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Symmetrization in Degenerate Parabolic Equations and Applications

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Abstract

The asymptotic behavior of the solution to a degenerate convection-reaction-diffusion equation is studied in two dimensions via symmetrization. Under some linking conditions between the terms of diffusion and reaction, the existence of a unique bounded time-global solution is given and a convergence result to the null function is proved.
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1 Introduction

The existence of a time-global solution to partial differential equations is easier to prove as soon as the space variable belongs to a ball of $\mathbb{R}^n$ and when, moreover, the local solution is radially symmetric. Indeed, in that case, the unknown may be considered as a solution of an equation with only one space variable, the radial one. Therefore to deal with the general problem it will be interesting to introduce, through the use of symmetrical rearrangements of a function (*cf* J. Mossino [11]), a symmetrized problem to which the first one can be compared.

Here, our purpose is to study the asymptotic behavior in time of the solution to the homogeneous Dirichlet problem

$$(PG) \begin{cases} u_t - \Delta\varphi(u) + div(\nabla P g(u)) + f(u) = 0, \\ u_{|\,\partial\Omega} = 0, \\ u(0,.) = u_0, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$.

Several authors have considered the time behavior of the solution of (PG) in the strictly parabolic case. Most of them treated this problem when the equation of (PG) is given by

$$u_t - \Delta u + f(t, x, u) = 0 \tag{1}$$

(See for instance, X.Y. Chen and P. Polacik [1], J.K. Hale and G. Raugel [8], A. Haraux [9], T.I. Zelenyak [15]). The strictly parabolic nonlinear case is studied in L. Simon [13], P. Polacik and K.P. Rybakowski [12] when f is an analytical function, and it is shown in [12] that this condition on f may be necessary as soon as f really depends on x. Now the degenerate parabolic problem (PG) has been considered by few authors. Assuming that a bounded solution to the problem (PG) exists, E. Feireisl and F. Simondon prove that the $\omega-$ limit set is a singleton under an analytical condition on f, firstly in a one dimensional framework [6] and then in several space dimensions with $\varphi(u) = u^m$ [7]. Under some linking condition between the functions f and φ, it is known that the problem (PG) has a unique bounded strong solution in one space dimension (F. Simondon [14]). Then in [3], M. Falliero and F. Simondon study the asymptotic behavior of this solution by application of H. Matano's works [10]. The previous results have been extended by M. Falliero and M. Madaune-Tort [5] for radially symmetric solutions in two dimensions.

The aim of this article is to prove via symmetrization the existence of a unique bounded solution to (PG) that converges to 0 as $t \to +\infty$.

Notations and hypotheses.
We assume that

$$(H_1) \begin{cases} P \in W^{1,\infty}(\Omega) \text{ and } \Delta P = 0, \\ u_0 \in L^\infty(\Omega) \text{ and } u_0 \geq 0, \\ \varphi \in C([0, +\infty[) \cap C^1(]0, +\infty[\,);\ \ \varphi(0) = 0 \text{ and } \forall x \in]0, +\infty[\,,\ \varphi'(x) > 0, \\ f \in C^1(\mathbb{R}),\ f \text{ is concave on } \mathbb{R}_+ \text{ and } f(0) = 0; \\ \exists \alpha \in [0, 1[,\ \exists (K_0, A_0) \in (\mathbb{R}_+^*)^2;\ \ \forall s \geq K_0,\ |f(s)| \leq A_0 \varphi(s)^\alpha; \\ g \text{ is Lipschitz on each bounded set of } \mathbb{R}. \end{cases}$$

For any measurable subset A of $\mathbb{R}^2$, we note $|A|$ the measure of A and Λ_2 is the measure of the unit ball of $\mathbb{R}^2$ ($\Lambda_2 = \pi$). Let $\widetilde{\Omega}$ be the open ball of $\mathbb{R}^2$ centered at the origin with boundary $\Gamma = \partial\,\widetilde{\Omega}$ and such that $\left|\widetilde{\Omega}\right| = |\Omega|$.

