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Asymptotic behavior of Kantorovich type operators [†]

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Abstract

We study a generalization of the Kantorovich operators firstly considered by Kacsó [3]. We put in relief some elementary properties of these operators and compute the asymptotic expansion for all their derivatives explicitly. As a particular case we obtain the expansion for the derivatives of the classical Kantorovich operators.

Keywords: Asymptotic approximation, simultaneous approximation, Bernstein operators, Kantorovich operators.

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1 Introduction

The Bernstein operators are one of the more classical examples of linear approximation process. Given $n \in \mathbb{N}$, $f \in \mathbb{R}^{[0,1]}$ and $x \in [0,1]$ the Bernstein operators are defined by

$$B_n f(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n}\right).$$

It is well known that if $f \in C[0,1]$ then $B_n f \to f$ uniformly in $[0,1]$. Furthermore, they present good properties of simultaneous approximation. That is to say, if $f \in C^k[0,1]$, it is possible to approximate not only the function f but also its derivatives and we thus have $D^i B_n f \to D^i f$, $i \in \{1, \ldots, k\}$.

When we are working with measurable functions instead of continuous ones, it is more useful a classical modification of the Bernstein operators introduced by Kantorovich. The Kantorovich operators are defined for $n \in \mathbb{N}$, $f \in L_1[0,1]$ and $x \in [0,1]$ by

$$K_n f(x) = (n+1) \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} f(z)dz$$

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and are connected with the Bernstein operators by means of the identity

$$K_n = DB_{n+1}S, \tag{1}$$

where $S : L_1[0,1] \to C[0,1]$ is the definite integral operator

$$Sf(x) = \int_0^x f(z)dz.$$

In fact, identity (1) makes possible to transfer several properties from the Bernstein operators to the Kantorovich ones. For instance, it allows to assert that the K_n operators have also good simultaneous properties.

With the aim of obtaining estimations of the approximation error for the derivatives of linear operators, Kacso [3] defines a generalization of the Kantorovich operators in the following way: given $q \in \mathbb{N}$, $n \in \mathbb{N}$,

$$K_{q,n} = D^q B_{n+1} S^q, \tag{2}$$

where $S^k = S \circ \cdots \circ S$ is the q-th iteration of the integration operator S. We will call $K_{q,n}$ the q-th Kantorovich operator. Indeed, the definition given by Kacsó is more general and it is useful for any linear operator and not only for the Bernstein ones.

It is well known [4] the following differentiation formula for the Bernstein operators: given $q \in \mathbb{N}$,

$$D^q B_n f(x) = n^{\underline{q}} \sum_{i=0}^{n-q} \binom{n-q}{i} x^i (1-x)^{n-q-i} \Delta^q_{\frac{1}{n}} f\left(\frac{i}{n}\right), \tag{3}$$

where Δ^q stands for the q-th forward difference and for any $x \in \mathbb{R}$, $x^{\underline{q}} = x(x-1)\cdots(x-q+1)$ is the falling factorial. It is not difficult to prove that

$$\Delta^q_{\frac{1}{n+1}} S^q f(x) = \int_{H^q_{n+1}(x)} f(z_q) dz_q \cdots dz_1, \tag{4}$$

where $H^q_a(z_0) = \{(z_1, \ldots, z_q) \in \mathbb{R}^q : \forall i \in \{1, \ldots, q\}, z_{i-1} \le z_i \le z_{i-1} + a\}$. From (3) and (4) it is straightforward to obtain the following integral representation of $K_{q,n}$:

$$\begin{aligned} &K_{q,n} f(x) = \\ &= (n+1)^{\underline{q}} \sum_{i=0}^{n-p+1} \binom{n-q+1}{i} x^i (1-x)^{n-q-i+1} \int_{H^q_{n+1}(\frac{i}{n+1})} f(z_k) dz_k \cdots dz_1. \end{aligned}$$

As it happens for K_n, identity (2) allows to deduce that $K_{q,n}$ are linear operators that preserve all convexities and present simultaneous approximation properties for all derivatives. Our purpose is to describe the way in which takes place such a convergence for the derivatives in a sense that we explain in the following paragraph.

Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of operators that approximates certain class of functions, $H \subseteq \mathbb{R}^I$, defined over an interval $I \subseteq \mathbb{R}$. An asymptotic formula of order $r \in \mathbb{N}$ for this sequence is an expression of the form

$$L_n f(x) = f(x) + \sum_{i=1}^{r} \frac{a_{f,i}(x)}{\phi(n)^i} + o(n^{-r}),$$

valid for all $f \in H$ differentiable enough and $x \in I$, where ϕ is a non decreasing and non vanishing function such that $\lim_{n\to\infty} \phi(n) = +\infty$ (usually $\phi(n) = n$ or $\phi(n) = n+1$). An asymptotic formula provides an exhaustive description of the convergence of the considered sequence of operators. The simplest example of asymptotic expression is a Voronovskaja formula for the sequence that usually adopts the form

$$\lim_{n\to\infty} \phi(n)\left(L_n f(x) - f(x)\right) = a_f(x)$$

that is actually an asymptotic expansion of order 1. When the sequence satisfies simultaneous approximation properties, it is also possible to consider the corresponding asymptotic expression for the convergence of the derivatives, namely

$$D^k L_n f(x) = D^k f(x) + \sum_{i=1}^{r} \frac{a_{f,k,i}(x)}{\phi(n)^i} + o(n^{-r}).$$

