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ω-conditioned divided differences to solve nonlinear equations.

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Abstract

In this paper, we study iterative processes of the Secant-type to solve nonlinear equations in the form $F(x) = 0$. To analyse the semilocal convergence of these methods, we generalize the usually (k, p)-Hölder continuous conditions and we consider ω-conditioned dividided differences.

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1 Introduction

We present iterative processes to solve nonlinear equations of the form

$$F(x) = 0, \tag{1}$$

which use divided differences instead of the derivative of F, as for example the Secant method ([1], [2], [3]). Remember that a bounded linear operator $[x, y; F] : \Omega \subseteq X \to Y$ is called a divided difference of first order for the operator F on the points x and y $(x \neq y)$ if the following equality holds:

$$[x, y; F](x - y) = F(x) - F(y). \tag{2}$$

To analyse the semilocal convergence of these iterative processes, conditions of the type:

$$\|[x, y; F] - [u, v; F]\| \leq k(\|x - u\|^p + \|y - v\|^p); \quad p \in [0, 1],$$

has been required. Observe that if $p = 1$ in the last inequality, F has a Lipschitz continuous divided difference [3] and, in other case, we say that F has a (k, p)-Hölder continuous divided difference [1].

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In this paper, we consider a class of Secant-like iterations given by

$$\begin{cases} x^{(-1)},\ x^{(0)} \quad \text{given} \\ y^{(n)} = \lambda x^{(n)} + (1-\lambda)x^{(n-1)}, \quad \lambda \in [0,1], \\ x^{(n+1)} = x^{(n)} - \left[y^{(n)}, x^{(n)}; F\right]^{-1} F(x^{(n)}) \end{cases} \tag{3}$$

and we study the semilocal convergence of these processes using ω-conditioned dividided differences, that is:

$$\|[x,y;F] - [u,v;F]\| \leq \omega(\|x-u\|, \|y-v\|); \quad x,y,u,v \in \Omega,$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function in its two arguments.

Finally, a numerical example is considered where the Secant-like iterations are applied.

2 Recurrence Relations

We establish the recurrence relations from which the convergence of (3) is proved later. Let $x^{(-1)}, x^{(0)} \in \Omega$ and assume

(I) $\|x^{(-1)} - x^{(0)}\| = \alpha$,

(II) there exists $L_0{}^{-1} = \left[y^{(0)}, x^{(0)}; F\right]^{(-1)}$ such that $\|L_0{}^{-1}\| \leq \beta$,

(III) $\|L_0{}^{-1} F(x^{(0)})\| \leq \eta$,

(IV) $\|[x,y;F] - [u,v;F]\| \leq \omega(\|x-u\|, \|y-v\|); \quad x,y,u,v \in \Omega$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing funtion in its two arguments.

(V) there exists a nondecreasing funtion $h : [0,1] \to \mathbb{R}$ such that, if $t \in [0,1]$, $\omega(tu_1, tu_2) \leq h(t)\omega(u_1, u_2)$.

This condition **(V)** is included in orden to can be more precise in certain bounds we shall carry out later on.

We denote

$$a_{-1} = \frac{\eta}{\eta + \alpha}, \quad a_0 = \beta\omega\left(\lambda\eta + (1-\lambda)\alpha, \eta\right),$$

and define the scalar sequence

$$a_n = f(a_{n-1})a_{n-1}h\left(f(a_{n-2})a_{n-2}\right), \quad n \geq 1, \tag{4}$$

where

$$f(x) = \frac{1}{1-x}.$$

As ${L_0}^{-1}$ exist, then $x^{(1)}$ is well defined and, from the initial hypotheses, it follows that

$$\|x^{(1)} - x^{(0)}\| = \|{L_0}^{-1}F(x^{(0)})\| \leq \eta = f(a_{-1})a_{-1}\|x^{(0)} - x^{(-1)}\|, \tag{5}$$

$$\|{L_0}^{-1}\|\omega\left(\lambda\|x^{(1)} - x^{(0)}\| + (1-\lambda)\|x^{(0)} - x^{(-1)}\|, \|x^{(1)} - x^{(0)}\|\right) \leq a_0.$$

