Monografías del Semin. Matem. García de Galdeano. **27:** 405–412, (2003).

Classification of shock models in system reliability

F. Mallor, J. Santos

Departamento de Estadística e Investigación Operativa, Universidad Pública de Navarra.

Abstract

Shock models in system reliability are usually defined by the time between two consecutive shocks, the damage caused by a shock, the system failure and the dependence relationship among the above elements. The main purpose of this work is to review and classify the large set of shock models defined and studied in the literature in the last three decades. Furthermore, we introduce a new model which generalizes some of the classical ones that arise when the system is governed by independent and identically distributed pairs, $\{(A_n, B_n)\}_{n=0}^{\infty}$, where A_n is the magnitude of the nth shock and B_n is the time between the $(n-1)$th and the nth shock.

Keywords: Reliability, Shock model.

AMS Classification: 90B25, 60K10.

1 Introduction

We consider systems subject to shocks that occur randomly in time. Shock models have been studied by several authors and provide a realistic formulation for modelling certain reliability systems situated in random environment. Various of the models collected here are physically motivated. For instance, the extreme and cumulative shock models may be appropriate descriptions for the fracture of brittle materials, such as glass, and for the damage due to the earthquakes or volcanic activity, respectively.

The way in which the time between two consecutive shocks, the damage caused by a shock, the system failure and the relationships among all these elements are modelled, characterizes a shock model. In the literature, two major types are distinguished depending on whether the effect of the shock on the system is independent of its arrival time or not. These principal models are collected in Section 2 and Section 3, respectively. In Section 4, we introduce a new shock model which generalizes some of the classical ones. Finally, we end with some remarks and conclusions.

2 Classic shock models with independence assumption

When independence between the shock effect and the arrival time is assumed, a sequence of surviving probabilities $\{\bar{P}_k\}$ is defined, being $\bar{P}_k$ the probability that the system still run after the kth shock. According to the time between consecutive shocks, these models are divided into four kinds:

- homogeneous Poisson process, that is, the times between two consecutive shocks are independent, identically distributed exponential random variables;

- non-homogeneous Poisson process, that is, a counting process null at the origin with independent increments where the probability of a shock in $(t+\Delta t]$ is $\lambda(t)\Delta t+o(\Delta t)$, while the probability of more than one shock in $(t+\Delta t]$ is $o(\Delta t)$;

- non-stationary pure birth process, that is, a Markov process where, given that k shocks have occurred in $(0,t]$, the probability of a shock in $(t,t+\Delta t]$ is $\lambda_k\lambda(t)\Delta t+o(\Delta t)$, while the probability of more than one shock in $(t,t+\Delta t]$ is $o(\Delta t)$;

- or renewal process, that is, the times between two consecutive shocks are independent and identically distributed random variables.

In the simplest case, homogeneous Poisson process, conditions on the prefixed sequence $\{\bar{P}_k\}$ are obtained to guarantee distribution properties of the survival function $\bar{H}(t)$. For more details, see Esary et al [6].

Some of these results are extended by A-Hameed and Proschan [1] in the case of the non-homogeneous Poisson process and by A-Hameed and Proschan [2], Klefsjö [13] and [14] in the case of the non-stationary pure birth process.

When the shocks occur according to a renewal sequence, Skoulakis [24] describes the system failure in such a way that generalizes the previous ones. In this model, we assume that the jth shock, independent of all else, has an intensity x randomly chosen from a distribution G_j, which is supported in $[0,1]$ and that it may cause the failure with probability x. This shock model also has the Ross' model [20, p. 22], the Råde's model [18] and the Nakagawa's model [17] as particular cases. In the renewal process case, the interest is focussed on the reliability function for a component and on the extension to multi-component systems.

We point out that, along these last years, several authors have incorporated elements to turn the system more realistic. For instance, Finkelstein and Zarudnij [9] add the concept of recovery to allow the system to eliminate the consequence of each shock in the following way: a r.v. τ is defined for each shock, which models the recovery time from the shock. Then, if a shock occurs before the recovery time from the previous one has elapsed, the system fails. For this particular model, Finkelstein and Zarudnij obtain the reliability function for times between consecutive shocks following a non-homogeneous Poisson process.

