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Qualitative Korovkin-type results on almost convergence

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Abstract

In this paper we state a general Korovkin-type result for the study of the almost convergence of general shape preserving approximation processes. We use the well-known notion of almost convergence introduced by Lorentz in 1948. Some applications are also shown.

Keywords: Korovkin-type result, almost convergence

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1 Introduction

The classical results of Korovkin [3] and the subsequent quantitative versions of Shisha and Mond [7], [8] on the convergence of a sequence of positive linear operators were formulated in [2] and [5] replacing the usual convergence by almost convergence. We recall here this notion that was introduced in 1948 by Lorentz [4]. As usual we denote by $\mathbb{R}^X$ the set of all real-valued functions on a set X, and by $C(X)$ the subset of the continuous ones. Let H_n be a sequence of linear operators defined on $C(X)$. Given $f \in C(X)$, we define $H_n f := H_n(f)$ to be almost convergent to g in $C(X)$, uniformly in X, ($H_n f \overset{\text{a.c.}}{\longrightarrow} g$), provided

$$\frac{1}{p}\sum_{n=v+1}^{v+p} H_n f(x),\ p, v \in \mathbb{N}$$

converges to $g(x)$ as $p \to \infty$, uniformly in v and uniformly in X. Pointwise almost convergence can be defined in an obvious way.

On the other hand, in [6] it was stated a Korovkin-type result on the convergence of sequences of operators that possess shape preserving properties much more general than positivity.

Our aim with this paper is to present qualitative results on almost convergence for these conservative approximation processes that we have just mentioned and that we detail in the next section together with the main result of this work. The rest of the sections are devoted to present applications of this result to the space of k-times differentiable functions defined on a compact subinterval of $\mathbb{R}$ and to the space $C_{2\pi}$ of all real-valued continuous 2π periodic functions on $\mathbb{R}$.

2 A general Korovkin-type result on almost convergence

In [6] it was presented an extension of the classical Korovkin theorem when non positive operators are considered. In this section we extend this theorem for the case of almost convergence.

Let $m \in \mathbb{N}$, let X be a compact subset of $\mathbb{R}^m$, let $B \subset \mathbb{R}^X$, let A be a subspace of $C(X)$ such that $A \subset B$, and let $L : B \to \mathbb{R}^X$ be a linear operator such that $L(A) \subset C(X)$.

Theorem 1 *Let $P = \{f \in B : Lf \geq 0\}$ and let C be a cone of A (i.e. if $\alpha \geq 0$ and $f, g \in C$, then $\alpha f, f + g \in C$). Assume that:*

(v.1) there exists $u \in A$ such that $Lu(x) = 1$ $\forall x \in X$.

(v.2) there exist $a_i, g_i \in C(X)$, $i = 1, 2, ..., m$, such that for all $z \in X$, the functions

$$\varphi_z := \sum_{i=1}^{m} a_i(z) g_i$$

belong to C, $L\varphi_z(x) \geq 0$ for all $x \in X$ and the equality is satisfied if and only if $z = x$.

(v.3) for each $f \in A$, there exists $\alpha = \alpha(f) \geq 0$ such that if $\beta > \alpha$, then $f + \beta\varphi_z \in C$.

Let $K_n : A \to B$ be a sequence of linear operators such that

(k.1) $K_n(P \cap C) \subset P$,

(k.2) $L(K_n g_i) \xrightarrow{a.c.} Lg_i$, $i = 1, \ldots, m$.

Then for all $f \in A$,

$$L(K_n f) \xrightarrow{a.c.} Lf.$$

Proof We organize the proof by several steps in a similar way as it is done in [1, Theorem 1.3]. For $g \in A$, $p, v \in \mathbb{N}$, we use the notation $t_p^v g(x) := \frac{1}{p}\sum_{n=v+1}^{v+p} K_n g(x)$.

