On the convergence of multivalued martingales in the limit

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Abstract

Nous présentons dans cet article une version multivoque d'un théorème du à Talagrand sur la convergence des martingales à la limite, notion beaucoup plus générale que celle de martingale.
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1 Preliminaries

The notion of "martingale in the limite" was first introduced by A.G.Mucci ([9]). The principal convergence theorem is due to Talagrand ([10])

Multivalued version of the Talagrand convergence theorem for the martingale in the limit was proved, with respect to the Mosco convergence, by Castaing et Ezzaki ([3], theorem 3.3. in [3]).

The aim of this paper is to extend this theorem to the linear convergence. This result is also a generalization of a theorem due to Choukairi ([4]) for the multivalued pramarts. The notion of martingale in the limit is a generalization of the notion of pramarts.

E-valued martingales in the limit

Let (Ω, Σ, P) be a probability space, E a Banach space and E^* its topological dual, $(\Sigma_n)_{n\geq 1}$ an increasing sequence of sub-σ-fields of Σ such that Σ is the σ-field generated by $\cup_{n\geq 1}\Sigma_n$.

Let $X_n : \Omega \to E$ be a random variable for each $n \in \mathbf{N}$. $(X_n, \Sigma_n)_{n\in\mathbf{N}}$ is said to be an *E-valued martingale in the limit* if there is a sequence $(h_n)_n$ of measurable and positive functions such that :

$$\begin{aligned} i)\ \lim_n \|h_n\| &= 0 \\ ii)\ \forall m &\geq n, \left\|E^{\Sigma_n}(X_n) - X_n\right\| \leq h_n \text{ almost everywhere} \end{aligned}$$

where $E^{\Sigma_n}(X_n)$ is the conditional expectation of X_n with respect to the σ-field Σ_n.

An other notion was introduced by Talagrand, lightly different of the "martingale in the limit" : the notion of MIL, which is a generalization of the "martingale in the limit"

$(X_n, \Sigma_n)_n$ is said to be a **MIL** if and only if, for each $\varepsilon > 0$, there is p such that, for each $n \geq p$:

$$P\left(\sup_q \left\{\left\|X_q - E^{\Sigma_q}(X_n)\right\|; p \leq q \leq n\right\} > \varepsilon\right) \leq \varepsilon.$$

Convergence results :

1) A real-valued MIL such that $\liminf E(|X_n|) < +\infty$ converges almost every where.

2) If $(X_n, \Sigma_n)_n$ is a $E-$valued MIL such that :

(i) $\liminf \int \|X_n\| \, dP < +\infty$

(ii) $x^* \circ X_n \to 0$ for each $x^* \in E^*$

then $\|X_n\| \to 0$ a.e.

Multivalued case.

Let (Ω, Σ, P) be a probability space, E a Banach space such that its topological dual E^* is strongly separable, $(\Sigma_n)_{n\geq 1}$ an increasing sequence of sub-σ-fields of Σ such that Σ is the σ-field generated by $\cup_{n\geq 1}\Sigma_n$.

The set of nonempty convex and weakly compact subsets of E will be denoted by $cw(E)$.

B and B^* are, respectively, the closed unit balls of E and E^*.

For each open subset U of E, we shall set

$U^- := \{C \in cw(E) : C \cap U \neq \emptyset\}$

and we shall denote by $\mathcal{E}$ the *Effrös σ-algebra* of $cw(E)$ that is the smallest $\sigma-$algebra over $cw(E)$ containing the class

$\{U^-, U \text{ open in } E\}$.

The two best-known functionals associated with an element C in $cw(E)$ are its distance functional and its support functional, defined by the familiar formulas:

$d(x, C) = inf\{\|x - y\|, y \in C\} \ (x \in E)$

$\delta^*(x^*, C) = \sup\{\langle x^*, y\rangle, y \in C\} \ (x^* \in E^*)$.

For any $C \in cw(E)$, we set

$|C| = \sup\{\|x\| : x \in C\}$.

A *random set* will be a multifunction $X : \Omega \longrightarrow cw(E)$ which is measurable with respect to the σ-fields Σ and $\mathcal{E}$.

We denote by $L^1_{cw(E)}(\Sigma)$ the space of all the random sets taking values in $cw(E)$ such that $\omega \to |X(\omega)|$ is integrable.

