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On a Transformation to Canonical Orbital Elements for the Two–Body Problem

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Abstract

Within the framework of the Hamiltonian Mechanics in extended phase space, the transformation from the enlarged Delaunay chart (as considered by Bond & Broucke and Bond & Janin in the late 1970s and early 1980s), leading to the construction of a canonical set of DS (Delaunay–Similar or Delaunay–Scheifele) elliptic Keplerian orbital elements proposed by Scheifele and Graf, is *generalized* so as to create a general, *unified pattern* for the systematic derivation of *any* set of elliptic DS orbital elements with respect to an *arbitrary independent variable* (some kind of anomaly–like parameter introduced by a generalized Sundman–type time–transformation). An adequate modification of the *generating function* of the canonical mapping takes into account the unspecified nature of the new phase variables.
Keywords: two–body problem, extended phase space, canonical transformations, Delaunay and Delaunay–Similar (DS) elements, time transformation, anomalies.
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1 Extended Phase Space and DS Elements

As a general rule in many mathematical and physical contexts, the time t is usually viewed and treated as a parameter completely different from the spatial–like coordinates (which are dependent variables). In the present paper we consider the generating function of a canonical transformation in an 8–dimensional, *extended phase space*, in which the dimensions corresponding to “space” and “time” are contemplated, *on an equal footing*, as similar coordinates. In such a case, some other adequate parameter (fictitious time or pseudo–time) should be used to replace t in the description of the evolution of the system (either in the configuration space or in the phase space); that reparametrizing pseudo–time will then take the role of the differentiation and integration parameter.

The *homogeneous canonical formalism* takes advantage of an elegant procedure due to *Poincaré* (1905, vol. I, Ch. 1, §12, pp. 13–16): the dimensionality of ordinary phase space is enlarged with two additional dimensions, giving rise to the concept of *extended phase space*: the physical time t is included as a dependent variable, a generalized coordinate in the canonical set, whose conjugate momentum is related to the original Hamiltonian (after a change of sign). Time is thus treated on a common basis with the other (spatial–like) coordinates, and the momentum conjugate to this "time–coordinate" is the negative of the original Hamiltonian in ordinary phase–space formulation. The extended–phase–space approach facilitates the introduction of new independent variables different from the physical time. Further details are found, e.g., in Stiefel & Scheifele (1971, §30, §34, §37), Scheifele (1970a, 1972), Goldstein (1980, §7.9, §8.4, and Ch. 8, Problem 32).

For the extended–phase–space canonical treatment, reduction and approximate analytical integration (with the help of canonical perturbation techniques) of perturbed Keplerian Hamiltonian systems related to the *zonal model* of gravitational geopotential in Artificial Satellite Theory, Scheifele and his collaborators (1970a, §2.2; 1970b; 1972, Part B; 1974) developed an analytical approach based on canonical transformations leading to the definition of new sets of eight dependent phase variables (resembling, in some sense, the classical Delaunay elements of the elliptic Kepler problem) and the use of new independent variables different from the physical time. *Elements of the motion* are to be understood as in Stiefel & Scheifele (1971, §18, pp. 83–84): quantities which, in the unperturbed problem, are constant or linear functions of the independent variable.

Reparametizing the motion with the new regularizing time parameter (introduced by means of a differential relation generalizing the time–transformation named after Sundman [1912, p. 127]), the new phase variables become *canonical orbital elements* with respect to that pseudo–time. A remarkable feature of Scheifele's approach is the fact that the Keplerian true anomaly (but also, in some cases, the eccentric anomaly) is incorporated into the theory so as to play a dual role: both as the independent variable and as a canonical element belonging to the new chart. The orbital variables created by Scheifele are widely known as *DS (Delaunay–Similar or Delaunay–Scheifele)* variables or elements.

As with the traditional construction of the classical Delaunay elements for the bound Kepler problem, the starting point for the standard development of the different transformations to DS sets (as usually presented by Scheifele and his co–workers) is based on the formulation of the Keplerian Hamiltonian in polar spherical coordinates –in the extended phase space– and its solution on integrating the corresponding Hamilton–Jacobi partial differential equation by separation of variables. The amount of work and effort at every intermediate calculation step and in the interpretation of results is considerable.

