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Conservative polynomial operators and discrete Sobolev-type products[†]

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Abstract

In this work we study the existence of linear shape preserving polynomial operators of the form $K(f) = \sum_{i=0}^{n} \lambda_i \langle f, Q_i \rangle_S Q_i$, where Q_i are orthogonal polynomials with respect to the inner product $\langle f, g \rangle_S = \int_{-1}^{1} f(x)g(x)dx + Mf'(c)g'(c)$ with $M > 0$ and $c \in [-1, 1]$.

We show some difficulties that appear when $c \in (-1, 1)$ and prove that for $c = \pm 1$ and for any $k \in \{1, \ldots, n\}$ there exist values λ_i such that $K|_{\mathbb{P}_k} = 1|_{\mathbb{P}_k}$ and K preserves the j-convexity for $j \in \{k, \ldots, n\}$.

Keywords: Sobolev-type inner product, conservative approximation.

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1 Introduction

The polynomial approximation using inner products is not used to preserve the shape properties of the functions. Nevertheless, there exist shape preserving polynomial operators whose proper functions are orthogonal polynomials with respect to certain inner product. These operators can be written in the form

$$K_n(f) = \sum_{i=0}^{n} \lambda_i \frac{\langle f, P_i \rangle}{\langle P_i, P_i \rangle} P_i, \tag{1}$$

where P_i are orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle$ and $\lambda_i > 0$ are the associated proper values. As an example, if we consider the Legendre inner product and the Jacobi one, then we obtain respectively the operators of Bernstein-Durrmeyer-Derriennic and the ones of Bernstein-Jacobi.

Let $C^i(I)$ be the set of all real valued functions defined on $I = [-1, 1]$ with continuous i-th derivative. We denote by D^i the i-th differential operator, and by $\deg(p)$ and $\mathrm{lc}(p)$ the degree and the leader coefficient of the polynomial $p = p(x)$ respectively.

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A function $f \in C^i(I)$ is said to be *i-convex* if $D^i f(x) \geq 0$ for all $x \in I$. A linear polynomial operator (LPO) K *preserves i-convexity* if it maps i-convex functions onto i-convex polynomials.

In [5] it is shown that, with the use of classical inner products, for any $n = 0, 1, \ldots,$ and any $j = 0, 1, \ldots, n$, there exist $\lambda_i > 0$ such that K_n preserves the i-convexities for $i = j, \ldots, n$ and it is the restriction of the identity operator to the spaces of polynomials of degree less than or equal to j, that we denote by $\mathbb{P}_j$.

As the Sobolev-type inner products consider the values of the function and its derivatives, then one could find them appropriate to preserve the sign of the derivatives (see [3, 4]).

In this work we consider inner products of the form

$$\langle f, g \rangle_S = \int_{-1}^{1} f(x) g(x) dx + M f'(c) g'(c) \tag{2}$$

where $M > 0$ and $c \in [-1, 1]$, and we prove that:

1) when $c = 1, -1$ one obtains results similar to the ones obtained with other Sobolev-type inner products.

2) for certain values of $c \in (-1, 1)$ it is not possible to find operators that preserve 2-convexity and/or 3-convexity.

Moreover, we show examples with certain values of $c \in (-1, 1)$ in which the best approximation of a convex function by polynomials of $\mathbb{P}_2$ is a concave one. Analogously with 3-convex functions in $\mathbb{P}_3$.

On the other hand, if in (1) the leader coefficient of P_n is positive and f is n-convex, then the inequality $\langle f, P_n \rangle \geq 0$ is a necessary and sufficient condition for $K_n(f)$ to be n-convex.

