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An improved quadrature method for integral equations with logarithmic kernel

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Abstract

In this paper we present a family of modified quadrature methods for the numerical approximation of integral equations of the first kind with logarithmic kernel. We prove the stability and the existence, under some smoothness assumptions for the exact solution, of an expansion in powers of the discretization parameter of the error. Using this expansion we deduce that a particular method reaches order three. Some comments on the use of Richardson extrapolation are given.

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1 The equation and its approximation

We are interested in solving numerically equations of the form

$$\mathsf{V}u := \int_0^1 \big[A(\,\cdot\,,t)\log(\sin^2(\pi(\,\cdot\, - t))) + K(\cdot,t)\big]\,u(t)\,dt = f, \qquad (1)$$

with $A, K \in \mathcal{C}^\infty(\mathbb{R}^2)$ and 1–periodic in both variables. We demand that $A(s,s) \neq 0$ for all s. Such equations appear when solving some boundary value problems in partial differential equations (Laplace, Helmholtz,...) on smooth domains of the plane by boundary element methods.

A natural frame for this kind of operator equations is that of periodic Sobolev spaces. To introduce these spaces we construct the following family of norms: let p be a trigonometric polynomial, for $s \in \mathbb{R}$ we consider

$$\|p\|_s := \Big[|\widehat{p}(0)|^2 + \sum_{m\neq 0} |m|^{2s} |\widehat{p}(m)|^2\Big]^{1/2}, \qquad \widehat{p}(m) := \int_0^1 p(s) \exp(-2\pi m\,\imath\, s)\, ds,$$

and H^s is defined as the completeness of trigonometric polynomials in the norm $\|\cdot\|_s$. Obviously H^0 can be identified with $L^2(0,1)$, whereas H^1 is simply $H^1(0,1)$ with periodic continuity conditions. The spaces H^s and H^{-s} form a dual pair when as dual product we take the extension of L^2–product

$$(f,g) := \widehat{f}(0)\overline{\widehat{g}(0)} + \sum_{m\neq 0} \widehat{f}(m)\overline{\widehat{g}(m)}, \qquad f \in H^s,\ g \in H^{-s}.$$

It is well known that $\mathsf{V} : H^s \to H^{s+1}$ is bounded [10]. Furthermore, it is a Fredholm operator of index 0, that is, injectivity is equivalent to the existence of bounded inverse. In the context of classical theory of pseudodifferential operators V is said to be a pseudodifferential operator of order -1 [9]. Given $N \in \mathbb{N}$, $h := 1/N$ we set $x_\alpha := \alpha h$, $\alpha \in \mathbb{R}$. The numerical method we propose is defined by the following scheme:

$$\left|\begin{array}{l} (u_0, u_1, \ldots, u_{N-1}) \in \mathbb{C}^N, \\ h\displaystyle\sum_{j=0}^{N-1}\Big(\epsilon[V(x_j, x_{i-1+\epsilon}) + V(x_j, x_{i+1-\epsilon})] + (1-\epsilon)[V(x_j, x_{i-\epsilon}) + V(x_j, x_{i+\epsilon})]\Big)\, u_j \\ \qquad\qquad = \epsilon[f(x_{i-1+\epsilon}) + f(x_{i+1-\epsilon})] + (1-\epsilon)[f(x_{i-\epsilon}) + f(x_{i+\epsilon})], \qquad i = 0, \ldots, N-1. \end{array}\right.$$

The unknowns u_j are approximations of the pointwise values $u(x_j)$. This class of methods extends a simpler family of quadrature methods studied in [6]. The parameter ϵ is free and can be chosen in $(0, 1/2]$. The choice $\epsilon = 0$ is not valid since it implies evaluating the kernel V on the diagonal. Notice that there is no use of any quadrature approximation to assemble the linear system and that it just requires to evaluate V in $2N^2$ different points.