Next, we prove the following recurrence relations that are satisfied by sequences (3) and (4). Then, by induction on n, the following items are shown for $n \geq 1$:

(i_n) $\exists\, {L_n}^{-1} = \left[y^{(n)}, x^{(n)}; F\right]^{-1}$ such that $\|{L_n}^{-1}\| \leq f(a_{n-1})\|{L_{n-1}}^{-1}\|$,

(ii_n) $\|x^{(n+1)} - x^{(n)}\| \leq f(a_{n-1})a_{n-1}\|x^{(n)} - x^{(n-1)}\|$,

(iii_n) $\|{L_n}^{-1}\|\ \omega(\lambda\|x^{(n+1)} - x^{(n)}\| + (1-\lambda)\|x^{(n)} - x^{(n-1)}\|, \|x^{(n+1)} - x^{(n)}\|) \leq a_n$

Assuming that $a_0 \leq a_{-1} \leq 1/2$ and $x^{(1)} \in \Omega$, by **(IV)** we obtain

$$\begin{aligned}\|I - {L_0}^{-1}L_1\| &\leq \|{L_0}^{-1}\|\|L_0 - L_1\| = \|{L_0}^{-1}\|\ \|\left[y^{(1)}, x^{(1)}; F\right] - \left[y^{(0)}, x^{(0)}; F\right]\| \\ &\leq \|{L_0}^{-1}\|\omega(\|y^{(1)} - y^{(0)}\|, \|x^{(1)} - x^{(0)}\|) \\ &\leq \|{L_0}^{-1}\|\omega(\lambda\|x^{(1)} - x^{(0)}\| + (1-\lambda)\|x^{(0)} - x^{(-1)}\|, \|x^{(1)} - x^{(0)}\|) \\ &\leq \beta\omega(\lambda\eta + (1-\lambda)\alpha, \eta) = a_0 < 1\end{aligned}$$

and, by the Banach lemma, ${L_1}^{-1}$ exists and

$$\|{L_1}^{-1}\| \leq f(a_0)\|{L_0}^{-1}\|.$$

Consequently iterate $x^{(2)}$ is well defined.

By (2) and (3), we have

$$F(x^{(1)}) = F(x^{(0)}) - \left[x^{(0)}, x^{(1)}; F\right](x^{(0)} - x^{(1)}) = \left(L_0 - \left[x^{(0)}, x^{(1)}; F\right]\right)(x^{(0)} - x^{(1)}).$$

Then, by **(IV)**, we get

$$\begin{aligned}\|F(x^{(1)})\| &\leq \|\left[x^{(0)}, x^{(1)}; F\right] - \left[y^{(0)}, x^{(0)}; F\right]\|\ \|x^{(1)} - x^{(0)}\| \\ &\leq \omega\left(\|x^{(0)} - y^{(0)}\|, \|x^{(1)} - x^{(0)}\|\right)\|x^{(1)} - x^{(0)}\| \\ &\leq \omega\left((1-\lambda)\|x^{(0)} - x^{(-1)}\|, \|x^{(1)} - x^{(0)}\|\right)\|x^{(1)} - x^{(0)}\| \\ &\leq \omega\left(\lambda\|x^{(1)} - x^{(0)}\| + (1-\lambda)\|x^{(0)} - x^{(-1)}\|, \|x^{(1)} - x^{(0)}\|\right)\|x^{(1)} - x^{(0)}\| \\ &\leq \omega\left(\lambda\eta + (1-\lambda)\alpha, \eta\right)\|x^{(1)} - x^{(0)}\|\end{aligned}$$

and then

$$\|x^{(2)} - x^{(1)}\| \leq \|{L_1}^{-1}\|\ \|F(x^{(1)})\| \leq f(a_0)\|{L_0}^{-1}\|\ \|F(x^{(1)})\| \leq f(a_0)\ a_0\ \|x^{(1)} - x^{(0)}\|.$$

