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Analysis of a Non-overlapping Domain Decomposition Method for Stokes Equations

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Abstract

In this note we extend the analysis for elliptic problems performed in [1] to saddle point problems like the Stokes equations. We use a non overlapping domain decomposition and the introduction of a penalty term. In a simply connected bounded domain $\Omega \subset \boldsymbol{R^2}$ with Lipschitz boundary, we decompose Ω into two non-overlapping Lipschitz subdomains Ω_1 and Ω_2 with $\Omega_1 \cap \Omega_2 = \emptyset$, and suppose that $\partial\Omega_i = \Gamma_i \cup \Gamma$ where Γ_i is the common boundary with Ω, $\Gamma_i = \partial\Omega \cap \partial\Omega_i$ and Γ is the interface with Ω_j, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The Stokes equations on Ω are solved via the following parallel process: For $n = 0, 1, 2, ...,$ given u_i^n, p_i^n we compute u_i^{n+1} and p_i^{n+1} $(i = 1, 2)$ such that

$$\left\{\begin{array}{lcll} -\Delta \mathbf{u}_i^{n+1} + \nabla p_i^{n+1} & = & \mathbf{f} & \text{in} \quad \Omega_i \\ \nabla \cdot \mathbf{u}_i^{n+1} & = & 0 & \text{in} \quad \Omega_i \\ \mathbf{u}_i^{n+1} & = & 0 & \text{on} \quad \Gamma_i \\ \dfrac{\partial \mathbf{u}_i^{n+1}}{\partial \mathbf{n}_{ij}} - p_i^{n+1}\mathbf{n}_{ij} & = & -\dfrac{1}{\epsilon}(\mathbf{u}_i^{n+1} - \mathbf{u}_j^n) \text{ on } \Gamma & \end{array}\right.$$

where $\mathbf{n}_{ij}$ is the outward normal vector on Γ pointing from Ω_i into Ω_j, $\epsilon > 0$ is a parameter that tends to cero and inforce the transmision conditions on the interface Γ and we stress that the pressures p_i do not longer have cero mean average. We present the convergence analysis of this technique and some numerical tests. An ampliation of this work will appear in [2].

Keywords: Stokes Problem, Parallel technique, Non Overlapping

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1 Introduction

In a simply connected bounded domain $\Omega \subset \boldsymbol{R^d}$ $(d = 2, 3)$ with a Lipschitz boundary $\partial\Omega$ and with $\mathbf{f} \in \left[L^2(\Omega)\right]^d$, we search for a velocity field $\mathbf{u} \in \left[H_0^1(\Omega)\right]^d$ and a pressure

$p \in L^2_0(\Omega)$ such that

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f}, \ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

In the classical mixed formulation of this problem we look for $(\mathbf{u}, p) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)$ with

$$(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega - (p, \nabla \cdot \mathbf{v})_\Omega - (\nabla \cdot \mathbf{u}, q)_\Omega = (\mathbf{f}, \mathbf{v})_\Omega, \tag{1}$$

for all $(\mathbf{v}, q) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)$. Now we decompose Ω into two non-overlapping Lipschitz subdomains Ω_1 and Ω_2 with $\Omega_1 \cap \Omega_2 = \emptyset$ (this choice is made to ease the exposition of the main ideas, but these can be extended to more than two subdomains). Suppose that $\partial\Omega_i = \Gamma_i \cup \Gamma$ where Γ_i is the common boundary with Ω, $\Gamma_i = \partial\Omega \cap \partial\Omega_i$ and Γ is the interface with Ω_j, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$, all of these boundaries are Lipschitz $(d-1)$-dimensional manifolds. Next, we consider the Sobolev spaces

$$\mathbf{X}_i = \left[H^1_0(\Omega_i; \Gamma_i)\right]^d = \{\mathbf{v} \in [H^1(\Omega_i)]^d \text{ s.t. } \mathbf{v}_{|\Gamma_i} = 0\}$$

normed by $|\mathbf{v}|^2_{1,\Omega_i} = (\nabla \mathbf{v}, \nabla \mathbf{v})_{\Omega_i}$ and the Hilbert spaces $M_i = L^2(\Omega_i)$ normed as usual. Now for $\epsilon > 0$ we consider the problem (P_ϵ):