Let H be the Hausdorff metric and τ_H be the topology associated to H.

The *linear topology* τ_L on $cf(E)$ is the topology generated by all sets of the form U^-, where U is an open subset of E, and all sets

$$H(x^*,\alpha) := \{C \in ckw(E) : \delta^*(x^*,C) < \alpha\}$$

where $x^* \in E^*$ is nonzero and $\alpha \in R$.

τ_L was first considered by Hess ([5])and studied by Beer([2]).

We have the following result.

Proposition 1.1. *Let $(C_n)_{n\in\mathbf{N}\cup\{\infty\}}$ be a sequence in $cf(E)$. Then: $C_\infty = \tau_L - \lim_n C_n$*

$$\Longleftrightarrow \begin{cases} (i)\ d(x,C_\infty) = \lim_n d(x,C_n), \forall x \in E \\ and\ (ii)\delta^*(x^*,C_\infty) = \lim_n \delta^*(x^*,C_n), \forall x^* \in E^*. \end{cases}$$

Reference [2],Theorem 3.4.

For any random set X, we put

$$S^1(X,\Sigma) = \{f \in L^1(\Sigma) : f \in X \text{ almost surely}\}.$$

In this definition, Σ may be replaced by any sub-$\sigma-$field of Σ.

$S^1(X,\Sigma)$ is closed if X is closed valued and it is non-empty if and only if the function $d(0,X) \in L^1_{\mathbb{R}}(\Omega,\Sigma,P)$.

The *multivalued integral* of X is defined, for each $A \in \Sigma$, by

$$\int_A X(\omega)\,P(d\omega) = cl\left[\left\{\int_A f(\omega)P(d\omega) : f \in S^1(X,\Sigma)\right\}\right].$$

For the basic properties of the multivalued integral, we refer the reader to [1].

Let X be an element of $\mathcal{L}^1_{cw(E)}(\Sigma)$, the *multivalued conditional expectation* of X with respect to Σ_n, denoted by $E^{\Sigma n}(X)$, is an element of $\mathcal{L}^1_{cw(E)}(\Sigma)$ such that

$$S^1\left(E^{\Sigma n}(X),\Sigma\right) = \left\{E^{\Sigma n}(f) : f \in S^1(X,\Sigma)\right\}.$$

2 Convergence theorem for multivalued martingales in the limit.

Definition 2.1. Let $X_n : \Omega \to cw(E)$ be a random set for each $n \in \mathbf{N}$. $(X_n,\Sigma_n)_{n\in\mathbf{N}}$ is said to be a *MIL taking values in* $L^1_{cw(E)}(\Sigma)$ if the following conditions hold :

(a) $\forall n \in \mathbf{N}$, X_n is Σ_n-measurable,

(b) $\forall\varepsilon > 0, \exists\, p \in \mathbb{N}^*$:

$$n \geq p \Rightarrow P\left[\sup_{p\leq q\leq n} h\left(X_q, E^{\Sigma_q}(X_n)\right) > \varepsilon\right] < \varepsilon.$$

The following theorem is an extension of the convergence theorem due to Talagrand ([10]):

Theorem 2.1. *Let* $(X_n, \Sigma_n)_{n\in\mathbf{N}}$ *be a MIL taking values in* $L^1_{cw(E)}(\Sigma)$ *such that:*

(i) $\sup_{n\in\mathbf{N}} \int_\Omega |X_n(\omega)| \, P(d\omega) < +\infty$

(ii) there is a random set $K : \Omega \to cw(E)$ *such that* $X_n(\omega) \subset K(\omega)$*, for each* n *and for each* ω*.*

Then, there is $X_\infty \in \mathcal{L}^1_{cw(E)}(\Sigma)$ *such that* $X_\infty = \tau_L - \lim_n X_n$ *almost surely.*

Proof

Let us first recall tree formulas deduced from the Hormander's formula and the properties of the Hausdorff distance.