1. It is verified that $L(K_n\varphi_z)(z) \xrightarrow{\text{a.c.}} 0$, $\forall z \in X$.

Indeed, applying the definition of almost convergence,

$$L(K_n\varphi_z)(z) \xrightarrow{\text{a.c.}} 0 \Longleftrightarrow L(t_p^v\varphi_z)(z) = \frac{1}{p}\sum_{n=v+1}^{v+p} L(K_n\varphi_z)(z) \to 0,$$

as $p \to \infty$, uniformly in v, so it suffices to observe that

$$\frac{1}{p}\sum_{n=v+1}^{v+p} L(K_n\varphi_z)(z) = \frac{1}{p}\sum_{n=v+1}^{v+p}\left(\sum_{i=1}^{m} a_i(z)L(K_n g_i)(z)\right)$$

$$= \sum_{i=1}^{m} a_i(z)\left(\frac{1}{p}\sum_{n=v+1}^{v+p} L(K_n g_i)(z)\right) \longrightarrow \sum_{i=1}^{m} a_i(z)Lg_i(z) = \varphi_z(z) = 0,$$

where we have used hypotheses $(v.2)$ and $(k.2)$.

2. $L(t_p^v u)$ is bounded on X.

In order to prove this assertion, let z_1, $z_2 \in X$ $(z_1 \neq z_2)$ and define $\varphi^* = \varphi_{z_1} + \varphi_{z_2}$. ¿From $(v.2)$, it is obvious that $L\varphi^*(x) > 0$, $\forall x \in X$. As X is a Haussdorf space, then there exist $\delta_1, \delta_2 > 0$, such that $B(z_1,\delta_1) \cap B(z_2,\delta_2) = \emptyset$. Let μ_1, μ_2 be the minimum values of φ_{z_1} y φ_{z_2} in $X - B(z_1,\delta_1)$ and $X - B(z_2,\delta_2)$ respectively, both necessarily positive. Defining $\mu = \min\{\frac{1}{2}\mu_1, \frac{1}{2}\mu_2\}$, one has that for all $x \in X$

$$Lu(x) = 1 \leq \mu^{-1}L\varphi^*(x).$$

Hence $\mu^{-1}\varphi^* - u \in P$. Moreover, according to hypothesis $(v.3)$, for a sufficiently small value of β, with $\beta \leq \mu$, we obtain $\beta^{-1}\varphi^* - u \in C$. Thus, $\beta^{-1}\varphi^* - u \in P \cap C$. Applying $(k.1)$, it is obtained that $L(K_n(\beta^{-1}\varphi^* - u)) \geq 0$ and directly $\beta^{-1}L(t_p^v\varphi^*) \geq L(t_p^v u)$. Using $(k.2)$, as $p \to \infty$, $\beta^{-1}L(t_p^v\varphi^*) = \beta^{-1}L(t_p^v\varphi_{z_1}) + \beta^{-1}L(t_p^v\varphi_{z_2}) \longrightarrow \beta^{-1}L\varphi_{z_1} + \beta^{-1}L\varphi_{z_2} = \beta^{-1}L\varphi^*$, uniformly in v. Thus, $L(t_p^v u)$ is bounded by a convergent and continuous function defined on a compact set. Therefore, $L(t_p^v u)$ is bounded.

3. Let $f_y \in A$, $y \in X$, be a family of functions for which $Lf_y(x)$ is a continuous function of $(x,y) \in X \times X$ and $Lf_y(y) = 0$, $\forall y \in X$. Then $L(K_n f_y)(y) \xrightarrow{\text{a.c.}} 0$ uniformly in $y \in X$.

This assertion is proved as follows: let $D = \{(y,y)/y \in X\}$ in $X \times X$ and let $y \in X$, then it is verified that for all $\epsilon > 0$ there exists a neighborhood V_y of the point (y,y) such that $|Lf_y(x)| < \epsilon$, $\forall (x,y) \in V_y$. Let $V = \cup_{y\in X} V_y$ and let F its complement in $X \times X$ which is clearly compact. Thus we can consider

$$m = \min_{(x,y)\in F} L(\varphi_y)(x),\ M = \max_{(x,y)\in F} Lf_y(x),$$

and write $|Lf_y(x)| < \epsilon + \frac{M}{m}L\varphi_y(x)$ for all $x, y \in X$. From $(v.3)$, for a sufficiently large β, it is verified that $\epsilon u + \beta\frac{M}{m}\varphi_y \pm f_y \in P \cap C$. Applying the hypothesis $(k.1)$, we obtain $\epsilon K_n u + \beta\dfrac{M}{m}K_n\varphi_y \pm K_n f_y \in P$. Thus,

$$-\epsilon L(K_n u)(y) - \beta\frac{M}{m}L(K_n\varphi_y)(y) < L(K_n f_y)(y) < \epsilon L(K_n u)(y) + \beta\frac{M}{m}L(K_n\varphi_y)(y),$$

and directly

$$-\epsilon L(t_p^v u)(y) - \beta\frac{M}{m}L(t_p^v\varphi_y)(y) < L(t_p^v f_y)(y) < \epsilon L(t_p^v u)(y) + \beta\frac{M}{m}L(t_p^v\varphi_y)(y).$$