Example 1. *Let $I = [-1, 1]$, let $W(I) = \left\{ f \in C(I) : \exists f'(\frac{1}{2}) \right\}$ endowed with the inner product*

$$\langle f, g \rangle_S = \int_{-1}^{1} f(x) g(x) dx + f'(\tfrac{1}{2}) g'(\tfrac{1}{2}) \tag{3}$$

and let Q_0, Q_1, Q_2 be the first three orthogonal polynomials with respect to (3) and with positive leader coefficient. An operator of the type $K_2(f) = \lambda_0 \langle f, Q_0 \rangle_S Q_0 + \lambda_1 \langle f, Q_1 \rangle_S Q_1 + \lambda_2 \langle f, Q_2 \rangle_S Q_2$ with $\lambda_i \in \mathbb{R}^+$ cannot preserve the 2-convexity. Indeed, it suffices to consider the function

$$f(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq \frac{1}{2} \\ \left(x - \frac{1}{2}\right)^4 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

We can find analogous results whenever for fixed values of $c \in (-1, 1)$ and M in (2) there exists n such that $Q_n(1) \leq 0$.

Example 2. *If* $c \in (-1,1)$ *and* $Q_n(1) \leq 0$ *we define the function*

$$f(x) = \begin{cases} 0 & if \ -1 \leq x \leq r \\ (x-r)^{n+2} & if \ \ r \leq x \leq 1 \end{cases}$$

where $r = \max\{c, \gamma\}$, γ *denoting the greatest root of* $Q_n(x)$ *in* $(-1,1)$. *Clearly,* $D^n f(x) \geq 0$ *for all* $x \in [-1,1]$, $f'(c) = 0$ *and* $Q_n(x)f(x) < 0$ *for all* $x \in (r,1]$. *Hence,* $\langle f, Q_n \rangle_S < 0$.

Example 3. *Let* $W(I) = \left\{ f \in C(I) : \exists f'(\frac{3}{4}) \right\}$ *endowed with the inner product (2) with* $c = \frac{3}{4}$ *and* $M = \frac{1}{2}$. *Let* Q_0, Q_1, Q_2 *and* Q_3 *be the first four orthogonal polynomials with positive leader coefficient. The operator* K_3 *defined by* $K_3(f) = \sum_{i=0}^{3} \lambda_i \langle f, Q_i \rangle Q_i$ *with* $\lambda_i \in \mathbb{R}^+$ *preserves the* 2*-convexity whenever* $(K_3(f))'' = \lambda_2 \langle f, Q_2 \rangle_S Q_2'' + \lambda_3 \langle f, Q_3 \rangle_S Q_3'' \geq 0$ *for all* 2*-convex function* f. *If we take*

$$f(x) = \begin{cases} 0 & if \ -1 \leq x \leq \frac{3}{4} \\ \left(x - \frac{3}{4}\right)^4 & if \ \ \frac{3}{4} \leq x \leq 1 \end{cases}$$

one can check easily that $\langle f, Q_2 \rangle_S$ *and* $\langle f, Q_3 \rangle_S$ *are negative, and* $Q_2''(1) > 0$, $Q_3''(1) > 0$. *Hence* $(K_3(f))''(1) = \lambda_2 \langle f, Q_2 \rangle_S Q_2''(1) + \lambda_3 \langle f, Q_3 \rangle_S Q_3''(1) < 0$.

Finally, in what follows, we prove that when $c = 1$ or $c = -1$, if we take any $M > 0$ in (2), then the existence of conservative operators of the type (1), associated to the corresponding system of orthogonal polynomials, is guaranteed.

2 Preliminaries.

Let us consider the space $C^n[-1,1]$ equipped with the inner product

$$\langle f, g \rangle_S = \int_{-1}^{1} f(x)g(x)dx + Mf'(1)g'(1) \tag{4}$$

with $M > 0$.

Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be the systems of monic orthogonal polynomials (**SMOP**) with respect to the inner products of Legendre and the one given in (4) respectively. For the sake of clarity we have written $\{Q_n\}_{n\geq 0}$ instead of $\left\{ Q_n^{(M)} \right\}_{n\geq 0}$.

Clearly, it holds that for $k = 0, 1$, $Q_k = P_k$.