Find $(\mathbf{u}_i, p_i) \in \mathbf{X}_i \times M_i$ with

$$(P_\epsilon) \quad \begin{cases} (\nabla \mathbf{u}_1, \nabla \mathbf{v}_1)_{\Omega_1} - (p_1, \nabla \cdot \mathbf{v}_1)_{\Omega_1} - (q_1, \nabla \cdot \mathbf{u}_1)_{\Omega_1} + \dfrac{1}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}_1)_{0,\Gamma} = (\mathbf{f}, \mathbf{v}_1)_{\Omega_1}, \\ (\nabla \mathbf{u}_2, \nabla \mathbf{v}_2)_{\Omega_2} - (p_2, \nabla \cdot \mathbf{v}_2)_{\Omega_2} - (q_2, \nabla \cdot \mathbf{u}_2)_{\Omega_2} + \dfrac{1}{\epsilon}(\mathbf{u}_2 - \mathbf{u}_1, \mathbf{v}_2)_{0,\Gamma} = (\mathbf{f}, \mathbf{v}_2)_{\Omega_2}, \end{cases}$$

for all $(\mathbf{v}_i, q_i) \in \mathbf{X}_i \times M_i$, $i = 1, 2$. This problem is the variational formulation of the following coupled partial differential equations

$$\begin{cases} -\Delta \mathbf{u}_1 + \nabla p_1 &= \mathbf{f} & \text{in } \Omega_1 \\ \nabla \cdot \mathbf{u}_1 &= 0 & \text{in } \Omega_1 \\ \mathbf{u}_1 &= 0 & \text{on } \Gamma_1 \\ \dfrac{\partial \mathbf{u}_1}{\partial \mathbf{n}_{12}} - p_1 \mathbf{n}_{12} &= -\dfrac{1}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2) & \text{on } \Gamma \end{cases} \qquad \begin{cases} -\Delta \mathbf{u}_2 + \nabla p_2 &= \mathbf{f} & \text{in } \Omega_2 \\ \nabla \cdot \mathbf{u}_2 &= 0 & \text{in } \Omega_2 \\ \mathbf{u}_2 &= 0 & \text{on } \Gamma_2 \\ \dfrac{\partial \mathbf{u}_2}{\partial \mathbf{n}_{21}} - p_2 \mathbf{n}_{21} &= -\dfrac{1}{\epsilon}(\mathbf{u}_2 - \mathbf{u}_1) & \text{on } \Gamma \end{cases}$$

where $\mathbf{n}_{ij}$ is the outward normal vector on Γ pointing from Ω_i into Ω_j and we stress that the pressures p_i do not longer have cero mean average. The apprpriated transmission conditions are enforced when $\epsilon \longrightarrow 0$ because we show that $\|\mathbf{u}_1 - \mathbf{u} - 2\|_{0,\Gamma} = \mathcal{O}(\epsilon)$.
The iteration process that we proposse is the following: For $n = 0, 1, 2, \ldots$, given $\mathbf{u}^n_1$, $\mathbf{u}^n_2$

we compute $\mathbf{u}_1^{n+1}$, $\mathbf{u}_2^{n+1}$ and p_1^{n+1}, p_2^{n+1} such that the following problems are satisfied

$$\begin{cases} -\Delta\mathbf{u}_1^{n+1} + \nabla p_1^{n+1} & = & \mathbf{f} & \text{in} \quad \Omega_1, \\ \nabla\cdot\mathbf{u}_1^{n+1} & = & 0 & \text{in} \quad \Omega_1, \\ \mathbf{u}_1^{n+1} & = & 0 & \text{on} \quad \Gamma_1, \\ \dfrac{\partial\mathbf{u}_1^{n+1}}{\partial\mathbf{n}_{12}} - p_1^{n+1}\mathbf{n}_{12} & = & -\dfrac{1}{\epsilon}(\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n}) \text{ on } \Gamma, \end{cases}$$
$$\begin{cases} -\Delta\mathbf{u}_2^{n+1} + \nabla p_2^{n+1} & = & \mathbf{f} & \text{in} \quad \Omega_2, \\ \nabla\cdot\mathbf{u}_2^{n+1} & = & 0 & \text{in} \quad \Omega_2, \\ \mathbf{u}_2^{n+1} & = & 0 & \text{on} \quad \Gamma_2, \\ \dfrac{\partial\mathbf{u}_2^{n+1}}{\partial\mathbf{n}_{21}} - p_2^{n+1}\mathbf{n}_{21} & = & -\dfrac{1}{\epsilon}(\mathbf{u}_2^{n+1} - \mathbf{u}_1^{n}) \text{ on } \Gamma. \end{cases}$$