Note that, if $x < 1$, $f(x)x \leq 1 \Leftrightarrow x \leq 1/2$. So, finally, from (i_1), (ii_1), (5) and $a_0 \leq a_{-1} \leq 1/2$, we have

$$\|L_1{}^{-1}\|\omega\left(\lambda\|x^{(2)} - x^{(1)}\| + (1-\lambda)\|x^{(1)} - x^{(0)}\|, \|x^{(2)} - x^{(1)}\|\right)$$

$$\leq f(a_0)\|L_0{}^{-1}\|$$
$$\omega\left(\lambda f(a_0)a_0\|x^{(1)} - x^{(0)}\| + (1-\lambda)f(a_{-1})a_{-1}\|x^{(0)} - x^{(-1)}\|, f(a_0)a_0\|x^{(1)} - x^{(0)}\|\right)$$

$$\leq f(a_0)\beta\omega\left(\lambda\eta + (1-\lambda)\alpha, \eta\right)h(f(a_{-1})a_{-1}) \leq f(a_0)a_0 h(f(a_{-1})a_{-1}) = a_1.$$

Now if we suppose that $\{a_n\}$ is decreasing, $x^{(n+1)} \in \Omega$ and (i_n) -(iii_n) are true for a fixed $n \geq 1$; we analogously prove (i_{n+1}) -(iii_{n+1}).

3 Semilocal convergence of Secant-like methods

We study the real sequence defined in (4) in order to obtain the convergence of sequence (3) in Banach spaces. It will be sufficient that $a_n < 1/2$ $(n \geq 0)$ and $\{x^{(n)}\}$ is a Cauchy sequence.

Lemma 3.1 *Let $\{a_n\}$ be the sequence defined in (4). If $a_1 < a_0 < a_{-1} < 1/2$, then the sequence $\{a_n\}$ is decreasing.*

Proof. We use mathematical induction on n. If we assume $a_n < a_{n-1} < a_{n-2}$, we have that $a_{n+1} < a_n$, since f and h are increasing:

$$a_{n+1} = f(a_n)a_n h\left(f(a_{n-1})a_{n-1}\right) \leq f(a_{n-1})a_{n-1}h\left(f(a_{n-2})a_{n-2}\right) = a_n. \qquad \blacksquare$$

Theorem 3.2 *Let $x^{(-1)}, x^{(0)} \in \Omega$ and $\lambda \in [0,1]$. Let us suppose (I)-(V) and the hypotheses of lemma 3.1 are satisfied. If $\overline{B(x^{(0)}, R)} \subseteq \Omega$, where $R = \frac{1-a_0}{1-2a_0}\eta$, then the sequence $\{x^{(n)}\}$ given by (3) is well defined, remains in $\overline{B(x^{(0)}, R)}$ and converges to a solution x^* of equation $F(x) = 0$ in $\overline{B(x^{(0)}, R)}$. Moreover the solution x^* is unique in $B(x^{(0)}, \tau) \cap \Omega$, where τ is the smallest positive root of $\beta\omega\left(\tau + (1-\lambda)\alpha, R\right) = 1$.*