Therefore

$$\left|L(t_p^v f_y)(y)\right| < \epsilon L(t_p^v u)(y) + \beta \frac{M}{m} L(t_p^v \varphi_y)(y).$$

¿From the steps 1 and 2 of this proof, there exists $U_0 \in \mathbb{R}$, upper bound of $L(t_p^v u)$, and $p_0 \in \mathbb{N}$, such that for all $p > p_0$

$$\left|L(t_p^v f_y)(y)\right| < \epsilon U_0 + \epsilon = \epsilon(U_0 + 1),$$

for all $v \in \mathbb{N}$ and for all $y \in X$.

4. Now we are in a position to prove the theorem. Given $f \in A$, we define the family

$$f_y = f - \frac{Lf(y)}{L\varphi^*(y)}\varphi^*, \ \forall y \in X.$$

The functions $Lf_y(x)$ are continuous in x and y with $Lf_y(y) = 0$, $\forall y \in X$ and satisfy the conditions of claim 3 above. Then we can assure that, for $p \to \infty$,

$$L(t_p^v f_y)(y) = L(t_p^v f)(y) - \frac{Lf(y)}{L\varphi^*(y)} L(t_p^v \varphi^*)(y) \longrightarrow 0,$$

uniformly in v and uniformly in y. Now the proof is over just observing that $L(t_p^v \varphi^*)(y) \to L\varphi^*(y)$ uniformly in v and y.

□

3 Applications to $C^k([0,1])$

Now we state two results for sequences of operators defined on $C^k([0,1])$. We omit the proofs since they are analogous to the ones of parallel results for the usual convergence that can be found in [6]. We shall denote $e_j(x) = x^j$.

Let $\sigma = \{\sigma_i\}_{i \geq 0}$ be a sequence with $\sigma_i \in \{-1, 0, 1\}$. Let $h, k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ with $0 \leq h < k$ and $\sigma_h \sigma_k \neq 0$. We define the following cones in $C^k([0,1])$:

$$C_{h,k}(\sigma) = \{f \in C^k([0,1]) : \ \sigma_i D^i f \geq 0, \ h \leq i \leq k\}.$$

Let $\Gamma = \{i : \ h \leq i < k, \ \sigma_i \neq 0, \ \sigma_{i+1} = 0, \ \sigma_i \sigma_{i+2} \neq -1\}$. If $\Gamma = \emptyset$, then we call $C_{h,k}(\sigma)$ a cone of type I, and if $\Gamma \neq \emptyset$, then we call $C_{h,k}(\sigma)$ a cone of type II.
We denote $\sigma^{[j]} = \{\sigma_i^{[j]}\}_{i \geq 0}$ with $\sigma_i^{[j]} = 0$ for $i \neq j$ and $\sigma_j^{[j]} = \sigma_j$.

Corollary 1 *Let $C_{h,k}(\sigma)$ be a cone of type I or II. Let $K_n : C^k([0,1]) \to C^k([0,1])$ be a sequence of linear operators.*
If $K_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma^{[k]})$ and $D^k(K_n e_j) \xrightarrow{a.c.} D^k e_j$ for every $j = h, ..., k+2$, then

$$D^k(K_n f) \xrightarrow{a.c.} D^k f \ \forall f \in C^k([0,1]).$$

Corollary 2 *Let $C_{h,k}(\sigma)$ be a cone of type II and let $r \in \Gamma$. Let $K_n : C^k([0,1]) \to C^k([0,1])$ be a sequence of linear operators.*
If $K_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma^{[r]})$ and $D^r(K_n e_j) \xrightarrow{a.c.} D^r e_j$ for every $j = h, h+1, ..., k$, then

$$D^r(K_n f) \xrightarrow{a.c.} D^r f \ \forall f \in C^k([0,1]).$$

4 Applications in $C_{2\pi}$

In [2, Theorem 5] it is shown a Korovkin-type condition for the almost convergence of positive operators defined on $C_{2\pi}$. It is stated as follows:

Theorem 2 *Let K_n be a sequence of positive linear operators defined on $C_{2\pi}$. The sequence $K_n(\psi)$ is almost convergent to ψ, uniformly on $[0,2\pi]$, for each $\psi \in C_{2\pi}$, if and only if $K_n(e_0)$, $K_n(\sin(\cdot))$ and $K_n(\cos(\cdot))$ are almost convergent respectively to e_0, $\sin(\cdot)$ and $\cos(\cdot)$, uniformly on $[0,2\pi]$.*

Now we present a generalization of this result considering operators non necessarily positive.