We remark that our method may be viewed as a variation of the Robin

2 Analysis of problem (P_ϵ)

Let us introduce the product spaces $\mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2$, $\mathbf{M} = M_1 \times M_2$ and denote by capital letters the elements (pairs) of $\mathbf{X}$ and $\mathbf{M}$. Then we norm $\mathbf{M}$ with $\|\mathbf{P}\|_{\mathbf{M}}^2 = \sum_{i=1}^2 \|p_i\|_{0,\Omega_i}^2$ and $\mathbf{X}$ via $((\mathbf{U},\mathbf{V}))_\epsilon = \sum_{i=1}^2 (\nabla\mathbf{u}_i, \nabla\mathbf{v}_i)_{\Omega_i} + \frac{1}{\epsilon}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}_1 - \mathbf{v}_2)_{0,\Gamma}$ i.e., the norm in $\mathbf{X}$ is given by $\|\mathbf{U}\|_\epsilon = ((\mathbf{U},\mathbf{U}))_\epsilon$. Next we define the forms $b(\mathbf{P},\mathbf{V}) = -\sum_{i=1}^2 (p_i, \nabla\cdot\mathbf{v}_i)_{\Omega_i}$, $F(\mathbf{V}) = \sum_{i=1}^2 (\mathbf{f}, \mathbf{v}_i)_{\Omega_i}$ and write problem (P_ϵ) in terms of the variational problem:

$$\begin{cases} \text{Find } (\mathbf{U},\mathbf{P}) \in \mathbf{X}\times\mathbf{M} \text{ such that} \\ ((\mathbf{U},\mathbf{V}))_\epsilon + b(\mathbf{P},\mathbf{V}) + b(\mathbf{Q},\mathbf{U}) = F(\mathbf{V}), \quad \forall\, (\mathbf{V},\mathbf{Q}) \in \mathbf{X}\times\mathbf{M}. \end{cases}$$

Now we consider the following symmetric and continuous, according to $\|\cdot\|_\epsilon$ and $\|\cdot\|_{\mathbf{M}}$, bilinear form on $\mathbf{X}\times\mathbf{M}$ given by

$$B_\epsilon(\mathbf{U},\mathbf{P};\mathbf{V},\mathbf{Q}) = ((\mathbf{U},\mathbf{V}))_\epsilon + b(\mathbf{P},\mathbf{V}) + b(\mathbf{Q},\mathbf{U})$$

for all pairs $(\mathbf{U},\mathbf{P}), (\mathbf{V},\mathbf{Q}) \in \mathbf{X}\times\mathbf{M}$. We have

Lemma 1 *There exists a positive constant γ independent of $\epsilon > 0$ such that for all $(\mathbf{U},\mathbf{P}) \in \mathbf{X}\times\mathbf{M}$*

$$S = \sup_{(\mathbf{V},\mathbf{Q})\in\,\mathbf{X}\mathrm{x}\mathbf{M}} \frac{|B_\epsilon(\mathbf{U},\mathbf{P};\mathbf{V},\mathbf{Q})|}{\|\mathbf{V}\|_\epsilon + \|\mathbf{Q}\|_{\mathbf{M}}} \;\geq\; \epsilon\,\gamma\,(\|\mathbf{U}\|_\epsilon + \|\mathbf{P}\|_{\mathbf{M}}).$$

As a consequence, given $\mathbf{f} \in [L^2(\Omega)]^d$ and for each $\epsilon > 0$ problem (P_ϵ) has a unique solution $(\mathbf{U}^\epsilon, \mathbf{P}^\epsilon) \in \mathbf{X}\times\mathbf{M}$.