For any u, measurable function on $\mathbb{R}_+ \times \Omega$, the decreasing rearrangement of u is the function u^* defined on $\mathbb{R}_+ \times [0, |\Omega|]$ by:

$$\begin{cases} u^*(t,\, s) = \inf \{\tau,\, \mu(t, \tau) \leq s\}\,,\ \text{if } 0 < s < |\Omega|\,, \\ u^*(t, |\Omega|) = \inf\ ess\,\{\, u(t, x);\ x \in \Omega\}\,, \\ u^*(t, 0) = \sup\ ess\,\{\, u(t, x);\ x \in \Omega\}\,. \end{cases}$$

with $\mu(t,\tau) = |\{u(t,.) > \tau\}|$.

Now, we introduce U the solution of the symmetrized problem

$$(P_s)\ \begin{cases} \dfrac{\partial U}{\partial t} - \Delta\varphi(U) + f(U) = 0 \text{ on } \mathbb{R}_+ \times \widetilde{\Omega}, \\ U_{|\,\Gamma} = 0, \\ U(0,.) = U_0, \end{cases}$$

where U_0 is the symmetrical rearrangement (decreasing along the radius) of u_0, defined by

$$\forall x \in \widetilde{\Omega},\ U_0(x) = u^*(0,\, \Lambda_2\, |x|^2).$$

2 Existence of a bounded solution

Let be $(u_{0,n})$ a decreasing sequence in $C^\infty(\,\overline{\Omega}\,)$ such that

$$\begin{cases} \varphi(u_{0,n}) \to \varphi(u_0) \text{ in } H^1(\Omega), \\ u_{0,n} \geq \dfrac{1}{n} \text{ on } \Omega \text{ and } u_{0,n}\ |_\Gamma = \dfrac{1}{n}, \\ -\Delta\varphi(u_{0,n}) + div(\nabla Pg(u_{0,n})) + f(u_{0,n}) = 0 \text{ on } \partial\Omega. \end{cases}$$

We deduce from [4] that for each $n \in \mathbb{N}^*$, the symmetrized problem

$$(P_{n,s})\ \begin{cases} \dfrac{\partial U_n}{\partial t} - \Delta\varphi(U_n) + f(U_n) = 0 \text{ on } \mathbb{R}_+ \times \widetilde{\Omega}, \\ U_n\ |_\Gamma = \dfrac{1}{n}, \\ U_n(0,.) = U_{0,n}\ , \end{cases}$$

with $U_{0,n}(x) = u^*_{0,\,n}(0, \Lambda_2\, |x|^2)$, has a unique solution such that

$$\forall T > 0,\ \ U_n \in C^{1,2}(\overline{Q_T}) \text{ and } \frac{\partial U_n}{\partial t} \in L^2(0,T; H^1_0(\widetilde{\Omega}))$$

where $Q_T =]0,T[\, \times \widetilde{\Omega}$.

Moreover, for each $n \in \mathbb{N}^*$, U_n is a positive and bounded function on $\mathbb{R}_+ \times \widetilde{\Omega}$ such that for each $t \in \mathbb{R}_+$, $U_n(t,.)$ is radially symmetric and decreases along the radius, this property being fulfilled by $U_{0,n}$. Then, denoting by R the radius of $\widetilde{\Omega}$, we know that the function v_n defined on $\mathbb{R}_+ \times [0,R]$ by $v_n(t,r) = U_n(t, r\cos\theta, r\sin\theta),\ \theta \in [0,2\pi]$ decreases on $[0,R]$. Therefore U_n coincides with its symmetrical rearrangement in the following sense

$$U_n(t, r\cos\theta, r\sin\theta) = U^*_n(t, \Lambda_2 r^2).$$

Now as the sequence of functions (U_n) decreases, (U_n) converges everywhere on $[0,+\infty[\times\widetilde{\Omega}$ to the solution U of the problem

$$(P_s)\begin{cases} \forall T>0,\ U\in C([0,T];\ L^2(\widetilde{\Omega})\,);\ \ \dfrac{\partial U}{\partial t}\in L^2(0,T;H^{-1}(\widetilde{\Omega})); \\ \\ \ \varphi(U)\in L^2(0,T;H_0^1(\widetilde{\Omega})); \\ \\ \text{p.p. } t\in]0,T[,\ \ \dfrac{\partial U}{\partial t}-\Delta\varphi(U)+f(U)=0\ \ \text{dans } D'(\widetilde{\Omega}); \\ \\ U(0,.)=U_0 \qquad \text{dans } \widetilde{\Omega}. \end{cases}$$

Moreover, the sequence $(\,U_n)$ is uniformly bounded on $\mathbb{R}_+\times\widetilde{\Omega}$ by $M=\sup ess\, U_1$. Therefore

$$\forall n\in\mathbb{N}^*,\ \sup ess\, U_n\leq M.$$

Then, we introduce the Lipschitz function $\widehat{g}$ defined by $\widehat{g}(x)=g(x)$ on $]-\infty,M]$ and $\widehat{g}(x)=M$ on $]M,+\infty[$.