In [2] we can find that

$$Q_n'(1) = \frac{P_n'(1)}{1 + M\, K_{n-1}^{(1,1)}(1,1)} > 0 \tag{5}$$

for each $n \geq 2$, where $K_n^{(r,s)}(x,y)$ represents $\frac{\partial^{r+s}}{\partial x^r \partial y^s} K_n(x,y)$, $K_n(x,y)$ denoting the n-th polynomial kernel associated to $\{P_n\}_{n\geq 0}$.

Definition 4. *A LPO K fixes or hold fixed the space* $\mathbb{P}_k$ *if* $K(p) = p$ *for all* $p \in \mathbb{P}_k$.

3 Integral formula for the Fourier coefficients.

Lemma 5. *i)* $\int_{-1}^{1} Q_n(x)dx = 0, \quad n \geq 1.$

ii) $MQ_n'(1) = -\int_{-1}^{1} xQ_n(x)dx, \quad n \geq 2.$

iii) $-\int_{-1}^{x} Q_n(t)dt = (1-x^2)\, q_{n-1}(x)\,, \quad n \geq 1,$
with $q_{n-1} \in \mathbb{P}_{n-1}$*,* $deg(q_{n-1}) = n-1$ *and* $lc(q_{n-1}) > 0$*.*

iv) $MQ_n'(1) = -\int_{-1}^{1} q_{n-1}(x)\,(1-x^2)\,dx, \quad n \geq 2.$

Proof. i) and ii) From the orthogonality of $\{Q_n\}_{n\geq 0}$,

$$0 = \langle 1, Q_n\rangle_S = \int_{-1}^{1} Q_n(x)dx, \quad n \geq 1,$$

$$0 = \langle x, Q_n\rangle_S = \int_{-1}^{1} xQ_n(x)dx + MQ_n'(1)\,, \quad n \geq 2.$$

iii) Trivially $\int_{-1}^{x} Q_n(t)dt \in \mathbb{P}_{n+1}$ and $\int_{-1}^{1} Q_n(t)dt = \int_{-1}^{-1} Q_n(t)dt = 0.$
iv) Integrating by parts in ii), taking into account iii),

$$\begin{aligned} MQ_n'(1) &= -\int_{-1}^{1} xQ_n(x)dx = -\left[x\int_{-1}^{x} Q_n(t)dt|_{-1}^{1} - \int_{-1}^{1}\left(\int_{-1}^{x} Q_n(t)dt\right)dx\right] \\ &= -\int_{-1}^{1} q_{n-1}(x)\left(1-x^2\right)dx. \end{aligned}$$

□

We will use this polynomial q_{n-1} for the sequel.

Lemma 6. *For* $n \geq 2$ *the n-th Fourier coefficient of a function* $f \in C^1[-1,1]$ *can be expressed as*

$$\langle f, Q_n\rangle_S = \int_{-1}^{1} [f'(x) - f'(1)]\, q_{n-1}(x)\left(1-x^2\right)dx.$$

Proof. It follows from Lemma 5 integrating by parts. □

4 On the zeros of $q_{n-1}(x)$

Lemma 7. *For* $n \geq 2$ *and for any polynomial* $p \in \mathbb{P}_{n-2}$

$$\int_{-1}^{1} p(x)\; q_{n-1}(x)\left(1-x^2\right)dx = -M\, p(1)\, Q_n'(1)$$

Proof. Define $P(x) := \int_{-1}^{x} p(t)\,dt$. From Lemma 6

$$0 = \langle P, Q_n \rangle_S = \int_{-1}^{1} (p(x) - p(1))\; q_{n-1}(x) \left(1 - x^2\right) dx$$

and, consequently, we can deduce from Lemma 5, iv)

$$\int_{-1}^{1} p(x)\; q_{n-1}(x) \left(1 - x^2\right) dx = p(1) \int_{-1}^{1} q_{n-1}(x) \left(1 - x^2\right) dx = -p(1)\, M\, Q_n'(1).$$

□

Corollary 8. *For each $n \geq 3$, $q_{n-1}(x)$ is orthogonal with respect to the inner product of Jacobi with weight $(1 - x^2)$ to any polynomial in $\mathbb{P}_{n-2}$ with a zero at $x = 1$.*

The following result shows a consequence of the aforementioned property.