For each C and for each D *in* $cw(E)$, we set :

$$H(C, D) = \sup_{x^* \in B^*} |\delta^*(x^*, C) - \delta^*(x^*, D)| \ \ (1)$$

or

$$H(C, D) = \sup_{x \in E} |d(x, C) - d(x, D)| \ \ (2)$$

and, also, for each $x \in E$,

$$d(x, C) = \sup_{x^* \in B^*} [< x^*, x > -\delta^*(x^*, C)] \ \ (3).$$

We deduce from (1) and definition 2.1 that, if $(X_n, \Sigma_n)_{n\in\mathbf{N}}$ is a bounded MIL taking values in $\mathcal{L}^1_{cw(E)}(\Sigma)$, then for each $x^* \in B^*$, $(\delta^*(x^*, X_n), \Sigma_n)_{n\in\mathbf{N}}$ is a real-valued bounded MIL. Using Talagrand theorem (theorem 4 in [10]), it follows that $(\delta^*(x^*, X_n), \Sigma_n)_{n\in\mathbf{N}}$ converges almost surely for each $x^* \in B^*$.

Then, we give an useful lemma proved by Christian Hess (see [6], Lemme 5.2).

Lemma 3.1. *Let* D^* *be a countable subset of* E^* *which is dense with respect to the Mackey topology and let* $(X_n)_{n\in\mathbf{N}}$ *be a sequence in* $L^1_{cw(E)}(\Sigma)$ *such that :*

(i) $\sup_{n\in\mathbf{N}} \int_\Omega |X_n(\omega)| \, P(d\omega) < +\infty$

(ii) there is a random set $K : \Omega \to cw(E)$ *and a set* N_0 *such that* $P(N_0) = 0$ *and* $X_n(\omega) \subset K(\omega)$ *for each* $(\omega, n) \in (\Omega \setminus N_0) \times \mathbf{N}$,

(iii) for each $x^* \in D^*$, $\delta^*(x^*, X_n)$ *converges almost surely.*

Then, there is a random set $X_\infty \in L^1_{cw(E)}(\Sigma)$ *such that, for each* $x^* \in E^*, \delta^*(x^*, X_n)$ *converges almost surely to* $\delta^*(x^*, X_\infty)$.

We recall that E^* is dense with respect to the Mackey topology if and only if E is separable.

Then, by the lemma below, there is a random set $X_\infty \in L^1_{cw(E)}(\Sigma)$ such that, for each $x^* \in E^*, \delta^*(x^*, X_n)$ converges almost surely to $\delta^*(x^*, X_\infty)$.

We deduce now from (2) that, for each $x \in E$, $(d(x, X_n), \Sigma_n)_{n\in\mathbf{N}}$ is a real-valued bounded MIL. Using Talagrand theorem, $(d(x, X_n), \Sigma_n)_{n\in\mathbf{N}}$ converges almost surely to a function f_x, for each $x \in E$.

Then, there is a subset N' of Ω such that $P(N') = 0$ and $\lim_n d(x, X_n(\omega)) = f_x(\omega)$ for each $\omega \notin N'$.

We proceed now to show that $f_x = d(x, X_\infty)$.

Let $D_0 = \left\{z_j^* : j \in N\right\}$ be a countable subset of B^* such that D_0 is dense with respect to the Mackey topology.

It follows from (3) that

$$d(x, X_n) = \sup_j \left[< z_j^*, x > -\delta^*(z_j^*, X_n)\right].$$

Then, for each $j \in N$, the sequence $\left(< z_j^*, x > -\delta^*(z_j^*, X_n)\right)_n$ converges almost surely to $< z_j^*, x > -\delta^*(z_j^*, X_\infty)$.

Using (3), we have, for each $j \in N$:

$$< z_j^*, x > -\delta^*(z_j^*, X_n) \leq d(x, X_n)$$

and a passage to the limite implies that :

$$d(x, X_\infty) \leq f_x \text{ almost surely}$$

for each $x \in E$.

Using the classical notations of the Mosco convergence, we set, for each $\omega \in \Omega$,

$$s - liX_n(\omega) = \{x \in E : \text{ there is a sequence } x_n$$

$$\text{which converges in norm to } x \text{ with } x_n \in X_n(\omega) \text{ for each } n\}.$$

Applying theorem 3.3. in [3], we have $X_\infty = s - liX_n$ almost surely.

By an inequality proved by Tsukada (theorem 2.2. in [11]), we have

$$\limsup_n d(x, X_n) \leq d(x, s - liX_n), \text{ for each } x \in E. \quad (4)$$

Then :

$$f_x \leq d(x, X_\infty) \text{ almost surely}$$

and we conclude that $X_\infty = \tau_L - \lim_n X_n$.

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