This drawback is softened if, form the start, the Kepler problem is given in the (enlarged) *polar nodal* chart (Deprit, 1981), from which the Keplerian *true anomaly* is easily made a coordinate–type phase variable, incorporating it as a DS canonical orbital element. Deprit's approach to the TR–map defining Scheifele's (1970b, 1972) DS set of Keplerian elements gave a clue to establish a general scheme (Floría, 1994) and a *unified pattern* for the *systematic derivation* of arbitrary sets of DS elements in a true–like anomaly.

A different approch to obtain DS elements can be taken. A generating function *of the second type* (Goldstein 1980, §9.1, pp. 383–384), depending on the old coordinates and the new momenta, allowed *Bond & Broucke* (1980) to perform a completely canonical transformation in extended phase space, from the (enlarged) Delaunay elliptic Keplerian elements to the set of DS canonical orbital variables applied by *Scheifele & Graf* (1974) to the oblateness perturbation problem in Artificial Satellite Theory. These Scheifele–Graf variables make up a set of Keplerian orbital elements with respect to the true anomaly. With the same generating function, *Bond & Janin* (1981) constructed analogous canonical orbital elements (of the Scheifele–Graf type) with respect to *any anomaly–like parameter*.

These authors focus on the Scheifele & Graf (1974) DS set. In line with some previous work (e.g., Floría 1994) concerning the general and systematic derivation of *a wide class of different DS sets* and their application to perturbed Keplerian systems, we will propose a *general expression* for the Bond–Broucke generating function and develop the transformation from (extended) Delaunay elements to *any possible DS set* of (unspecified) canonical orbital variables. In so doing, the canonical sets of phase variables due to Scheifele and his co–workers fit into our unified pattern. As in Bond & Janin (1981), the new Keplerian orbital elements use an arbitrary anomaly–type parameter as the independent variable.

2 From Delaunay Elements to Generic DS Variables

The Delaunay set, $(l_D, g_D, h_D; L_D, G_D, H_D)$, enlarged by the canonically conjugate pair of phase variables $(t; T)$, the time t and the negative of the total energy, is the starting point for the transition to a new set of generic DS variables in the 8–dimensional phase space. Inspired by Bond & Broucke (1980) and Bond & Janin (1981), a *more general functional structure* for the generating function of a canonical transformation to *unspecified DS Keplerian orbital elements* is given. In Delaunay formulation, a Keplerian Hamiltonian,

$$\mathcal{H}_0 \;\equiv\; \mathcal{H}_0\,(\,-\,,\,-\,,\,-\,;\,L_D\,,\,-\,,\,-\,)\;=\;-\,\mu^2/(\,2\,L_D^2\,)\;, \tag{1}$$

is cyclic in five Delaunay variables: with respect to the physical time, the Delaunay set is a *set of canonical Keplerian elements* (Stiefel & Scheifele 1971, §18). Some Keplerian relations for the standard orbital elements $a \equiv a(L_D)$, $e \equiv e(L_D, G_D)$, $p \equiv p(G_D)$, $I(G_D, H_D)$, $\omega(g_D)$ and $\Omega(h_D)$, the distance r (length of the position vector of the mobile), and the eccentric and true anomalies, $E \equiv E(r; L_D, G_D)$ and $f \equiv f(r; L_D, G_D)$, are

$$L_D \;=\; \sqrt{\mu\,a}\;, \quad G_D^2 \;=\; \mu\,a\,\left(1\,-\,e^{\,2}\right) \;=\; \mu\,p\;, \quad H_D \;=\; G_D\,\cos I\;, \tag{2}$$

$$e^2 = 1 - \left(G_D^2/L_D^2\right), \quad p = G_D^2/\mu, \quad g_D = \omega, \quad h_D = \Omega, \tag{3}$$

$$r = a\,(1 - e\cos E), \quad r = p/(1 + e\cos f), \tag{4}$$

$$l_D = \Phi(E) = E - e\sin E, \quad dl_D/dE = d\Phi(E)/dE = 1 - e\cos E = r/a. \tag{5}$$