We introduce next the consistency error of problem (P_ϵ) as an approximation of the Stokes equations in variational form. This error is the result of plugging the solution of (1) into (P_ϵ).

Lemma 2 *Let* $(\mathbf{u}, p)$ *be the solution of the Stokes problem and* $\mathbf{U} = (\mathbf{u}_{|\Omega_1}, \mathbf{u}_{|\Omega_2})$, $\mathbf{P} = (p_{|\Omega_1}, p_{|\Omega_2})$. *Then, we consider the consistency error of problem* (P_ϵ) *via*

$$\begin{aligned} G(\mathbf{V}) &= ((\mathbf{U},\mathbf{V}))_\epsilon + b(\mathbf{P},\mathbf{V}) - F(\mathbf{V}) \\ &= \sum_{i=1}^{2}(\nabla\mathbf{u},\nabla\mathbf{v}_i)_{\Omega_i} - \sum_{i=1}^{2}(p,\nabla\cdot\mathbf{v}_i)_{\Omega_i} - \sum_{i=1}^{2}(\mathbf{f},\mathbf{v}_i)_{\Omega_i} \end{aligned}$$

for all $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{X}$. *Then, assuming* $\mathbf{u} \in [H^2(\Omega) \cap H_0^1(\Omega)]^d$ *and* $p \in H^1(\Omega)$, *let* $\mathbf{n}_{1,2} = \mathbf{n}$, *we have*

$$G(\mathbf{V}) = \int_\Gamma (\partial_\mathbf{n}\mathbf{u} - p\,\mathbf{n}) \cdot (\mathbf{v}_1 - \mathbf{v}_2)\, d\sigma$$

and therefore

$$|G(\mathbf{V})| \leq \|\partial_\mathbf{n}\mathbf{u} - p\,\mathbf{n}\|_{0,\Gamma}\|\mathbf{v}_1 - \mathbf{v}_2\|_{0,\Gamma}.$$

Now we can estimate the error in approximating the variational formulation of the Stokes Equations with problem (P_ϵ)

Lemma 3 *Suppose that* $\mathbf{u} \in [H^2(\Omega) \cap H_0^1(\Omega)]^d$ *and* $p \in H^1(\Omega)$ *is the solution to the Stokes problem. For each* $\epsilon > 0$ *let* $(\mathbf{U}^\epsilon, \mathbf{P}^\epsilon) \in \mathbf{X} \times \mathbf{M}$ *be the unique solution of problem* (P_ϵ), *with* $\mathbf{U}^\epsilon = (\mathbf{u}_1^\epsilon, \mathbf{u}_2^\epsilon)$ *and* $\mathbf{P}^\epsilon = (p_1^\epsilon, p_2^\epsilon)$. *Let* $c(\mathbf{u}, p) = \|\partial_\mathbf{n}\mathbf{u} - p\,\mathbf{n}\|_{0,\Gamma}$, $\mathbf{U} = (\mathbf{u}_{|\Omega_1}, \mathbf{u}_{|\Omega_2})$ *and construct*

$$\pi^\epsilon = p_1^\epsilon \chi_{\Omega_1} + p_2^\epsilon \chi_{\Omega_2} - \frac{1}{|\Omega|}\left(\int_{\Omega_1} p_1^\epsilon + \int_{\Omega_2} p_2^\epsilon\right).$$

Then

$$\|\mathbf{U} - \mathbf{U}^\epsilon\|_\epsilon \leq c(\mathbf{u}, p)\sqrt{\epsilon} \quad \text{and} \quad \|p - \pi^\epsilon\|_{0,\Omega} \leq c(\mathbf{u}, p)\sqrt{\epsilon}.$$

As a consequence we have

$$\sum_{i=1}^{2} |\mathbf{u} - \mathbf{u}_i^\epsilon|_{1,\Omega_i} \leq c(\mathbf{u}, p)\sqrt{\epsilon} \quad \text{and} \quad \|\mathbf{u}_1^\epsilon - \mathbf{u}_2^\epsilon\|_{0,\Gamma} \leq c(\mathbf{u}, p)\, \epsilon.$$