In order to analyze the method we write it in a more compact form. Let δ_z be the Dirac delta distribution at point z. We set

$$S_h := \Big\{ \sum_{j=0}^{N-1} u_j \delta_{x_j} \,\Big|\, u_j \in \mathbb{C} \Big\} \subset H^{-1}, \qquad T_h := \Big\{ r_h \in H^1 \,\Big|\, r_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_1, \quad \forall i \Big\},$$

a discrete space of Dirac deltas and the continuous piecewise linear functions over the grid $\{x_i\}$ respectively. We also define the inner discrete product given by

$$\langle f, g\rangle_h := \frac{h}{2} \sum_{j=0}^{N-1} \Big[f(x_{j+\epsilon})\overline{g(x_{j+\epsilon})} + f(x_{j-\epsilon})\overline{g(x_{j-\epsilon})} \Big] \approx (f,g) = \int_0^1 f(t)\overline{g(t)} dt.$$

Then, our method is equivalent to

$$\left| \begin{array}{l} u_h \in S_h, \\ \langle \mathsf{V} u_h, r_h \rangle_h = \langle f, r_h \rangle_h, \qquad \forall r_h \in T_h, \end{array} \right. \tag{2}$$

in the following sense: if $(u_0, u_1, \ldots, u_{N-1}) \in \mathbb{C}^N$ is the solution of the quadrature method,

$$u_h := h \sum_{j=0}^{N-1} u_j \delta_{x_j} \tag{3}$$

is the solution of (2). From this new point of view, the method can be seen as a nonconforming 'qualocation' method (qualocation is a portmanteau word for quadrature modified collocation and names a class of numerical methods for integral and pseudodifferential equations) with a discrete space of Dirac deltas as trial, instead of the commonly used periodic splines (see [8] and references therein).

2 Stability

Foreword. In the sequel C (with posible sub- and superscripts) denotes a constant independent of h and of any quantity it is multiplied by. □

We first introduce some elements and technical results necessary for the proof of the main result of this section. Consider

$$\Lambda g := \int_0^1 \log(\sin^2(\pi(\,\cdot\, - t)))g(t)\, dt \tag{4}$$

and

$$S_h \ni \xi_h := h \sum_{j=0}^{N-1} \delta_{x_j} \approx 1.$$

It can be easily verified that

$$\|1 - \xi_h\|_{-1} \leq Ch. \tag{5}$$

Lemma 1 *For all $\epsilon \neq 0$ there exists $C > 0$ (depending on ϵ) such that*

$$|\langle \Lambda(\xi_h - 1), r_h \rangle_h| \leq Ch \|r_h\|_{-1}, \qquad \forall r_h \in T_h.$$

Proof. Consider the 1–periodic function $\eta \to E_h(\eta)$

$$E_h(\eta) := h \sum_{j=0}^{N-1} \log(\sin^2(\pi h(j - \eta))) - \int_0^1 \log(\sin^2(\pi s))\, ds.$$

Obviously, E_h is the error of approximating $\int_0^1 \log(\sin^2(\pi t))dt$ by a composite rectangular rule. For $\eta \notin \mathbb{Z}$, we have [1]

$$|E_h(\eta)| \leq C_\eta h.$$

Applying this bound, it follows that

$$\begin{aligned}
|\langle \Lambda(\xi_h - 1), r_h\rangle_h| &= |\langle E_h(\,\cdot\,/h), r_h\rangle_h| \\
&\le \tfrac{1}{2}\Big| E_h(\epsilon) h \sum_{j=0}^{N-1} \overline{r_h(x_{j+\epsilon})}\,\Big| + \tfrac{1}{2}\Big| E_h(-\epsilon) h \sum_{j=0}^{N-1} \overline{r_h(x_{j-\epsilon})}\,\Big| \\
&\le Ch|\widehat{r}_h(0)| \le C'h\|r_h\|_{-1}.
\end{aligned}$$

□

Let us consider the isomorphisms

$$\mathsf{D} + \mathsf{J} : H^0 \to H^{-1}, \qquad \mathsf{D}^{-1} + \mathsf{J} : H^{-1} \to H^0,$$

where $\mathsf{J}u := \widehat{u}(0)$, $\mathsf{D}u = u'$ and

$$(\mathsf{D}^{-1}u)(x) = \int_0^x \left(u(t) - \int_0^1 u(s)ds \right) dt.$$

Note that these maps are inverse of each other.