For each $n\in\mathbb{N}^*$, we consider the approximating problem

$$(PG_n)\begin{cases} \dfrac{\partial u_n}{\partial t}-\Delta\varphi(u_n)+div(\nabla P\widehat{g}(u_n))+f\,(u_n)=0, \\ u_{n\,|\,\partial\Omega}=\dfrac{1}{n}, \\ u_n(0,.)=u_{0,n}. \end{cases}$$

This problem has a unique solution such that

$$\forall T>0,\ u_n\in C^{1,2}([0,T]\times\overline{\Omega}) \text{ and } \frac{\partial u_n}{\partial t}\in L^2(0,T;H_0^1(\Omega)).$$

For each $n\in\mathbb{N}^*$, we put

$$d(s)=\Lambda_2\, s,\ k_n(t,s)=\int_0^s u_n^*(t,\sigma)\,d\sigma \text{ and } K_n(t,\,s)=\int_0^s U_n^*(t,\,\sigma)d\sigma.$$

Let be L the operator defined by

$$L(Y)(t,s)=\frac{\partial Y}{\partial t}(t,s)-\Lambda_2 s\frac{\partial}{\partial s}\left(\varphi\left(\frac{\partial Y}{\partial s}\right)\right)(t,s)+\int_0^s f\left(\frac{\partial Y}{\partial s}\right)(t,\,\sigma)d\sigma.$$

Following [11], [2], we first prove the

Lemma 1 $\quad \forall\, n \in \mathbb{N}^*,\ L(k_n) \leq 0\ \ et\ L(K_n) = 0.$

Now, as f is concave, we can compare the functions k_n and K_n. Indeed:

Lemma 2 $\quad \forall\, n \in \mathbb{N}^*,\ \forall\,(t,s) \in \mathbb{R}_+ \times [0, |\Omega|],\ k_n(t,s) \leq K_n(t,s).$

We deduce from the last lemma the

Proposition 3

The problem (PG) has a unique bounded solution u on $\mathbb{R}_+ \times \Omega$ such that

$$\forall t \in \mathbb{R}_+,\ \forall p \in [1, +\infty],\ \|\, u(t, .)\|_{L^p(\Omega)} \leq \|U(t, .)\|_{L^p(\widetilde{\Omega})},$$

where U is the unique solution of the symmetrized problem (P_s).

Proof.

Since for each $t \in \mathbb{R}_+$ the functions $u_n^*(t,\, .)$ and $U_n^*(t,\, .)$ are nonnegative on $[0, |\Omega|]$ and $u_n^*(t,\, .)$ decreases, Lemma 2 implies

$\forall\, n \in \mathbb{N}^*,\ \forall\, p \geq 1,\ \forall\,(t,s) \in \mathbb{R}_+ \times [0, |\Omega|],$

$$\int_0^s |u_n^*(t,\, \sigma)|^p\, d\sigma \leq \int_0^s |U_n^*(t,\, \sigma)|^p\, d\sigma.$$

When $p \to +\infty$, we deduce from this inequality that the sequence (u_n) is uniformly bounded by M because M is the upper bound of (U_n) on $\mathbb{R}_+ \times \widetilde{\Omega}$.
Therefore, for each $n \in \mathbb{N}^*$, $\widehat{g}(u_n) = g(u_n)$ and we can prove that (u_n) converges everywhere on $\mathbb{R}_+ \times \Omega$ to the solution u of the problem (PG).

3 Convergence to 0

In this section, we give some conditions that ensure the convergence of the solution $u(t, .)$ of the problem (PG) to the null function as $t \to +\infty$.