A transformation $(t, l_D, g_D, h_D; T, L_D, G_D, H_D) \longrightarrow (\psi, l, g, h; \Psi, L, G, H)$, from the Delaunay elements (in extended phase space) to a generic set of Delaunay–Similar variables, is defined by a *generating function* $S \equiv S_{DS}(t, l_D, g_D, h_D; \Psi, L, G, H)$,

$$S = t\,L + \left[\mu/\sqrt{2\,L}\right] l_D + \mathcal{F}(\Psi, L, G, H)\, Z(t) + g_D\, G + h_D\, H, \tag{6}$$

more general than the generator (Bond & Broucke 1980; Bond & Janin 1981) yielding the Scheifele–Graf (1974) canonical set. Here $\mathcal{F}(\Psi, L, G, H)$ is an *arbitrary function* of the new canonical momenta, and $Z(t)$ is some *unspecified* function of t. With the notations

$$Z'(t) \equiv dZ(t)/dt, \quad \mathcal{F}_{(\Psi,L,G,H)} \equiv \partial\mathcal{F}/\partial(\Psi, L, G, H) = \nabla_{(\Psi,L,G,H)}\mathcal{F}, \tag{7}$$

the generating relations derived from S lead to the *implicit transformation equations*

$$\psi = \partial S/\partial\Psi = \mathcal{F}_\Psi\, Z(t) \implies Z(t) = \psi/\mathcal{F}_\Psi, \tag{8}$$

$$l = \partial S/\partial L = t - \left[\mu/(2L)^{3/2}\right] l_D + \mathcal{F}_L\, Z(t), \text{ generalized Kepler's equation,} \tag{9}$$

$$g = \partial S/\partial G = g_D + \mathcal{F}_G\, Z(t) = g_D + (\mathcal{F}_G/\mathcal{F}_\Psi)\,\psi, \tag{10}$$

$$h = \partial S/\partial H = h_D + \mathcal{F}_H\, Z(t) = h_D + (\mathcal{F}_H/\mathcal{F}_\Psi)\,\psi, \tag{11}$$

$$L_D = \partial S/\partial l_D = \mu/\sqrt{2\,L} \Rightarrow L = \mu^2/\left(2\,L_D^2\right), \tag{12}$$

$$T = \partial S/\partial t = L + \mathcal{F}(\Psi, L, G, H)\,(dZ/dt) \implies \mathcal{F} = (T - L)/Z'(t), \tag{13}$$

$$G_D = \partial S/\partial g_D = G, \quad H_D = \partial S/\partial h_D = H. \tag{14}$$

In the DS chart, with t as the independent variable, the homogeneous Hamiltonian is

$$\mathcal{H}_0 \longrightarrow (\mathcal{H}_0)_h = T - \left[\mu^2/\left(2\,L_D^2\right)\right] \implies \widetilde{\mathcal{H}}_h = \mathcal{F}(\Psi, L, G, H)\, Z'(t), \tag{15}$$

whose numerical value is zero (Bond & Broucke 1980, p. 358; Poincaré 1905, §12).

3 Introduction of a New Independent Variable

The functional structure of Hamiltonian (15) is simplified if one reparametrizes the motion in terms of new independent variables τ, anomaly–like parameters adequately introduced by generalized Sundman–type (1912, p. 127) *differential time–transformations*,

$$t \longrightarrow \tau : dt = \tilde{f}\, d\tau, \; t' \equiv dt/d\tau = \tilde{f}, \; \tilde{f} \text{ being a function of the } \textit{new} \text{ variables.} \tag{16}$$

With τ as the *fictitious time* (Scheifele 1970a; Stiefel & Scheifele 1971, §34), $\widetilde{\mathcal{H}}_h$ becomes

$$\mathcal{K}_0 = \tilde{f}\,\widetilde{\mathcal{H}}_h = \mathcal{F}(\Psi, L, G, H)\,(dZ/dt)\,\tilde{f}, \tag{17}$$

which can be appropriately simplified if some conditions are imposed on the unspecified time–related functions $\widetilde{f}$ and $Z(t)$. *For our Keplerian system*, take $\widetilde{f}$ and $Z(t)$ such that