Ageing is another element that has been incorporated to some models. Fan et al [7] include this ageing notion to a compound Poisson process shock, $P(\lambda, x)$, that is, a shock model where the shocks have random magnitudes x and arrive according to an homogeneous Poisson process of rate λ. In the compound Poisson process, a shock is fatal to the system with probability $1-\exp(-x)$, where x is the shock's magnitude. The incorporation of ageing is carried out by means of a constant δ, the rate of ageing, in such a way that the probability of failure due to a shock of magnitude x arriving at time u is $1-\exp(-\delta u - x)$. The reliability function is also obtained and an extension to multi-component systems is provided.

3 Classic shock models with a dependence structure

When there exists dependence between the effect of the shock and its arrival time, the damage caused by a shock is modelled in the Fan's way, that is, by a random variable representing the shock's magnitude. Three principal models are considered: extreme shock model, where the system breaks down as soon as the magnitude of an individual shock exceeds some given level; cumulative shock model, where the system fails when the cumulative shock magnitude exceeds some given level and run shock model, where the system works until k consecutive shocks with critical magnitude occur. However, Agrafiotis and Tsoukalas [3] define a shock model that is an extension of the cumulative shock model: the system failure depends on the cumulative damage of those shocks with a magnitude exceeding some pre-specified threshold.

The general setup in these three main shock models is a family $\{(A_n, B_n)\}_{n=0}^{\infty}$ of i.i.d. two-dimensional vectors where A_n represents the magnitude of the nth shock and B_n the time between the $(n-1)$st and the nth shock or, alternatively, the time between the nth and the $(n+1)$st shock, called model I and model II respectively. Model II differs significantly from model I in that the magnitude A_n of the nth shock affects future events, that is, the time interval B_n until the $(n+1)$st shock. Moreover, there exists a first shock at time $t=0$ in model II while A_0 and B_0 are assumed to be zero in the model I.

Let T be the time to the system failure and $\{N(t), t \geq 0\}$, the counting process generated by the renewal sequence $\{B_n\}_{n=0}^{\infty}$. Then, for a fixed threshold $z > 0$, we have that, in the extreme damage case, $T \leq t \Leftrightarrow \max\{A_n | 0 \leq n \leq N(t)\} > z$; in the cumulative damage case, $T \leq t \Leftrightarrow \sum_{n=0}^{N(t)} A_n > z$; in the run case, where z is the level which defines a shock as critical, $T \leq t \Leftrightarrow \min\{n | A_{n-j} > z,\ j = 0, 1, \ldots, k-1\} \leq N(t)$.

Under the model I and model II, the two first failure models were studied by Shanthikumar and Sumita [22], [23], [25] and Gut [10] and [11], providing the reliability function, two first moments and some results about the asymptotic behaviour of the system failure time. In the case when the interarrival time B_n has infinite mean, Anderson, [4] and [5], obtains asymptotic results for the model I.

Igaki et al [12] extend the extreme and cumulative shock model under the model I incorporating the influence of the external system state after the nth shock, J_n, on the correlation structure between the shock magnitudes and the shock interarrival times. More concretely, a trivariate stochastic process $\{A_n, B_n, J_n\}_{n=0}^{\infty}$ is defined satisfying the following Markov property for all $n \in \{0, 1, 2, \dots\}$ and $j \in S = \{1, 2, \dots, N\}$:

$$P\{A_{n+1} \leq a, B_{n+1} \leq b, J_{n+1} = j | A_0, \dots, A_n, B_0, \dots, B_n, J_0, \dots, J_n\} \\ = P\{A_{n+1} \leq a, B_{n+1} \leq b, J_{n+1} = j | J_n\}$$

Also, temporally homogeneity is assumed so that the right hand side of the above equation is independent of n. That is, the system state changes after each shock according to a Markov process and the joint distribution of $\{A_n, B_n\}$ is affected by transitions of the system state.

The third failure model was recently introduced by Mallor and Omey [15] obtaining properties of the distribution function of the system failure time and the limit behaviour when k tends to infinity or when the probability of entering a critical set tends to zero. All these models are summarized in the Table 1.