3 Discrete problem and error estimates

We suppose that the domain Ω is polygonal and take for $h > 0$ an admissible and regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ formed by polygons ($d = 2$) or polyhedra ($d = 3$) elements such that Γ is formed by faces or sides of elements K in $\mathcal{T}_h$. Then we use $\mathcal{T}_h^i = \mathcal{T}_h \cap \overline{\Omega_i}$, for $i = 1, 2$. These triangulations of $\overline{\Omega_i}$ are compatible on Γ, i.e., they share the same edges on Γ. For the triangulation $\mathcal{T}_h$ we consider finite element subspaces (V_h, P_h) of $([H_0^1(\Omega)]^d, L_0^2(\Omega))$ satisfying the discrete inf-sup condition of Ladyzhenskya-Brezzi-Babuška on Ω. Now we consider the discrete solution $(\mathbf{u}_h, p_h) \in V_h \times P_h$ of the discrete version of the Stokes problem posed on $V_h \times P_h$ and assume that, when the solution $(\mathbf{u}, p)$ to the continuous Stokes problem in Ω satisfies $\mathbf{u} \in \left[H^{k+1}(\Omega) \cap H_0^1(\Omega)\right]^d$ and $p \in H^k(\Omega) \quad (k \geq 1)$, then

$$|\mathbf{u}_h - \mathbf{u}|_{1,\Omega} + \|p_h - p\|_{0,\Omega} \leq C_0\, h^k \tag{2}$$

for some constant $C_0 = C_0(\mathbf{u}, p)$. Now, based on $\mathcal{T}_h^i$, use finite element subspaces of $(\mathbf{X}_i, M_i)$, denoted by $(\mathbf{X}_{i,h}, M_{i,h})$, such that each pair $(\mathbf{Y}_{i,h}, N_{i,h})$, where $\mathbf{Y}_{i,h} = \mathbf{X}_{i,h} \cap [H_0^1(\Omega_i)]^d$ and $N_{i,h} = M_{i,h} \cap L_0^2(\Omega_i)$ also satisfies the discrete inf-sup condition on Ω_i. For instance we could use the restriction of the spaces V_h and P_h to each of the Ω_i. Set now $\mathbf{X}_h = \mathbf{X}_{1,h} \times \mathbf{X}_{2,h}$ and $\mathbf{M}_h = M_{1,h} \times M_{2,h}$ and pose the discrete version of (P_ϵ), that we denote by $(P_{\epsilon,h})$:

$$\begin{cases} \text{Find } (\mathbf{U}_h^\epsilon, \mathbf{P}_h^\epsilon) \in \mathbf{X}_h \times \mathbf{M}_h \text{ such that} \\ ((\mathbf{U}_h^\epsilon, \mathbf{V}_h))_\epsilon + b(\mathbf{P}_h^\epsilon, \mathbf{V}_h) + b(\mathbf{Q}_h, \mathbf{U}_h^\epsilon) = F(\mathbf{V}_h), \quad \forall (\mathbf{V}_h, \mathbf{Q}_h) \in \mathbf{X}_h \times \mathbf{M}_h. \end{cases}$$

The existence and uniqueness of solution for $(P_{\epsilon,h})$ is carried out as for (P_ϵ) and we have the estimates

Theorem 4 *Let $\mathbf{u} \in \left[H^{k+1}(\Omega) \cap H_0^1(\Omega)\right]^d$ and $p \in H^k(\Omega)$ $(k \geq 1)$ be the solution to the Stokes problem in Ω and for each $h > 0$ let $(\mathbf{u}_h, p_h) \in V_h \times P_h$ solve the discrete Stokes problem on $V_h \times P_h$. Now consider $\mathbf{U}_h = (\mathbf{u}_{h|\Omega_1}, \mathbf{u}_{h|\Omega_2}) \in \mathbf{X}_h$, $\mathbf{P}_h = (p_{h|\Omega_1}, p_{h|\Omega_2}) \in \mathbf{M}_h$. For each $\epsilon > 0$ let $(\mathbf{U}_h^\epsilon, \mathbf{P}_h^\epsilon) \in \mathbf{X}_h \times \mathbf{M}_h$ solve $(P_{\epsilon,h})$ and write $\mathbf{U}_h^\epsilon = (\mathbf{u}_{1,h}^\epsilon, \mathbf{u}_{2,h}^\epsilon)$ and $\mathbf{P}_h^\epsilon = (p_{1,h}^\epsilon, p_{2,h}^\epsilon)$. Now construct*