Proof. It is clear that $a_n < 1/2$. We then prove that $x^{(n)} \in B(x^{(0)}, R)$ for $n \geq 1$ and $\{x^{(n)}\}$ is a Cauchy sequence. Thus, for arbitrary positive integers m and n, we consider

$$\|x^{(n+m)} - x^{(n)}\| \leq \|x^{(n+m)} - x^{(n+m-1)}\| + \|x^{(n+m-1)} - x^{(n+m-2)}\| + \cdots + \|x^{(n+1)} - x^{(n)}\|$$

$$\leq f(a_{n+m-2})a_{n+m-2}\cdots f(a_{n+1})a_{n+1}\, f(a_n)a_n\|x^{(n+1)} - x^{(n)}\|$$
$$+ f(a_{n+m-3})a_{n+m-3}\cdots f(a_{n+1})a_{n+1}f(a_n)a_n\|x^{(n+1)} - x^{(n)}\|$$
$$+ \cdots + f(a_n)a_n\|x^{(n+1)} - x^{(n)}\| + \|x^{(n+1)} - x^{(n)}\|$$

$$= \left[\prod_{j=n}^{n+m-2} f(a_j)a_j + \prod_{j=n}^{n+m-3} f(a_j)a_j + \cdots + f(a_n)a_n + 1\right] \|x^{(n+1)} - x^{(n)}\|. \quad (6)$$

Now, by lemma 3.1 since f is increasing, $\{a_j\}$ is decreasing ($a_j < a_0$), we have

$$f(a_j)a_j < f(a_0)a_0 = \frac{a_0}{1-a_0} = \Delta < 1.$$

Thus, for $n \geq 1$:

$$\|x^{(n+m)} - x^{(n)}\| < \left[\Delta^{m-1} + \Delta^{m-2} + \cdots + \Delta + 1\right] \|x^{(n+1)} - x^{(n)}\|$$

$$< \frac{1-\Delta^m}{1-\Delta} \; \Delta^n \|x^{(1)} - x^{(0)}\|.$$

If $n = 0$, by (6),

$$\|x^{(m)} - x^{(0)}\| < \frac{1-\Delta^m}{1-\Delta} \|x^{(1)} - x^{(0)}\|,$$

from this, we deduce

$$\|x^{(m)} - x^{(0)}\| < \frac{\eta}{1-\Delta} = R.$$

Consequently, for all n, $x^{(n)} \in B(x^{(0)}, R)$ and the sequence $\{x^{(n)}\}$ is well defined. Secondly, $\{x^{(n)}\}$ is a Cauchy sequence and has a limit x^* in $\overline{B(x^{(0)}, R)}$. Thirdly, we see that x^* is a zero of F. Since

$$\begin{aligned} &\|F(x^{(n)})\| \\ &\quad \leq \omega\left(\lambda\|x^{(n)} - x^{(n-1)}\| + (1-\lambda)\|x^{(n-1)} - x^{(n-2)}\|, \|x^{(n)} - x^{(n-1)}\|\right) \|x^{(n)} - x^{(n-1)}\|, \end{aligned}$$

and $\|x^{(n)} - x^{(n-1)}\| \to 0$ as $n \to \infty$, we obtain $F(x^*) = 0$.

Finally, to prove the uniqueness of solution x^*, we assume that z^* is another root of (1) in $B(x^{(0)}, \tau) \cap \Omega$ and consider the operator $A = [z^*, x^*; F]$. We have

$$\|{L_0}^{-1}A - I\| \leq \|{L_0}^{-1}\| \; \|A - L_0\| \leq \|{L_0}^{-1}\| \|[z^*, x^*; F] - \left[y^{(0)}, x^{(0)}; F\right]\|$$

$$\leq \beta\omega\left(\|z^* - y^{(0)}\|, \|x^* - x^{(0)}\|\right) \leq \beta\omega\left(\|z^* - x^{(0)}\| + \|x^{(0)} - y^{(0)}\|, \|x^* - x^{(0)}\|\right)$$

$$< \beta\omega\left(\tau + (1-\lambda)\alpha, R\right) = 1$$

and the operator A is then invertible, and consequently $z^* = x^*$. ∎

Remark. Note that the operator F is differentiable when the divided differences are Lipschitz o (k,p)-Hölder continuous for all $x, y, u, v \in \Omega$ [3]. But, under condition **(IV)**, F is differentiable if $\omega(0,0) = 0$. Therefore, if $\omega(0,0) \neq 0$, theorem is true for non-differentiable operators.