Proposition 9. *For $n \geq 2$, the zeros of $q_{n-1}(x)$, $y_1, \ldots, y_{n-1}$ say, are real, simple and at least $n-2$ of them, $y_1, \ldots, y_{n-2}$ say, are on $(-1, 1)$ and $y_{n-1} \in (-1, +\infty)$.*

Proof. Let $y_1, \ldots, y_k$ denote the different zeros of $q_{n-1}(x)$ of odd multiplicity which are in $(-1, 1)$. Define $r(x) = (x - y_1) \ldots (x - y_k)$, then the polynomial $q_{n-1}(x)\, r(x)\, (x - 1)$ does not change its sign in the interval $(-1, 1)$. Hence

$$\int_{-1}^{1} q_{n-1}(x)\, r(x)\, (x - 1) \left(1 - x^2\right) dx \neq 0.$$

From the previous corollary it follows that $\deg(r) = k \geq n-2$. This implies that all the zeros of $q_{n-1}(x)$ are real, simple and at the most one of these lies outside the interval $(-1, 1)$. Hence, $q_{n-1}(x) = a_{n-1}(x - y_1)(x - y_2) \cdots (x - y_{n-1})$, with $y_1, \ldots, y_{n-2} \in (-1, 1)$ and $a_{n-1} > 0$.

On the other hand, by Lemma 7 and (5), one has that

$$\int_{-1}^{1} q_{n-1}(x)\, (x - y_1) \ldots (x - y_{n-2}) \left(1 - x^2\right) dx = -\prod_{i=1}^{n-2} (1 - y_i)\, M Q_n'(1) < 0,$$

and consequently

$$\int_{-1}^{1} a_{n-1} (x - y_1)^2 \ldots (x - y_{n-2})^2 (x - y_{n-1}) \left(1 - x^2\right) dx < 0,$$

so $x - y_{n-1}$ takes negative values. In particular $-1 - y_n < 0$. □

5 The main result

Lemma 10. *For $i \geq 2$ and $j > i$, there exist constants $A_{ij} > 0$ such that* *a) the polynomials $q_{i-1}(x) \pm A_{ij} q_{j-1}(x)$ have exactly $i-1$ simple roots and a last sign positive in $(-1,1)$ or have exactly $i-2$ simple roots and a last sign negative in $(-1,1)$.*

b) $Q_i'(1) \pm A_{ij} Q_j'(1) \geq 0$.

Proof. a) It follows from proposition 9 and the fact that the roots of a polynomial are continuous with respect to their coefficients. See [1].

b) From (5), $Q_j'(1) > 0$ so A_{ij} in a) can be chosen in such a way that $Q_i'(1) > A_{ij} Q_j'(1)$. □

Corollary 11. *For $i \geq 2$ and $j > i$, there exist exactly $(x_{jk})_{k=1}^{i-1}$ and $(y_{jk})_{k=1}^{i-1}$ points in $(-1,1]$ such that*

$$(q_{i-1}(x) + A_{ij} q_{j-1}(x)) \prod_{k=1}^{i-1} (x - x_{jk}) \geq 0, \text{ and } (q_{i-1}(x) - A_{ij} q_{j-1}(x)) \prod_{k=1}^{i-1} (x - y_{jk}) \geq 0.$$

Lemma 12. *Let us suppose that for a polynomial $q(x)$ there exists a set of points $(y_j)_{j=1}^{k}$ in $I = [a,b]$ such that $q(x) \prod_{j=1}^{k} (x - y_j) \geq 0$ for all $x \in I$. Then for each k-convex function $f \in C^k[a,b]$ it is verified that $(f-p)(x)q(x) \geq 0$ for all $x \in I$, where $p(x)$ is the polynomial of $\mathbb{P}_{k-1}$ which interpolates f at the points y_j.*