For $i \in \mathbb{N}_0 \cup \{+\infty\}$ we define the linear operators $L_i : C_{2\pi} \to C_{2\pi}$ as follows: let $\phi \in C_{2\pi}$ and let a_i, b_i be the Fourier coefficients associated to ϕ, then $L_0\phi(x) = \frac{a_0}{2}$ and for $i \neq 0$,

$$L_i\phi(x) = \frac{a_0}{2} + \sum_{k=1}^{i} a_k \cos(kx) + b_k \sin(kx).$$

Let h, k, m be integers with $0 \leq h \leq k < m$. Let us consider the following cones in $C_{2\pi}$:

$$C_{2\pi}^{h,k} = \{\phi \in C_{2\pi} : L_i\phi \geq 0, h \leq i \leq k\}.$$

Corollary 3 *Let $P_m = \{\phi \in C_{2\pi} : L_m\phi \geq 0\}$. Let $K_n : C_{2\pi} \to C_{2\pi}$ be a sequence of linear operators.*
If $K_n(C_{2\pi}^{h,k} \cap P_m) \subset P_m$ and $L_m(K_n\psi) \xrightarrow{a.c.} L_m\psi$ for all $\psi \in \{e_0, \sin(i\cdot), \cos(i\cdot) : i = 1, 2, \ldots, k+1\}$, then

$$L_m(K_n\phi) \xrightarrow{a.c.} L_m\phi \ \forall \phi \in C_{2\pi}.$$

Proof We apply Theorem 1 with $A = B = C_{2\pi}$, $L = L_m$, $C = C_{2\pi}^{h,k}$, where we consider $C_{2\pi} = C(X)$, being X identified with the compact set $S = \{z \in \mathbb{R}^2 / |z| = 1\}$.
The condition $(v.1)$ is trivially verified by $u = e_0 \in C_{2\pi}$.
Let $z \in X$, we define $\varphi_z(x) = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(k+1)!} - \cos(x-z) - \frac{1}{2!}\cos^2(x-z) - \frac{1}{3!}\cos^3(x-z) - \cdots - \frac{1}{(k+1)!}\cos^{(k+1)}(x-z)$ $\forall x \in X$. As the function φ_z admits a Fourier expansion with no null coefficients till order $(k+1) \leq m$, then for all $x \in X$, $L_m\varphi_z(x) = \varphi_z(x) \geq 0$, and the equality is satisfied if and only if $x = z$. Moreover, if $h > 0$, then $L_i(\varphi_z(x)) = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(k+1)!} - \cos(x-z) - \frac{1}{2!}\cos^2(x-z) - \frac{1}{3!}\cos^3(x-z) - \cdots - \frac{1}{i!}\cos^i(x-z) \geq 0$, $\forall x \in X$, $\forall i \in \{h, h+1, ..., k\}$, so $\varphi_z \in C_{h,k}^{2\pi}$, and if $h = 0$, then $L_0\varphi_z(x) = 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{(k+1)!} > 0$ trivially. Hence, $(v.2)$ is verified.
Now we show that condition $(v.3)$ is also satisfied. Given $\phi \in C_{2\pi}$, for $i = h, h+1, \ldots, k$, let $m_i = \min\{(L_i\phi)(x),\, x \in X\}$. Now if we consider $\beta = \max_{h \leq i \leq k}\{(k+1)!|m_i| + 1\}$, then for $i = h, h+1, ..., k$

$$L_i(\beta\varphi_z + \phi)(x) = \beta(L_i\varphi_z)(x) + (L_i\phi)(x) \geq |m_i|(k+1)!(L_i\varphi_z)(x) + (L_i\varphi_z)(x) + m_i$$

$$\geq |m_i| + \frac{1}{(k+1)!} + m_i \geq \frac{1}{(k+1)!} > 0,$$

and therefore $\beta\varphi_z + \phi \in C_{2\pi}^{h,k}$. The rest of the conditions are verified obviously. $\square$

Remark 2 Theorem 2 appears considering $m = \infty$ and $k = 0$.

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