4 Numerical example

We consider the following boundary value problem:

$$\begin{cases} \dfrac{d^2x(t)}{dt^2} + x(t)^{1+p} + x(t)^2 = 0, \quad p \in [0,1] \\ x(0) = x(1) = 0. \end{cases} \tag{7}$$

Firstly, we apply the semilocal convergence result given above to approximate the solution of the equation (7). Secondly, we show how the speed of convergence for (3) varies along with λ.

We divide the interval $[0,1]$ into subintervals and let $l = 1/n$. We denote the points of subdivision by $t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$, with the corresponding values of the function $x_0 = x(t_0) = 0, \ldots, x_{n-1} = x(t_{n-1}),\ x_n = x(t_n) = 0$. We first approximate the second derivative $x''(t)$ in the differential equation by

$$x''(t_i) \approx (x_{i+1} - 2x_i + x_{i-1})/l^2, \quad i = 1, 2, \ldots, n-1.$$

Discretizing the differential equation we obtain the following system of non-linear equations

$$\begin{cases} 2x_1 - l^2 x_1^{1+p} - l^2 x_1^2 - x_2 = 0, \\ -x_{i-1} + 2x_i - l^2 x_i^{1+p} - l^2 x_i^2 - x_{i+1} = 0, \quad i = 2, \ldots, n-2, \\ -x_{n-2} + 2x_{n-1} - l^2 x_{n-1}^{1+p} - l^2 x_{n-1}^2 = 0. \end{cases} \tag{8}$$

We have an operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $F(x) = H(x) - l^2 g(x)$, where

$$H = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} x_1^{1+p} + x_1^2 \\ x_2^{1+p} + x_2^2 \\ \vdots \\ x_{n-1}^{1+p} + x_{n-1}^2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

Then, we apply theorem 3.2 to find a solution x^* of the equation $F(x) = 0$.

We have

$$F'(x) = H - l^2(1+p) \begin{pmatrix} x_1^p & 0 & \cdots & 0 \\ 0 & x_2^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1}^p \end{pmatrix} - 2l^2 \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1} \end{pmatrix}$$

In this case, we consider

$$[u, v; F] = \int_0^1 F'(u + t(v-u))\, dt.$$

So we study the value $\|F'(x) - F'(u)\|$ to obtain a bound for $\|[x, y; F] - [u, v; F]\|$. For all $x, u \in \mathbb{R}^{n-1}$ with $|x_i| > 0$, $|u_i| > 0$, $(i = 1, 2, \cdots .n-1)$ and taking into account the max-norm it follows

$$\|F'(x) - F'(u)\| = \|\mathrm{diag}\{l^2(1+p)\ (u_i^p - x_i^p) + 2l^2(u_i - x_i)\}\|$$

$$= \max_{1\le i\le n-1} |l^2(1+p)\ (u_i^p - x_i^p) + 2l^2(u_i - x_i)|$$

$$\le (1+p)l^2\ [\max_{1\le i\le n-1} |u_i - x_i|\ ]^p + 2l^2\|u - x\| = (1+p)l^2\|u-x\|^p + 2l^2\|u - x\|.$$

Therefore

$$\|[x,y;F] - [u,v;F]\| \le \int_0^1 \|F'\left(x + t(y-x)\right) - F'\left(u + t(v-u)\right)\| \, dt$$

$$\le l^2 \int_0^1 \left((1+p)\|(1-t)(x-u) + t(y-v)\|^p + 2\|(1-t)(x-u) + t(y-v)\|\right) dt$$

$$\le l^2(1+p) \int_0^1 \left((1-t)^p\|x-u\|^p + t^p\|y-v\|^p\right) dt + 2l^2 \int_0^1 \left((1-t)\|x-u\| + t\|y-v\|\right) dt$$

$$\le l^2\left(\|x-u\|^p + \|y-v\|^p + \|x-u\| + \|y-v\|\right)$$

Consider $\omega(u_1,u_2) = l^2\left(u_1^p + u_2^p + u_1 + u_2\right)$, $h(t) = t^p$. Now we apply the iteration (3) for $\lambda = 0$ to approximate the solution of $F(x) = 0$