Proposition 4

Under the hypotheses (H_1) and the conditions

$$(H_2) \begin{cases} \varphi^{-1} \text{ is Hölder continuous with order } \gamma \in \,]0, 1[\,, \\ \exists A_1 > 0,\ \exists\, m_f > 0,\ \forall x \in [0, A_1],\ -f(x) \leq m_f\, \varphi(x), \end{cases}$$

there exists a real μ such that if the open set Ω fulfills the assumption $|\Omega| \leq \mu$ then

$$\forall p \in [1, +\infty[, \lim_{t \to +\infty} \|u(t, .)\|_{L^p(\Omega)} = 0.$$

Proof.
First, we prove the convergence property in $L^1(\Omega)$.

Let n be in $\mathbb{N}^*$. We compare K_n to the function Z defined on $]0, +\infty[\times[0, |\Omega|]$ by

$$Z(t, s) = \int_0^s \varphi^{-1}\left(C\sigma^{-(1/2)}(1+t)^{-\lambda}\right) d\sigma$$

where we choose C such that $\varphi^{-1}\left(C\sigma^{\beta-1}\right) \geq M$ in order to have

$$Z(0, .) \geq K_n(0, .).$$

Let T be fixed in $]0, +\infty[$. We prove that there exists an integer n_T such that

$$n \geq n_T \Longrightarrow \forall t \in [0, T], \ \frac{\partial Z}{\partial s}(t, |\Omega|) \geq \frac{\partial K_n}{\partial s}(t, |\Omega|).$$

Lastly, thanks to the condition (H_2), we can choose a real $\lambda > 0$ such that $L(Z) \geq 0$, as soon as the measure of Ω, $|\Omega|$, is small enough..
Therefore

$$\forall T > 0, \exists n_T \in \mathbb{N}^*; \ n \geq n_T \Longrightarrow K_n \leq Z \text{ on } [0, T] \times [0, |\Omega|].$$

Now as (U_n^*) converges everywhere on $[0, +\infty[\times]0, |\Omega|[$ to U^*,

$$\forall T > 0, \ K \leq Z \text{ on } [0, T] \times [0, |\Omega|],$$

where K is defined on $\mathbb{R}_+ \times [0, |\Omega|]$ by $K(t, s) = \displaystyle\int_0^s U^*(t, \sigma) d\sigma$.
Finally $K \leq Z$ on $[0, +\infty[\times[0, |\Omega|]$. Then

$$\exists c \geq 0, K \leq c(1+t)^{-\lambda\gamma} \text{ on }]0, +\infty[\times[0, |\Omega|]. \tag{2}$$

Hence, the convergence property in $L^1(\Omega)$,

$$\lim_{t \to +\infty} \int_\Omega u(t, x) dx = 0.$$

Now let be $p > 1$. Thanks to the upper bound (2) and an Hölder's inequality, the convergence property is proved in $L^p(\Omega)$.

Proposition 5

Under the hypotheses (H_1), if f is a nonnegative function on $\mathbb{R}_+$, then

$$\forall p \in [1,+\infty[, \lim_{t \to +\infty} \|u(t,.)\|_{L^p(\Omega)} = 0.$$

Proof.
We prove that 0 is the only element of the ω-limit set of the solution U of the symmetrized problem (P_s). Indeed, let be U the solution of the problem (P_s) and $\omega(U_0)$ its ω-limit set. Let $W \in \omega(U_0)$.

We know that W is radially symmetric. Then, we introduce the function w defined on $[0, R]$ by

$$w(r) = W(r\cos\theta, r\sin\theta) \text{ where } \theta \in [0, 2\pi].$$

We put $F = f \circ \varphi^{-1}$. Then, w is a solution of the stationary equation

$$\frac{d}{dr}\left(r\frac{dw}{dr}\right) = rF(w).$$

Consequently

$$\frac{R}{2}(w')^2(R) + \frac{1}{2}\int_0^R (w')^2(\tau)d\tau = -\int_0^R \widehat{F}(w)dr,$$

where $\widehat{F}$ is defined on $\mathbb{R}$ by $\widehat{F}(x) = \displaystyle\int_0^x F(\tau)d\tau$.

We deduce from this last inequality that if f fulfills the assumption $f \geq 0$ on $\mathbb{R}_+$ Then $w' = 0$ on $[0, R]$ and, therefore $w = 0$ on $[0, R]$.

Hence, thanks to Proposition 3 of Section 2,

$$\forall p \in [1,+\infty[, \lim_{t \to +\infty} \|u(t,.)\|_{L^p(\Omega)} = 0.$$

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