Proof. It suffices to consider the expression of the error given by

$$f(x) - p(x) = \frac{D^k f(\alpha(x))}{k!} (x - y_1)(x - y_2) \cdots (x - y_k) \text{ con } \alpha(x) \in I.$$

Thus $(f(x) - p(x))q(x) \geq 0$ for all $x \in I$. □

Theorem 13. *For each $i \geq 1$ and $j > i$, there exist constants $A_{ij} > 0$ such that for all i-convex function f*

$$\langle f, Q_i \rangle_S \geq A_{ij} \left| \langle f, Q_j \rangle_S \right|.$$

Proof. For $i \geq 2$ we consider A_{ij} of Lemma 10. In this way we can write

$$\begin{aligned}\langle f, Q_i - A_{ij} Q_{ij} \rangle_S = & \int_{-1}^{1} (f'(x) - f'(1)) (q_{i-1}(x) - A_{ij} q_{j-1}(x)) \left(1 - x^2\right) dx = \\ & \int_{-1}^{1} (f'(x) - f'(1) - p(x)) (q_{i-1}(x) - A_{ij} q_{j-1}(x)) \left(1 - x^2\right) dx + \\ & \int_{-1}^{1} p(x) (q_{i-1}(x) - A_{ij} q_{j-1}(x)) \left(1 - x^2\right) dx,\end{aligned}$$

where $p(x) \in \mathbb{P}_{i-2}$ interpolates the function $f'(x) - f'(1)$ at the points $\{y_1, y_2, \ldots, y_{i-1}\}$ given in Corollary 11. As $f'(x) - f'(1)$ is $(i-1)$-convex, by Lemma 12

$$\int_{-1}^{1} (f'(x) - f'(1) - p(x)) (q_{i-1}(x) - A_{ij} q_{j-1}(x)) \left(1 - x^2\right) dx \geq 0.$$

On the other hand, for all $x \in [-1,1]$, $f'(x) - f'(1) - p(x) = \frac{D^i f(\xi_x)}{i!} \prod_{k=1}^{i-1} (x - y_k)$ with $\xi_x \in (-1,1)$. Hence $-p(1) = \frac{D^i f(\xi_1)}{i!} (1-y_1) \cdots (1-y_{i-1}) \geq 0$ and from Lemma 10 and Lemma 7 we deduce that

$$\int_{-1}^{1} p(x) \left(q_{i-1}(x) - A_{ij} q_{j-1}(x)\right) \left(1-x^2\right) dx = -p(1)\, M \left(Q_i'(1) - A_{ij} Q_j'(1)\right)$$

is greater than or equal to zero.

Finally, it holds $\langle f, Q_i - A_{ij} Q_{ij} \rangle_S \geq 0$.

If $i = 1$, $Q_1 = P_1$ has a root in $(-1,1)$ and proceeding as in Lemma 10 there exist constants $A_{1j} > 0$ such that, on one hand, the polynomials $Q_1 - A_{1j} Q_j$ have exactly one simple root and a last sign positive in $(-1,1)$, and on the other hand $Q_1'(1) - A_{1j} Q_j'(1) \geq 0$. Let $f \in C^1[-1,1]$ be an increasing function.

$$\langle f, Q_1 - A_{1j} Q_j \rangle_S = \int_{-1}^{1} f(x) \left(Q_1(x) - A_{1j} Q_j(x)\right) dx + M\, f'(1) \left(Q_1'(1) - A_{1j} Q_j'(1)\right).$$

If $p \in \mathbb{P}_0$ interpolates f at the zero of $Q_1(x) - A_{1j} Q_j(x)$ in $(-1,1)$, then by Lemma 12

$$\int_{-1}^{1} f(x) \left(Q_1(x) - A_{1j} Q_j(x)\right) dx = \int_{-1}^{1} \left(f(x) - p(x)\right) \left(Q_1(x) - A_{1j} Q_j(x)\right) dx \geq 0,$$

since, by the orthogonality of $\{Q_n\}_{n\geq 0}$, $\int_{-1}^{1} p(x)\, Q_n(x)\, dx = 0$, $n \geq 1$ and from $M\, f'(1) \left(Q_1'(1) - A_{1j} Q_j'(1)\right) \geq 0$, it holds $\langle f, Q_i - A_{ij} Q_{ij} \rangle_S \geq 0$.