$$\widetilde{f}\ (dZ/dt)\ =\ 1\ \Longrightarrow\ \mathcal{K}_0\ =\ \mathcal{F}\,(\Psi\,,\,L\,,\,G\,,\,H)\ \equiv\ 0\ \text{(along solutions)} \tag{18}$$

in the phase space of the new variables. But $\mathcal{F}\,(\Psi\,,\,L\,,\,G\,,\,H)$ still remains unspecified. Specific *choices* of $\mathcal{F}$ (Floría 1994) produce particular DS sets, and *a wide class of DS sets of canonical orbital elements* can be defined. The simple structure of a *canonical solution, parametrized by* the new pseudo–time τ, to the Hamilton equations generated by $\mathcal{K}_0$,

$$(\Psi, L, G, H)'\ =\ -\,\partial\mathcal{F}/\partial\,(\psi, l, g, h)\ =\ -\,\nabla_{(\psi,l,g,h)}\mathcal{F}\ =\ \mathbf{0} \tag{19}$$

$$\Longrightarrow\ (\Psi, L, G, H)\ =\ (\Psi_0, L_0, G_0, H_0)\,, \tag{20}$$

$$(\psi, l, g, h)'\ =\ \partial\mathcal{F}/\partial\,(\Psi, L, G, H)\ =\ \mathcal{F}_{(\Psi,L,G,H)} \tag{21}$$

$$\Longrightarrow\ (\psi, l, g, h)\ =\ (\mathcal{F}_\Psi, \mathcal{F}_L, \mathcal{F}_G, \mathcal{F}_H)\ \tau\ +\ (\psi_0, l_0, g_0, h_0)\,, \tag{22}$$

where $(\psi_0, l_0, g_0, h_0, \Psi_0, L_0, G_0, H_0)$ are integration constants, shows that these DS phase variables are a set of Keplerian *elements of the motion* with respect to τ. In particular,

$$\psi'\ \equiv\ d\psi/d\tau\ =\ \partial\mathcal{K}_0/\partial\Psi\ =\ \mathcal{F}_\Psi\ \Longrightarrow\ \psi\ =\ \mathcal{F}_\Psi\,\tau\ +\ \text{const.} \tag{23}$$

$$\Longrightarrow\ d\psi/dt\ =\ (d\psi/d\tau)\,(d\tau/dt)\ =\ \mathcal{F}_\Psi\,(d\tau/dt)\ =\ \mathcal{F}_\Psi/\widetilde{f}\,, \tag{24}$$

and the generalized anomaly ψ must be *consistent* with the choice of the reparametrizing function $\widetilde{f}$ (Bond & Janin 1981, p. 161). Some Keplerian elements are now rewritten as

$$a\ =\ L_D^2/\mu\ =\ \mu/(\,2\,L\,)\,,\quad n\ =\ \sqrt{\mu/a^{\,3}}\ =\ (\,2\,L\,)^{3/2}/\mu\,, \tag{25}$$

$$e^{\,2}\ =\ 1\ -\ (p/a)\ =\ 1\ -\ \left(G_D^2/L_D^2\right)\ =\ 1\ -\ \left(2\,L\,G^2/\mu^2\right)\,. \tag{26}$$

From the generating relations and formulae for the Delaunay variables, Eq. (9) becomes

$$t\ =\ l + \left[\mu/\,(2L)^{3/2}\right][\,E - e\sin E\,] - \ (\mathcal{F}_L/\mathcal{F}_\Psi)\,\psi\,,\ \ \textit{generalized Kepler equation}, \tag{27}$$

from which the time transformation is developed to obtain $\widetilde{f}$ for each special choice of ψ.

4 Study of the Reparametrizing Function

As in Bond & Janin (1981, §4), the total derivative of Eq. (27) with respect to τ, say

$$\begin{aligned}\widetilde{f}\ &=\ t'\ \equiv\ dt/d\tau\ =\ l'\ +\ \left[\mu/(\,2\,L\,)^{3/2}\right]'\ \Phi(E)\ +\ \left[\mu/(\,2\,L\,)^{3/2}\right]\Phi'(E)\\ &\quad -\ (\mathcal{F}_L/\mathcal{F}_\Psi)'\,\psi\ -\ (\mathcal{F}_L/\mathcal{F}_\Psi)\,\psi'\,,\end{aligned} \tag{28}$$

