq \emptyset$ excludes the case where $\alpha = \mathcal{L}\left(C\right)$ that corresponds to the Minkowski difference if A and C are two closed, convex sets, $\operatorname{int}(C)$ being nonempty and A or C being bounded (see proposition 2). In the following section, we study this particular case and, more generally, we give Lipschitz results on the Minkowski difference.

3 Lipschitz properties of the Minkowski difference.

In this section, let us consider the general case where E is a normed vector space (with finite or infinite dimension).

3.1 The bounded case.

For any nonempty subset A of E, we introduce the value $\rho(A)$, belonging to $\mathbb{R}_+ \cup \{+\infty\}$, called inradius, defined as follow:

$$\rho(A) = \sup \{r > 0 : \exists a \in A \quad B(a,r) \subset A\}.$$

Our main result is based on the following lemma:

Lemma 1 *Let A be a convex, bounded, subset of E such that $\operatorname{int}(A)$ is nonempty. Then, we have the following property: for any $\varepsilon > 0$, there exists $\tau > 0$ such that*

$$\forall x \in A \quad \exists y \in B(x,\varepsilon) : B(y,\tau) \subset A.$$

If a and $R > 0$ are such that $B(a,R) \subset A$, we can give the following evaluation of τ :

$$\tau = \min\left(R, \frac{R\varepsilon}{\operatorname{diam}(A) - R}\right).$$

We are now in position to state the following theorem:

Theorem 3 *Let us define the function $F : CB(E) \times CB(E) \to CB(E) \cup \{\emptyset\}$ as follows:*

$$F(A,C) = A \overset{*}{-} C.$$

If $\operatorname{int}\left(A \overset{*}{-} C\right) \neq \emptyset$ then F is Lipschitz on the neighbourhood

$$B\left(A, \alpha\rho\left(A \overset{*}{-} C\right)\right) \times B\left(C, \alpha\rho\left(A \overset{*}{-} C\right)\right)$$

of (A,C) with rate

$$\frac{\operatorname{diam}(A) + (4\alpha - 1)\rho\left(A \overset{*}{-} C\right)}{(1-2\alpha)\rho\left(A \overset{*}{-} C\right)}$$

for any α in $\left]0, \frac{1}{6}\right]$.

The quality of the Lipschitz constant depends on its value and on the size of the neighbourhood of $A \times C$ on which you want the estimate.

We have a quite good estimate of the smallest Lipschitz constant:

Example 1 *Let us consider the case where* $E = \mathbb{R}^2$. *For* R, R', *such that* $R' > R > 0$, *set* $A = \bar{co}\left(\{(0,0)\} \cup B\left((0,R'),R\right)\right)$ *where* $\bar{co}$ *represents the closed convex hull. For any* $\varepsilon \in [0,R]$, *set* $C_\varepsilon = B\left((0,0),\varepsilon\right)$. *It is easy to see that, for* $\varepsilon' \in [0,R']$:

$$d_H\left(A \overset{*}{-} C_\varepsilon, A \overset{*}{-} C_{\varepsilon'}\right) = \frac{R'}{R} d_H\left(C_\varepsilon, C_{\varepsilon'}\right).$$

On the other hand, Theorem 3 states that a Lipschitz constant is

$$\frac{\left(\operatorname{diam}(A) + (4\alpha - 1)\rho\left(A \overset{*}{-} C_0\right)\right)}{(1-2\alpha)\rho\left(A \overset{*}{-} C_0\right)} = \frac{R' + 4\alpha R}{(1-2\alpha)R}.$$

We have:

$$\inf_{\alpha \in \left]0,\frac{1}{6}\right]} \frac{\left(\operatorname{diam}(A) + (4\alpha - 1)\rho\left(A \overset{*}{-} C_0\right)\right)}{(1-2\alpha)\rho\left(A \overset{*}{-} C_0\right)} = \frac{R'}{R}.$$

3.2 The unbounded case.

$K(A)$ denotes the recession (or asymptotic cone) of A.
We state a proposition analogous to Lemma 1 in the unbounded case:

Proposition 5 *Let* A *be a nonempty subset of* E *such that* $K(A)$ *has a nonempty interior. Then we have that for any* $\varepsilon > 0$ *there exists* $\tau > 0$ *such that*

$$\forall x \in A \quad \exists y \in B(x,\varepsilon) : B(y,\tau) \subset A.$$

If a *and* $R > 0$ *are such that* $B(a,R) \subset K(A)$, *then we can take* $\tau = \frac{\varepsilon R}{\|a\|}$ *and* $y = x + \frac{\varepsilon}{\|a\|}a$, *in the case where* $a \neq 0$.
If $a = 0$ *then* $A = E$, *the result is then obtained with any* $\tau > 0$ *and any* y *in* A.

We are now in position to state a Lipschitz result in the unbounded case.

Theorem 4 *Let us define the mapping* $F : C(E) \times C(E) \to C(E) \cup \{\emptyset\}$ *as follows:*

$$F(A,C) = A \overset{*}{-} C.$$

If $\operatorname{int}\left(K\left(A \overset{*}{-} C\right)\right) \neq \emptyset$, *then* F *is Lipschitz on the neighbourhood* $B\left(A,\frac{R}{2}\right) \times B\left(C,\frac{R}{2}\right)$ *of* (A,C) *with rate* $\frac{\|a\|}{R}$, *where* a *and* R *are such that* $B(a,R) \subset K\left(A \overset{*}{-} C\right)$.

The following example shows that we obtain a good estimation of the Lipschitz constant.

Example 2 *Let us place in the case where* $E = \mathbb{R}^2$.
Set $f(x) = |x|$ *for any* x *in* $\mathbb{R}$ *and* $A =$*epi(f). For any* $\varepsilon \geq 0$, *let us consider* $C_\varepsilon = \varepsilon B$. *It is clear that* $d_H(C_\varepsilon, C_{\varepsilon'}) = |\varepsilon - \varepsilon'|$. *We can verify that*

$$A \overset{*}{-} C_\varepsilon = A + \left(0, \sqrt{2}\varepsilon\right)$$

and then

$$d_H\left(A \overset{*}{-} C_\varepsilon, A \overset{*}{-} C_{\varepsilon'}\right) = \sqrt{2} d_H(C_\varepsilon, C_{\varepsilon'}).$$

On the other hand, $B\left((0, \sqrt{2}), 1\right) \subset K\left(A \overset{*}{-} C_0\right)$. *Then, applying Theorem 4, we obtain that*

$$d_H\left(A \overset{*}{-} C_\varepsilon, A \overset{*}{-} C_{\varepsilon'}\right) \leq \sqrt{2} d_H(C_\varepsilon, C_{\varepsilon'})$$

for any C_ε, $C_{\varepsilon'}$ *in* $B\left(C_0, \frac{1}{2}\right)$.

Remark 1 *If* $\operatorname{int}\left(K\left(A \overset{*}{-} C\right)\right)$ *is empty, the multi-valued map* F, *with* $F(A', C') = A' \overset{*}{-} C'$, *may be discontinuous at* (A, C). *Indeed, let us consider the particular case where* $E = \mathbb{R}^3$. *Let*

$$\begin{aligned} \psi : \left[0, \tfrac{\pi}{2}\right[&\to \mathbb{R} \\ \theta &\longmapsto \tfrac{\tan\theta}{\cos\theta} \end{aligned}$$

the polar equation of the parabole $P_0 : y = x^2$. *Set* $A = \bar{co}\,(A_1 \cup A_2)$ *where* $A_1 = \{(0, 0, z) : z \in [-1, 1]\}$ *and* $A_2 = \left\{(\psi(\theta)\cos(\theta), \psi(\theta)\sin(\theta), 0) : 0 \leq \theta < \frac{\pi}{2}, \right\}$. *For any* $\varepsilon \geq 0$, *let us consider* $C_\varepsilon = \varepsilon B$. *If* $0 < \varepsilon < 1$, *we have:*

$$d_H\left(A \overset{*}{-} C_\varepsilon, A \overset{*}{-} C_0\right) = +\infty.$$

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