$$\pi_h^\epsilon \;=\; p_{1,h}^\epsilon \chi_{\Omega_1} + p_{2,h}^\epsilon \chi_{\Omega_2} - \frac{1}{|\Omega|}\Big(\int_{\Omega_1} p_{1,h}^\epsilon + \int_{\Omega_2} p_{2,h}^\epsilon\Big) \tag{3}$$

then, the following error estimate hold

$$\|\mathbf{U}_h - \mathbf{U}_h^\epsilon\|_\epsilon \;\leq\; C\,(h^k + \sqrt{\epsilon}) \tag{4}$$

$$\|p_h - \pi_h^\epsilon\|_{0,\Omega} \;\leq\; C\,(h^k + \sqrt{\epsilon}) \tag{5}$$

where $C = C(\mathbf{u}, p)$ is a positive constant just depending on $(\mathbf{u}, p)$. As a consequence of (4) we have

$$\sum_{i=1}^{2} |\mathbf{u}_h - \mathbf{u}_{i,h}^\epsilon|_{1,\Omega_i} \leq C\,(h^k + \sqrt{\epsilon}) \quad \textit{and} \quad \|\mathbf{u}_{1,h}^\epsilon - \mathbf{u}_{2,h}^\epsilon\|_{0,\Gamma} \leq C\,(\sqrt{\epsilon}\, h^k + \epsilon).$$

Via the triangular inequality, we give the main result of this section

Theorem 5 *Let $\mathbf{u} \in \left[H^{k+1}(\Omega) \cap H_0^1(\Omega)\right]^d$ and $p \in H^k(\Omega)$, $(k \geq 1)$ be the solution to the Stokes problem in Ω. For each $h > 0$ and $\epsilon > 0$ let $(\mathbf{U}_h^\epsilon, \mathbf{P}_h^\epsilon) \in \mathbf{X}_h \times \mathbf{M}_h$ solve $(P_{\epsilon,h})$ with finite dimensional spaces of accuracy $k \geq 1$ and write $\mathbf{P}_h^\epsilon = (p_{1,h}^\epsilon, p_{2,h}^\epsilon)$. Then construct π_h^ϵ as in (3). The following bounds hold*

$$\sum_{i=1}^{2} |\mathbf{u} - \mathbf{u}_{i,h}^\epsilon|_{1,\Omega_i} + \frac{1}{\sqrt{\epsilon}}\|\mathbf{u}_{1,h}^\epsilon - \mathbf{u}_{2,h}^\epsilon\|_{0,\Gamma} \;\leq\; C\,(h^k + \sqrt{\epsilon}) \tag{6}$$

$$\|p - \pi_h^\epsilon\|_{0,\Omega} \;\leq\; C\,(h^k + \sqrt{\epsilon}) \tag{7}$$

where $C = C(\mathbf{u}, p, \mathbf{f})$ is a positive constant just depending on the data. When $\epsilon = O(h^{2k})$ we have

$$\sum_{i=1}^{2} |\mathbf{u} - \mathbf{u}^{\epsilon}_{i,h}|_{1,\Omega_i} + \|p - \pi^{\epsilon}_h\|_{0,\Omega} \leq C\ h^k \quad \textit{and} \quad \|\mathbf{u}^{\epsilon}_{1,h} - \mathbf{u}^{\epsilon}_{2,h}\|_{0,\Gamma} \leq C\ h^{2\,k}.$$