System failure \ (A_n, B_n)	Extreme damage / Critical region	Cumulative damage $\sum A_i > z$	k consecutive shocks with $(A, B) \in R$
$\{(A_n, B_n)\}_{n=0}^{\infty}$ i.i.d pairs of correlated variables Model I	D.F. of T_z [22] Moments of T_z [22], [11] Properties of T_z [23] Asymptotic behaviour •$E[B] < \infty$ [22], [11] •$E[B] = \infty$ [4] Random environment [12]	D.F. of T_z [25], [3] Moments of T_z [25], [3] Properties of T_z [25] Asymp. behaviour •$E[B] < \infty$ [25], [10], [3] •$E[B] = \infty$ [5] Random environment [12]	D.F. of T_z [15] Moments [15] Asymp. behaviour [15]
$\{(A_n, B_n)\}_{n=0}^{\infty}$ i.i.d pairs of correlated variables Model II	D.F. of T_z [22] Moments of T_z [22] Properties of T_z [23] Asymp. behaviour •$E[B] < \infty$ [3]	D.F. of T_z [25] Moments de T_z [25] Properties of T_z [25] Asymp. behaviour •$E[B] < \infty$ [25], [10]	
$\{(A_n, B_n)\}_{n=0}^{\infty}$ independent but not necessarily identically distributed pairs of correlated variables Model I	Asymp. behaviour [11]		

Table 1: Correlated effect and interarrival shock time

4 Definition of a new general model

We consider a system subject to shocks. Let A_n denote the magnitude of the nth shock and B_n the time between the $(n-1)$st and the nth shock. Let $R \subseteq \mathbb{R}$ be a prefixed real subset. We say that the nth shock is a *critical shock* if $A_n \in R$. Also, we say that k consecutive shocks form a *critical run* of length k if all of them are critical and they are not contained in any sequence of $k+1$ consecutive critical shocks. We define a *complete run* of length $k+1$ as a critical run of length k immediately followed by a non critical shock.

In order to model the damage caused by a shock, we introduce a new set of random variables d_j, $j = 0, 1, \ldots$. The variable d_0 represents the damage due to a non critical shock and we assume it to be zero. For $j \geq 1$, the variable d_j represents the damage caused by a shock when it occupies the jth place in a critical run. That is, the nth shock causes a damage d_j if

$$\begin{aligned} A_{n-i} \in R, \quad \text{for } i = 0, 1, \ldots, j-1 \quad \text{and} \quad A_{n-j} \notin R, \qquad & \text{for } n \geq j \geq 1; \\ A_n \notin R, \qquad & \text{for } j = 0. \end{aligned}$$

The system fails as soon as the accumulated damage due to the random variables d_j's exceeds a fixed threshold $z > 0$.

For our model we suppose that the defined random variables verify the following stochastic assumptions:

a) $\{d_j\}_{j=0}^{\infty}$ is a family of nonnegative and independent but not necessarily identically distributed random variables. We also assume that $d_0 = 0$;

b) for each $j \geq 0$, $\{(A_n, B_n, d_j)\}_{n=1}^{\infty}$ are nonnegative, independent random vectors, all of them equally distributed as the random vector (A, B, d_j);

c) we do not impose any condition of independence among the variables A, B and d_j for all $j \geq 0$.

In brief, our model is governed by a sequence of random vectors of three correlated variables which represent the magnitude of the shock, the intershock time and the damage caused by the shock, respectively. Note that this general shock model extends the model I in the cases of cumulative damage, extreme damage and run damage.

The distribution function of the system failure time and its mean value are provided via Laplace transforms in Mallor and Santos [16].

5 Final remarks and conclusions

The models where a probability is attached to each shock bring together the cumulative and extreme shock models defining adequately the sequence $\{\bar{P}_k\}$ and are studied as particular cases by A-Hameed and Proschan [1] and Esary et al [6]. Let F_j be the distribution function of the damage caused by the jth shock, then for the cumulative shock model $\bar{P}_k = F_1 * F_2 * \cdots * F_k(z)$ and for the extreme shock model $\bar{P}_k = \prod_{j=1}^{k} F_j(z)$.