In the case of the Kantorovich operators, K_n, the complete asymptotic expansions for the sequence and for the first derivative have been obtained by Abel [2] for any order r. The main goal of this paper is to explicitly compute the complete asymptotic formula for all derivatives of the q-th Kantorovich operators for each $q \in \mathbb{N}$. The basic tool that we will use in order to reach such an objective is the study of the properties of the expansion for the derivatives of the Bernstein operators already done by López-Moreno et al. [5,6,7].

2 Asymptotic expansion for the derivatives of the Bernstein operators.

The complete asymptotic expansion of the Bernstein operators was computed in an explicit way by Abel [1,2]. The expression obtained by Abel is written in terms of the Stirling numbers of the first and second kind, denoted by S_i^j and σ_i^j, respectively, and defined by

$$x^{\underline{i}} = \sum_{j=0}^{i} S_i^j x^j, \quad x^i = \sum_{j=0}^{i} \sigma_i^j x^{\underline{j}} \ ,$$

where $x^{\underline{i}}$ is the falling factorial and $x^{\underline{0}} = 1$. Later, López-Moreno [5,6] proved that the asymptotic expression of the Bernstein operators presents good properties with respect to the derivatives. More precisely he obtains the following:

Theorem 1 (López-Moreno [5, Teorema 3.2]). *Let $r \in \mathbb{N}$ be even, let $k \in \mathbb{N}_0$ and let $f \in \mathbb{R}^{[0,1]}$ be bounded and $k+r$ times differentiable at $x \in [0,1]$. Then,*

$$D^k B_n f(x) = P_{f,k,r,n}(x) + o(n^{-\frac{r}{2}}),$$

and it is satisfied that $P_{f,k,r,n}(x) = D^k P_{f,0,r,n}(x)$. Furthermore, $P_{f,k,r,n}(x)$ has the following explicit representation,

$$\begin{aligned} P_{f,k,r,n}(x) = \sum_{\alpha=0}^{\frac{r}{2}} n^{-\alpha} \sum_{s=(\alpha-k)^+}^{2\alpha} D^{k+s} f(x) \sum_{i=0}^{\alpha,s} x^{s-i} \times \\ \times \sum_{j=\alpha,k+i}^{k+s} \binom{k+s}{j} \binom{j-i}{k} \frac{k!}{(k+s)!} (-1)^{k+s-j} S_{j-i}^{j-\alpha} \sigma_j^{j-i}, \end{aligned} \tag{5}$$

where $z^+ = z$ when $z > 0$ and vanishes otherwise.

As a corollary of this result it is also possible to compute the Voronovskaja formula for all derivatives of the Bernstein operators.

Corollary 2 (López-Moreno [5,Corolario 3.3]). *Let $f \in \mathbb{R}^I$ be bounded and $k+2$ times differentiable, $k \in \mathbb{N}$, at $x \in [0,1]$. Then,*

$$\begin{aligned} & \lim_{n\to\infty} 2n \left(D^k B_n f(x) - D^k f(x) \right) = \\ & = \; -k(k-1) D^k f(x) + k D^{k+1} f(x)(1-2x) + D^{k+2} f(x) x(1-x). \end{aligned}$$

3 Asymptotic and Voronovskaja formulae for the derivatives of the q-th Kantorovich operator.

In the following result we use identity (1) and Theorem 1 in order to calculate the asymptotic expression for the derivatives of the q-th Kantorovich operator.

Theorem 3. *Let $r \in \mathbb{N}$ be even, let $q \in \mathbb{N}$, $k \in \mathbb{N}_0$ and let $f \in L_1[0,1]$ be $k+r$ times differentiable at $x \in [0,1]$. Then,*

$$D^k K_{q,n} f(x) = P^{(q)}_{f,k,r,n}(x)$$

and it is satisfied that $P^{(q)}_{f,k,r,n}(x) = P_{S^q f,k+q,r,n+1}(x) = D^k P^{(q)}_{f,0,r,n}(x)$, with $P_{S^q f,k+q,r,n+1}(x)$ defined as in Theorem 1. Furthermore, the following explicit expression holds true,

$$\begin{aligned} P^{(q)}_{f,k,r,n}(x) = \sum_{\alpha=0}^{\frac{r}{2}} (n+1)^{-\alpha} \sum_{s=(\alpha-k-q)^+}^{2\alpha} D^{k+s} f(x) \sum_{i=0}^{\alpha,s} x^{s-i} \times \\ \times \sum_{j=\alpha,k+q+i}^{k+q+s} \binom{k+q+s}{j} \binom{j-i}{k+q} \frac{(k+q)!}{(k+q+s)!} (-1)^{k+q+s-j} S_{j-i}^{j-\alpha} \sigma_j^{j-i}. \end{aligned} \tag{6}$$