4 Iteration process

We search for the solution of $(P_{\epsilon,h})$ via the following parallelizable technique: For $n = 0, 1, 2, ...,$ given $\mathbf{u}^n_1 = \mathbf{u}^{\epsilon,n}_{1,h}$ and $\mathbf{u}^n_2 = \mathbf{u}^{\epsilon,n}_{2,h}$ we compute $\mathbf{u}^{n+1}_1 \in \mathbf{X}_{1,h},\ \mathbf{u}^{n+1}_2 \in \mathbf{X}_{2,h}$ and $p^{n+1}_1 \in M_{1,h},\ p^{n+1}_2 \in M_{2,h}$ such that the following problem $(P^n_{\epsilon,h})$ is satisfied

$$\begin{cases} (\nabla \mathbf{u}^{n+1}_1, \nabla \mathbf{v}_1)_{\Omega_1} - (p^{n+1}_1, \nabla \cdot \mathbf{v}_1)_{\Omega_1} - (q_1, \nabla \cdot \mathbf{u}^{n+1}_1)_{\Omega_1} + \dfrac{1}{\epsilon}(\mathbf{u}^{n+1}_1 - \mathbf{u}^n_2, \mathbf{v}_1)_{0,\Gamma} = (\mathbf{f}, \mathbf{v}_1)_{\Omega_1}, \\ \\ (\nabla \mathbf{u}^{n+1}_2, \nabla \mathbf{v}_2)_{\Omega_2} - (p^{n+1}_2, \nabla \cdot \mathbf{v}_2)_{\Omega_2} - (q_2, \nabla \cdot \mathbf{u}^{n+1}_2)_{\Omega_2} + \dfrac{1}{\epsilon}(\mathbf{u}^{n+1}_2 - \mathbf{u}^n_1, \mathbf{v}_2)_{0,\Gamma} = (\mathbf{f}, \mathbf{v}_2)_{\Omega_2} \end{cases}$$

for all $(\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{X}_h$ (we drop the indices ϵ and h when not needed). We obtain the following geometric rate of convergence

Theorem 6 *Let $\mathbf{U}^{\epsilon}_h = (\mathbf{u}^{\epsilon}_{1,h}, \mathbf{u}^{\epsilon}_{2,h}) \in \mathbf{X}_h$ and $\mathbf{P}^{\epsilon}_h = (p^{\epsilon}_{1,h}, p^{\epsilon}_{2,h}) \in \mathbf{M}_h$ be the solution of $(P_{\epsilon,h})$ and $\mathbf{U}^{\epsilon,n}_h = (\mathbf{u}^{\epsilon,n}_{1,h}, \mathbf{u}^{\epsilon,n}_{2,h}) \in \mathbf{X}_h$, $\mathbf{P}^{\epsilon,n}_h = (p^{\epsilon,n}_{1,h}, p^{\epsilon,n}_{2,h}) \in \mathbf{M}_h$ be the solution of $(P^n_{\epsilon,h})$. Let us define π^{ϵ}_h and $\pi^{\epsilon,n}_h$ as in (3). Then, starting off the iterative process, for instance, with $\mathbf{u}^{0,\epsilon}_{i,h} = 0$, there exists a positive constant C_0 such that for each $\epsilon, h > 0$ and all $n \geq 0$*

$$\begin{aligned} \sum_{i=1}^{2} |\mathbf{u}^{\epsilon,n+1}_{i,h} - \mathbf{u}^{\epsilon}_{i,h}|_{1,\Omega_i} &\leq \frac{\mathcal{P}\,\|\mathbf{f}\|_{0,\Omega}}{\sqrt{\epsilon}\,(1 + 2\,C_0\,\epsilon)^{n/2}}, \\ \|\pi^{\epsilon,n+1}_h - \pi^{\epsilon}_h\|_{0,\Omega} &\leq \frac{\mathcal{P}\,\|\mathbf{f}\|_{0,\Omega}}{\epsilon\,(1 + 2\,C_0\,\epsilon)^{n/2}} \end{aligned}$$

for some constant $\mathcal{P}$ proportional to the constant in Poincare's Inequality.