As before, different elements are incorporated to get a more realistic model such as the assumption of a random threshold instead of a prefixed z and the stochastically decrease of the sequence $\{F_k\}_{k=1}^{\infty}$.

Interarrival shock time \ System failure	General failure model $\bar{P}_k = \prod_{j=1}^{k}(1-p_j)$	Cumulative shock damage X_i i.i.d	Cumulative shock damage independent X_i and $F_i(z)$ decreasing in i	Extreme shock damage
Homogeneous Poisson process of rate λ $\bar{H}(t) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \bar{P}_k$	Sufficient conditions on $\bar{P}_k$ $\bar{H}(t) : IFR, IFRA$ $NBU, NBUE, DMRL$ $h(t) : PF_2, PF_n$ and dual [6] $\bar{H}(t) : HNBUE, HNWUE$ [14] $h(s+t) : SC_n$ [6] • Ageing [7] • Recovery [9]	$\bar{P}_k = F^{(k)}(z)$ $\bar{H}(t) : IFRA, IFR$ $h(t) : PF_2$ [6] • Random threshold [6] $\bar{H}(t) : IFR, IFRA, NBU$ $h(t) : PF_2$ • l kinds of shocks ind. [6] $\bar{H}(t) : IFRA$	$\bar{P}_k = F_1 * \cdots * F_k(z)$ $\bar{H}(t) : IFR, IFRA$ [6] $h(t) : PF_2$ [6] • Random threshold [6] $\bar{H}(t) : NBU$ • Dependence [6] $\bar{H}(t) : IFRA$	•Independent X_i and $F_i(x)$ decreas. in i $\bar{H}(t) : IFR$ [6] • Ageing $\bar{H}(t) : IFR$ [6] and dual [6] • Random threshold [6] $h(t)$: log-concave
Non-homogeneous Poisson process $\bar{H}(t) = \sum_{k=0}^{\infty} \frac{(\Lambda(t))^k}{k!} e^{-\Lambda(t)} \bar{P}_k$	Sufficient conditions on $\bar{P}_k$ and $\Lambda(t)$ $\bar{H}(t) : IFR, IFRA$ $NBU, NBUE, DMRL$ $h(t) : PF_2$ and dual [1] $\bar{H}(t) : HNBUE, HNWUE$ [14] • m components [21] • Recovery [9]		• m components in series $\bar{H}(t) : IFRA$ • m components in series with random threshold $\bar{H}(t) : IFRA, NBU$ and dual [1]	$\bar{H}(t) : IFRA$ [19]
Non-stationary Pure birth process $\bar{H}(t) = \sum_{k=0}^{\infty} s_k(t)\bar{P}_k$ $h(t) = \sum_{k=0}^{\infty} s_k(t)\lambda_k\lambda(t)p_{k+1}$ $s_k(t) = P\{k$ shocks in $[0,t]\}$	Conditions on $\bar{P}_k, \lambda_k, \lambda(t)$ $\bar{H}(t) : IFR, IFRA$ $NBU, NBUE, DMRL$ $h(t) : PF_2$ and dual [2] $\bar{H}(t) : HNBUE, HNWUE$ [13] $\bar{H}(t) : IFRA, DFRA$ [13]			
Renewal process	Reliability function for p_k • indep. of k [18] • dep. of k [17] • indep. of k and random [20] • dep. of k and random [24]	Asymp. behaviour of T_z [25]		

Table 2: Independence between the effect and interarrival shock time

Ross [19] extends these two failure modes defining a new damage function D_t such that:

- $D_t(x_1, \ldots, x_n, \ldots, \mathbf{0})$ represents the damage at time t if exactly n shocks having magnitudes $x_1, \ldots, x_n$ have occurred by time t, with $\mathbf{0} = (0, 0, \ldots)$;

- D_t is nondecreasing in each of its arguments for $t \geq 0$;

- $D_t(x_1, \ldots, x_n, \mathbf{0}) = D_t(x_{i_1}, \ldots, x_{i_n}, \mathbf{0})$ whenever $(i_1, \ldots, i_n)$ is a permutation of 1, 2, $\ldots$, n for all n.