thanks to the Hamilton equations [(19) and (21)], the constant orbital eccentricity e in the pure Kepler problem, Formula (25) for the semi–major axis a, and Eq. (5), reads

$$\begin{aligned}\widetilde{f}\ &=\ \mathcal{F}_L\ +\ \left[\mu/(\,2\,L\,)^{3/2}\right](d\,l_D/d\,\tau)\ -\ (\mathcal{F}_L/\mathcal{F}_\Psi)\,\mathcal{F}_\Psi\\ &=\ \left[\mu/(\,2\,L)^{3/2}\right][\,1\ -\ e\,\cos E\,]\,(dE/d\tau)\ =\ \left[r/\sqrt{2\,L}\right]\,(dE/d\tau)\,.\end{aligned} \tag{29}$$

From Eq. (23), $d\psi = \mathcal{F}_\Psi d\tau$, derivatives with respect to τ are rewritten as derivatives with respect to ψ, and we obtain the *reparametrizing function* in the time transformation (16):

$$\begin{aligned} dE/d\tau &= (dE/d\psi)\,(d\psi/d\tau) \Rightarrow \widetilde{f} = \left(r/\sqrt{2L}\right)[\partial\mathcal{F}\,(\Psi, L, G, H)\,/\partial\Psi]\,(dE/d\psi)\,, && (30)\\ Z(t) &= \psi/\mathcal{F}_\Psi \implies dZ(t)/d\tau = \psi'/\mathcal{F}_\Psi = 1 \implies Z(t) = \tau + \text{const.} && (31) \end{aligned}$$

As a conclusion, "in order to evaluate $\widetilde{f}$ for a particular anomaly ψ, *the eccentric anomaly E must be found as a function $E = E(\psi)$ of ψ* (Bond & Janin 1981, p. 165)". According to the preceding developments, a **summary of final general results** reads:

• After the completely canonical transformation to *generic DS phase variables* derived from the generating function (6), the fundamental relations $\psi \longrightarrow E = E(\psi)\,,\ dE/d\psi\,,$

$$\begin{aligned} t &= l + \frac{\mu}{(\,2L\,)^{3/2}}\,[\,E - e\,\sin E\,] - \left(\frac{\mathcal{F}_L}{\mathcal{F}_\Psi}\right)\psi\,, \ \text{generalized Kepler's equation}\,, && (32)\\ dt &= \widetilde{f}\,d\tau,\ \widetilde{f} = \frac{r}{\sqrt{2L}}\,\frac{\partial\mathcal{F}\,(\Psi, L, G, H)}{\partial\Psi}\,\frac{dE}{d\psi}\,, \ \text{change of independent variable}\,, && (33) \end{aligned}$$

complete the introduction of canonical orbital elements, *depending on any kind of new independent variable*, for the Keplerian system governed by (1). Appropriate selections of $\mathcal{F}(\Psi, L, G, H) = \mathcal{K}_0$ are related to special instances of elliptic DS sets (Floría 1994). Our considerations involve three *unspecified* mathematical and dynamical objects: the generalized anomaly $\psi = \mathcal{F}_\Psi Z(t) = \mathcal{F}_\Psi \tau + \text{const.}$ (which also appears as a DS variable), and the functions $\widetilde{f}$ (in the differential relation (16) defining the time transformation $t \to \tau$) and $\mathcal{F}(\Psi, L, G, H) = \mathcal{K}_0$ (attached, in each particular case, to a specific DS set).

5 Some Remarks on Certain Special Cases

The most commonly used sets of DS variables, as obtained and applied by Scheifele and his collaborators (see papers by these authors; also, Deprit 1981, and Floría 1994), lead to simple expressions for $\mathcal{K}_0$ (which would be reflected in some subsequent simplification in the resulting final formulae when dealing with these simple cases). For instance:

$$\begin{aligned} \mathcal{K}_0 &= \Psi \implies (\mathcal{F}_\Psi, \mathcal{F}_L, \mathcal{F}_G, \mathcal{F}_H) = (\,1\,, 0\,, 0\,, 0\,)\,, && (34)\\ \mathcal{K}_0 &= G\,\Psi - \left(\Psi^2/2\right) \implies (\mathcal{F}_\Psi, \mathcal{F}_L, \mathcal{F}_G, \mathcal{F}_H) = (\,G - \Psi\,, 0\,, \Psi\,, 0\,)\,, && (35)\\ \mathcal{K}_0 &= \Psi - \left(\mu/\sqrt{2L}\,\right) \implies (\mathcal{F}_\Psi, \mathcal{F}_L, \mathcal{F}_G, \mathcal{F}_H) = \left(1\,, \mu/(\,2L\,)^{3/2}, 0\,, 0\right). && (36) \end{aligned}$$