We choose $p = 1/2$, $n = 10$, then (8) gives 9 equations. Since a solution of (7) would vanish at the end points and be positive in the interior, a reasonable choice of initial approximation seems to be $10\sin\pi t = z^{(-1)}(t)$. This is, $z_i^{(-1)} = z^{(-1)}(t_i) = 10\sin\pi t_i$, $z^{(0)}(t_i) = z^{(-1)}(t_i) - 10^{-5}, \quad i = 1,2,...9.$

Using the Method (3) for ($\lambda = 0$), after two iterations we obtain

$$z^{(1)} = \begin{pmatrix} 2.453176290658909 \\ 4.812704101582601 \\ 6.8481873135861 \\ 8.252997367741953 \\ 8.75737771678512 \\ 8.252997367741953 \\ 6.8481873135861 \\ 4.812704101582601 \\ 2.453176290658909 \end{pmatrix}, \ z^{(2)} = \begin{pmatrix} 2.404324055268407 \\ 4.713971539035271 \\ 6.7003394962933925 \\ 8.066765882171131 \\ 8.556329565792526 \\ 8.066765882171131 \\ 6.7003394962933924 \\ 4.713971539035271 \\ 2.404324055268407 \end{pmatrix}.$$

Then we take $x^{(-1)} = z^{(1)}$, and $x^{(0)} = z^{(2)}$. With the notation of theorem 3.2 we can easily obtain the following result:

$$\alpha = 0.201048,\ \beta = 15.319,\ \eta = 0.0346555$$
$$a_1 = 0.0638626 < a_0 = 0.133313 < a_{-1} = 0.14703 < 1/2$$
$$R = 0.0409552,\ \tau = 4.0272$$

All the hypotheses of theorem 3.2 are now satisfied. Consequently, iteration (3) converges to unique solution x^* in $\overline{B(x^{(0)},\tau)}$ of equation $F(x) = 0$.

We obtain the vector x^* as the aproximate solution of system (8) :

$$x^* = \begin{pmatrix} 2.394640794786742 \\ 4.694882371216001 \\ 6.672977546934751 \\ 8.033409358893319 \\ 8.520791423704788 \\ 8.033409358893319 \\ 6.67297754693475 \\ 4.694882371216 \\ 2.394640794786742 \end{pmatrix}.$$

Finally, we apply (3) for different values of the parameter λ.

The following table contains the errors $\|x^* - z^{(n)}\|$ for the iterates $z^{(n)}$ generated by the Secant-like methods for different λ.

n	$\lambda = 0$	$\lambda = 0.7$	$\lambda = 0.99$
1	2.36586×10^{-1}	2.36585×10^{-1}	2.36585×10^{-1}
2	3.55381×10^{-2}	1.55367×10^{-2}	5.96404×10^{-3}
3	8.82627×10^{-4}	1.37074×10^{-4}	5.55811×10^{-6}
4	3.50678×10^{-6}	7.33869×10^{-8}	4.1064×10^{-11}
5	3.50793×10^{-10}	3.41061×10^{-13}	7.10543×10^{-15}
6	7.10543×10^{-15}	3.55271×10^{-15}	
7	5.32907×10^{-15}		

As we can see in table, iteration (3) converges faster to x^*, when λ is increased.

Note that, in this example, the convergence can not be guaranteed by classical studies, where used divided differences are Lipschitz or Hölder continuous, whereas we can do it by the technique presented in this paper.

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