By taking $D_t(x_1, \ldots, x_n, \mathbf{0}) = \max\{x_1, \ldots, x_n\}$ or $D_t(x_1, \ldots, x_n, \mathbf{0}) = \sum_{i=1}^{n} x_i$, we obtain the cumulative and the extreme shock model respectively. All these models are summarized in the Table 2.

In this work, we have presented the main shock reliability models studied in the literature which, as far as we know, have not been analysed jointly. So, this task facilitates us to see what has been done and, what is more important, what is still to be done.

References

[1] A-Hameed, M.S. and Proschan, F. (1973). Nonstationary shock models. *Stochastic Processes and their Applications*, Vol. 1, No. Oct.; 383-404.

[2] A-Hameed, M.S. and Proschan, F. (1975). Shock models with underlying birth process. *Journal of Applied Probability* 12; 18-28.

[3] Agrafiotis, G.K. and Tsoukalas, M.Z. (1995). On excess-time correlated cumulative processes. *Journal of the Operational Research Society* 46; 1269-1280.

[4] Anderson, K.K. (1987). Limit theorems for general shock models with infinite mean intershock times. *Journal of Applied Probability* 24; 449-456.

[5] Anderson, K.K. (1988). A note on cumulative shock models. *Journal of Applied Probability* 25; 220-223.

[6] Esary, J.D., Marshall, A. W. and Proschan, F. (1973). Shocks models and wear processes. *The Annals of Probability* Vol. 1, No.17; 627-649.

[7] Fan J., Ghurke S.G. and Levine R.A. (2000). Multicomponent lifetime distributions in the presence of ageing. *Journal of Applied Probability* 37, 521-533.

[8] Feller, W. (1968). *An Introduction to Probability Theory and its Applications*, Vol. 1. New York. Wiley & Sons.

[9] Finkelstein M.S. and Zarudnij V.I. (2001). A shock process with non-cumulative damage. *Reliability Engineering & System Safety* 71, 103-107.

[10] Gut, A. (1990). Cumulative shock models. *Advances in Applied Probability* 22; 504-507.

[11] Gut, A. (1999). Extreme shock models. *Extremes* 2:3; 295-307.

[12] Igaki, N., Sumita, U. and Kowada, M. (1995). Analysis of Markov renewal shock models. *Journal of Applied Probability* 32; 821-831.

[13] Klefsjö, B. (1981). Survival under the pure birth shock model. *Journal of Applied Probability* 18; 554-560.

[14] Klefsjö, B. (1981). HNBUE survival under some shock models. *Scandinavian Journal of Statistics* 8; 39-47.

[15] Mallor, F. and Omey, E. (2001). Shocks, Runs and Random Sums. *Journal of Applied Probability* 38; 438-448.

[16] Mallor, F. and Santos, J. Reliability of systems subject to shocks with a stochastic dependence for the damages. *Submitted.*

[17] Nakagawa, T. (1979). Further results of replacement problem of a parallel system in random environment. *Journal of Applied Probability* 16; 923-926.

[18] Råde, L. (1976). Reliabilty systems on random environment. *Journal of Applied Probability* 34; 407-410.

[19] Ross, S.M. (1981). Generalized Poisson shock models. *The Annals of Probability* 9; 896-898.

[20] Ross, S.M. (1983). *Stochastic Processes.* New york; John Wiley & Sons.

[21] Savits T.H. (1988). Some multivariate distributions derived from a non-fatal shock model. *J. Appl. Prob.* **25**, 383-390.

[22] Shanthikumar, J.G. and Sumita, U. (1983). General shock models associated with correlated renewal sequences. *Journal of Applied Probability* 20; 600-614.

[23] Shanthikumar, J.G. and Sumita, U. (1984). Distribution properties of the system failure time in a general shock model. *Journal of Applied Probability* 16; 363-377.

[24] Skoulakis, G. (2000). A general shock model for a reliability system. *Journal of Applied Probability* 37; 925-935.

[25] Sumita, U. and Shanthikumar, J.G. (1985). A class of correlated cumulative shock models. *Advances in Applied Probability* 17; 347-366.