Proof. If $f \in L_1[0,1]$ is $k+r$ times differentiable at x then, it is immediate that $S^q f \in C^q[0,1]$ is $k+q+r$ times differentiable at x so that we can apply Theorem 1 to check that the asymptotic expansion of $D^k K_{q,n} f(x) = D^{k+q} B_{n+1} S^q f(x)$ is $P_{S^q f, k+q, r, n+1}(x)$, that is to say, we have $P^{(q)}_{f,k,r,n}(x) = P_{S^q f, k+q, r, n+1}(x)$ from which (6) follows immediately. On the other hand, from the properties with respect to the derivatives of the expansion of the Bernstein operators, we have $P^{(q)}_{f,k,r,n}(x) = P_{S^q f, k+q, r, n+1}(x) = D^k P_{S^q, q, r, n+1}(x) = D^k P^{(q)}_{f,0,r,n}(x)$. □

Now, let us obtain the Voronovskaja formulae for the derivatives of the q-th Kantorovich operators.

Theorem 4. *Let $f \in L_1[0,1]$ be $k+2$ times differentiable at $x \in [0,1]$ for $k \in \mathbb{N}_0$. Then,*

$$\lim_{n\to\infty} 2n\left(D^k K_{q,n} f(x) - D^k f(x)\right) = x(1-x)D^{k+2} f(x)$$
$$+(k+q)(1-2x)D^{k+1} f(x) - \frac{(k+q)(k+q-1)}{2} D^k f(x).$$

Proof. From Theorem 1 we have $P_{g,0,2,n}(x) = g(x) + \frac{x(1-x)}{2n} D^2 g(x)$ and $P_{g,s,2,n}(x) = D^s P_{g,0,2,n}(x)$, for any $s \in \mathbb{N}_0$ and $g \in C[0,1]$ differentiable enough at x. Then

$$P^{(q)}_{f,k,2,n}(x) = P_{S^q f, k+q, 2, n+1}(x) = D^{k+q} P_{S^q f, 0, 2, n+1}(x)$$
$$= D^{k+q}\left(S^q f + \frac{t(1-t)}{2(n+1)} D^2 S^q f\right)(x) = D^k f + D^{k+q}\left(\frac{t(1-t)}{2(n+1)} D^2 S^q f\right)(x).$$

To compute the last summand in the above identity, let us make use of the Leibniz formula for the differentiation of the product of two functions, $D^s(g\cdot h) = \sum_{i=0}^{s}\binom{s}{i} D^i g D^{s-i} h$. Then,

$$D^{k+q}\left(t(1-t)D^2 S^q f\right)(x) = \sum_{i=0}^{k+q}\begin{pmatrix} k+q \\ i \end{pmatrix} D^i\left(t(1-t)\right)(x) D^{k+q-i} D^2 S^q f(x)$$
$$= x(1-x)D^{k+2} f(x) + (k+q)(1-2x)D^{k+1} f(x) - \frac{(k+q)(k+q-1)}{2} D^k f(x),$$

which inserted in the expression for $P^{(q)}_{f,k,2,n}(x)$ given above ends the proof. □

Once we have obtained the Voronovskaja formulae for all derivatives of the operators we can prove the following 'little o' saturation theorem:

Theorem 5. *Given $f \in C^{k+2}[0,1]$, we have*

$$D^k K_{q,n} f(x) - D^k f(x) = o(n^{-1})$$
$$\Updownarrow$$
$$\begin{cases} f \in \mathbf{span}\{1, t, \ldots, t^{k-1}, \dfrac{1}{t^{q-1}}, \dfrac{1}{(1-t)^{q-1}}\}, & \textit{if } q \geq 2, \\ f \in \mathbf{span}\{1, t, \ldots, t^{k-1}, \log(t), \log(1-t)\}, & \textit{if } q = 1. \end{cases}$$

Proof. From the proof of Theorem 4 we know that

$$\lim_{n\to\infty} 2n\left(D^k K_{q,n} f(x) - D^k f(x)\right) = D^{k+q}\left[t(1-t)D^2 S^q f\right](x).$$

From this identity it is obvious that $D^k K_{q,n} f(x) - D^k f(x) = o(n^{-1})$ if and only if f is a solution of the differential equation

$$D^{k+q}\left[t(1-t)D^2 S^q f\right](x) = 0.$$

Thus, it suffices to compute a fundamental system for such an equation. □

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