This last option is just the one occurring in the study by Scheifele & Graf (1974), which is at the root of the articles by Bond & Broucke (1980) and Bond & Janin (1981) whose generalization motivated the present research.

Appart from this freedom in the choice of the dependent DS variables, both ψ and $\widetilde{f}$ remain unspecified. On the basis of the preceding scheme, particular selections of the anomaly–type parameter ψ lead to the determination of the functional structure and dependence of the reparametrizing function $\widetilde{f}$. In line with Bond & Janin (1981, §4), we summarize the results and conclusions pertaining to some significant choices for ψ.

5.1 Eccentric anomaly

$$\psi \;=\; E \;\Longrightarrow\; dE/d\psi \;=\; 1\,, \tag{37}$$
$$\widetilde{f} \;=\; \left(r/\sqrt{2\,L} \right) \mathcal{F}_{\Psi}\,, \tag{38}$$
$$t \;=\; \left\{ l \,+\, \left[\frac{\mu}{(\,2\,L\,)^{3/2}} \,-\, \frac{\mathcal{F}_L}{\mathcal{F}_{\Psi}} \right] E \right\} \,-\, \frac{\mu}{(\,2\,L\,)^{3/2}}\; e\,\sin E\,. \tag{39}$$

5.2 True anomaly

$$\psi \;=\; f \;\Longrightarrow\; dE/df \;=\; r/\sqrt{a\,p}\,, \tag{40}$$
$$\widetilde{f} \;=\; \left(r^2/\sqrt{\mu\,p} \right) \mathcal{F}_{\Psi} \;=\; \left(r^2/G \right) \mathcal{F}_{\Psi} \;=\; \left(r^2/G_D \right) \mathcal{F}_{\Psi}\,, \tag{41}$$
$$t \;=\; \left\{ l \,-\, \frac{\mathcal{F}_L}{\mathcal{F}_{\Psi}}\, f \right\} \,+\, \frac{\mu}{(\,2\,L\,)^{3/2}}\; (\,E \,-\, e\,\sin E\,)\,. \tag{42}$$

5.3 Mean anomaly

$$\psi \;=\; l_D \;=\; \Phi(E) \;\Longrightarrow\; dE/dl_D \;=\; a/r\,, \tag{43}$$
$$\widetilde{f} \;=\; \left[\mu/(\,2\,L\,)^{3/2} \right] \mathcal{F}_{\Psi}\,, \tag{44}$$
$$t \;=\; l \,+\, \left[\frac{\mu}{(\,2\,L\,)^{3/2}} \,-\, \frac{\mathcal{F}_L}{\mathcal{F}_{\Psi}} \right] l_D\,. \tag{45}$$

5.4 Length of orbital arc (Brumberg 1992; Floría 1997)

$$\psi \;=\; \sigma \;\Longrightarrow\; d\sigma/dE \;=\; a\,\sqrt{1 \,-\, e^2\,\cos^2 E}\,, \tag{46}$$
$$\widetilde{f} \;=\; \mathcal{F}_{\Psi}\, r^{1/2}/\sqrt{2\,\mu \,-\, 2\,L\,r} \;=\; \mathcal{F}_{\Psi}\, r^{1/2}/\sqrt{\mu\,(\,1 \,+\, e\,\cos E\,)}\,, \tag{47}$$
$$t \;=\; \left\{ l \,-\, \frac{\mathcal{F}_L}{\mathcal{F}_{\Psi}}\, \sigma \right\} \,+\, \frac{\mu}{(\,2\,L\,)^{3/2}}\; (\,E \,-\, e\,\sin E\,)\,. \tag{48}$$

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