Analogously one can prove that $\langle f, Q_i + A_{ij} Q_{ij} \rangle_S \geq 0$. □

Corollary 14. *For $n \geq 0$, if $f \in C^1[-1,1]$ is n-convex, then $\langle f, Q_n \rangle_S \geq 0$.*

For $c = -1$ and for any value of M in (4) the same properties are obtained analogously.

Finally, the result stated in the previous theorem is used by F. J. Muñoz and V. Ramírez in [5] as a sufficient condition to prove the existence of conservative operators associated to a given inner product.

Theorem 15. *(F. J. Muñoz and V. Ramírez [5]) Let F be certain space of real functions defined on a bounded interval $I = [a,b]$ equipped with an inner product $\langle \cdot, \cdot \rangle$ and satisfying also that $\mathbb{P}_n \subset F$. Let $\{Q_i\}_{i=0}^{n}$ the polynomial orthogonal system with respect to $\langle \cdot, \cdot \rangle$, with $gr(Q_i) = i$ and positive leader coefficient. Let us suppose that given $m \in \{0, 1, \ldots, n\}$, there exist $A_{i,j} > 0$ such that $\langle f, Q_i \rangle \geq A_{ij} \left|\langle f, Q_j \rangle\right|$ for all i-convex function f, with $i = m, \ldots, n-1$ and $j > i$ and such that $\langle f, Q_n \rangle \geq 0$ provided f is n-convex. Then there exist values $\lambda_i > 0$ with $1 = \lambda_0 = \ldots = \lambda_m \geq \lambda_{m+1} \geq \ldots \geq \lambda_n > 0$ such that the operator $K_{n,m}(f) = \sum_{i=0}^{n} \lambda_i \langle f, Q_i \rangle Q_i$ preserves the i-convexities for $i = m, \ldots, n$ and holds fixed $\mathbb{P}_m$.*

Remark 16. *Theorem 13 states that the set of possible values for the constant m of Theorem 15 is $\{1, \ldots, n\}$, and consequently we cannot guarantee the preservation of positivity, as it is shown in the following example.*

Example 17. *Let* $\langle f,g\rangle_S=\int_{-1}^{1}f(x)g(x)dx+f'(1)g'(1)$. *Let* $Q_0(x)=\frac{1}{\sqrt{2}}$ *and* $Q_1(x)=\sqrt{\frac{3}{5}}x$ *be the first two orthogonal polynomials. Let us see that there exist no constant* $B>0$ *such that the operator* $K(f)=\langle f,Q_0\rangle_S Q_0+B\langle f,Q_1\rangle_S Q_1$ *preserves positivity. Indeed, it suffices to define* $f(x):=x^{2n}$, *with* $n\geq 1$, *and check that* $K(f)(-1)=\frac{1}{2n+1}-B\frac{6}{5}n<0$, *for* $n:=n(B)$ *sufficiently large.*

References

[1] Henrici, P., Applied and Computational Complex Analysis, Vol. I Wiley, New York,(1974)

[2] F. Marcellán, T. E. Pérez, and M. A. Piñar, On zeros of Sobolev-type orthogonal polynomials. Rendiconti di Matematica, Serie VII. Vol. 12, Roma (1992), 455-473.

[3] A. Molina, F. J. Muñoz, Aproximación conservativa asociada a sistemas de polinomios ortogonales, *Actas EMA2000,* (2000), 655-662.

[4] A. Molina, F. J. Muñoz, Aproximación conservativa con productos escalares tipo Sobolev, Actas V^{as} Jornadas Zaragoza-Pau de Matemática Aplicada y Estadística, (1997), 461-468.

[5] F. J. Muñoz, V. Ramirez, Orthogonal polynomials and conservative approximation, J. of Comp. and Appl. Math.,**49** (1993), 217-223.