Via the triangular inequality we obtain the final bound

Theorem 7 *Let $\mathbf{u} \in \left[H^{k+1}(\Omega) \cap H^1_0(\Omega)\right]^d$ and $p \in H^k(\Omega)$, for $k \geq 1$, be the solution to the Stokes problem in Ω. For each $h > 0$ and $\epsilon > 0$ let $(\mathbf{U}^{\epsilon,n}_h, \mathbf{P}^{\epsilon,n}_h) \in \mathbf{X}_h \times \mathbf{M}_h$ $(n \geq 1)$ solve the iteration problem $(P^n_{\epsilon,h})$ starting off the iteration with $\mathbf{U}^{\epsilon,0}_h = 0$, and using finite element spaces of accuracy $k \geq 1$. Then the following bounds hold for all $n \geq 0$*

$$\sum_{i=1}^{2} |\mathbf{u} - \mathbf{u}^{n+1,\epsilon}_{i,h}|_{1,\Omega_i} \leq C\ (h^k + \sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}\,(1 + 2\,C_0\,\epsilon)^{n/2}}) \tag{8}$$

$$\|p - \pi^{n+1,\epsilon}_h\|_{0,\Omega} \leq C\ (h^k + \sqrt{\epsilon} + \frac{1}{\epsilon\,(1 + 2\,C_0\,\epsilon)^{n/2}}) \tag{9}$$

where $C = C(\mathbf{u}, p)$ is a positive constant just depending on $(\mathbf{u}, p)$. When $\epsilon = O(h^{2k})$ and n large enough we obtain error bounds $O(h^k)$ for velocity and pressure

$$\sum_{i=1}^{2} |\mathbf{u} - \mathbf{u}_{i,h}^{n,\epsilon}|_{1,\Omega_i} + \|p - \pi_h^{n,\epsilon}\|_{0,\Omega} \leq C\, h^k$$

where $C = C(\mathbf{u}, p)$ is a positive constant just depending on $(\mathbf{u}, p)$.

5 Numerical experiments

We use a known solution of the incompressible Stokes equations to compute the error between the exact solution and the numerical approximation in the case $k = 1$. In this test $\Omega = (0,1) \times (0,1)$ and the boundary condition is $\mathbf{u} = 0$ on the boundary $\partial\Omega$ of Ω. The exact solution is

$$\begin{aligned} u(x,y) &= -\cos(2\pi x)\,\sin(2\pi y) + \sin(2\pi y) \\ v(x,y) &= \sin(2\pi x)\,\cos(2\pi y) - \sin(2\pi x) \\ p(x,y) &= 2\pi(-\cos(2\pi x) + \cos(2\pi y)) \end{aligned}$$

and we take viscosity $\nu = 1$. We consider the interface Γ as the line $y = 0.5$ and then $\Omega_1 = (0,1) \times (0,0.5)$ and $\Omega_2 = (0,1) \times (0.5,1)$. Next, we consider a uniform triangular mesh of mesh size $h = h_x = h_y$, take $\epsilon = h^2$ and use $\mathbf{P}_1$ finite elements with the Brezzi-Pitkaranka stabilization technique for computing the solutions $\mathbf{u}_{i,h}$ and $p_{i,h}$ on each Ω_i. Then we construct the approximated velocity field $\mathbf{u}_h$ and pressure $\pi_h \in L_0^2(\Omega)$ via

$$\begin{cases} \mathbf{u}_h = \mathbf{u}_{i,h}, & \text{in } \Omega_i \\ \mathbf{u}_h = (\mathbf{u}_{1,h} + \mathbf{u}_{2,h})/2, & \text{on } \partial\Gamma, \\ \pi_h = p_{1,h}\,\chi_{\Omega_1} + p_{2,h}\,\chi_{\Omega_2} - \dfrac{1}{|\Omega|}\Big(\displaystyle\int_{\Omega_1} p_{1,h} + \int_{\Omega_2} p_{2,h}\Big), & \text{in } \Omega \end{cases}$$

where $|\Omega| = 1$. Finally we compute the errors $eu(h) = (\sum_{i=1}^{2} \int_{\Omega_i} |\nabla(u_h - u_{ih}^{n,\epsilon})|^2\,dx)^{1/2}$ and $ep(h) = \|p - p_h\|_{0,\Omega}$. The following table shows the values obtained for these measures.

wesh	$16 \times 16 \quad (h = 1/16)$	$32 \times 32 \quad (h = 1/32)$	$64 \times 64 \quad (h = 1/64)$
eu(h)	0.4600	0.13413	0.0412
ep(h)	0.5773	0.1942	0.066

Indeed, an order of convergence slightly larger that 1 is obtained on this example.

References

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