A key point to the stability result is the definition of discrete versions of these maps. To do this, we set $B_1(x) := x - 1/2$ for $x \in [0, 1)$, and denote by $\underline{B}_1$ its 1–periodic extension. The discrete operators

$$J_h u := \widehat{u}(0)\xi_h, \qquad \chi_h u := (1 + h\underline{B}_1(\cdot/h))\widehat{u}(0).$$

can be understood as approximations of J. Furthermore, if

$$T_h^0 := \{u_h \in H^0 :\ u_h|_{(x_i,x_{i+1})} \in \mathbb{P}_0, \quad \forall i\}.$$

is the space of piecewise constant functions on the grid $\{x_i\}$, then

$$\mathsf{D} + J_h : T_h^0 \to S_h, \qquad \mathsf{D}^{-1} + \chi_h : S_h \to T_h^0,$$

satisfy $(\mathsf{D} + J_h)^{-1} = \mathsf{D}^{-1} + \chi_h$ and are therefore reciprocal inverses (see the proof of [3] Proposition 16). Since there exist c and C, independent of h, such that

$$c\|(\mathsf{D} + J_h)v_h\|_{-1} \le \|v_h\|_0 \le C\|(\mathsf{D} + J_h)v_h\|_{-1}, \qquad \forall v_h \in S_h, \tag{6}$$

then both sequences of maps are uniformly bounded.

Proposition 2 *Let A be a pseudodifferential operator of the following form*

$$\mathsf{A}g := \text{p.v.} \int_0^1 \cot \pi(\,\cdot\, - t) g(t) dt + \int_0^1 \left[A_0(\,\cdot\,, t) \log(\sin^2 \pi(\,\cdot\, - t)) + A_1(\,\cdot\,, t) \right] g(t)\, dt,$$

(p.v. stands for the Cauchy principal value), where A_0 and A_1 are smooth and periodic. Then, for h small enough, there exists $\beta > 0$ independent of h such that

$$\sup_{r_h \in T_h} \frac{|\langle \mathsf{A}v_h, r_h\rangle_h|}{\|r_h\|_{-1}} \ge \beta \|v_h\|_0, \qquad \forall v_h \in T_h^0.$$

Proof. This inf–sup condition is a consequence of the convergence of the 'qualocation' method for A with T_h^0 and T_h as respective trial and test spaces [7, 5]. $\square$

Theorem 3 *Let $\epsilon \neq 0, 1/2$. Then for h small enough there exists $\beta > 0$, independent of h, such that*

$$\sup_{r_h \in T_h} \frac{|\langle \mathsf{V} u_h, r_h \rangle_h|}{\|r_h\|_{-1}} \geq \beta \|u_h\|_{-1}, \qquad \forall u_h \in S_h. \tag{7}$$

Proof. Suppose first that the coefficient $A(s,s)$ (see (1)) is constant. Without loss of generality, we can suppose $A(s,s) \equiv 1$. Let $u_h \in S_h$ and $v_h := (\mathsf{D}^{-1} + \chi_h)u_h$ and notice that $\widehat{u}_h(0) = \widehat{v}_h(0)$. Then

$$\begin{aligned} \mathsf{V} u_h &= \mathsf{V}(\mathsf{D} + J_h)(\mathsf{D}^{-1} + \chi_h)u_h \\ &= \mathsf{V}(\mathsf{D} + \mathsf{J})v_h + \mathsf{V}(J_h - \mathsf{J})v_h \\ &= \mathsf{V}(\mathsf{D} + \mathsf{J})v_h + \widehat{u}_h(0)\mathsf{V}(\xi_h - 1). \end{aligned}$$

Since $\mathsf{K} := \mathsf{V} - \Lambda : H^{-1} \to H^1$ is bounded (see [4] and recall the definition of Λ in (4)), it is clear by Lemma 1 that for all $r_h \in T_h$

$$\begin{aligned} |\langle \mathsf{V}(\xi_h - 1), r_h \rangle_h| &\leq C_1 h \|r_h\|_{-1} + |\langle \mathsf{K}(\xi_h - 1), r_h \rangle_h| \\ &\leq C_1 h \|r_h\|_{-1} + C_2 \|\mathsf{K}(\xi_h - 1)\|_1 \|r_h\|_{-1} \\ &\leq C_1 h \|r_h\|_{-1} + C_3 \|\xi_h - 1\|_{-1} \|r_h\|_{-1} \leq C_4 h \|r_h\|_{-1}, \end{aligned}$$

where we have also applied that $|\langle u, r_h \rangle_h| \leq C \|u\|_1 \|r_h\|_{-1}$ for all $u \in H^1$ and $r_h \in T_h$ (see [4] Corollary 2) and (5).

On the other hand it can be easily seen that $\mathsf{V}(\mathsf{D} + \mathsf{J})$ fulfills the hypotheses of Lemma 2. Thus, gathering all the previous inequalities and applying (6) we obtain that for all $u_h \in S_h$

$$\sup_{r_h \in T_h} \frac{|\langle \mathsf{V} u_h, r_h \rangle_h|}{\|r_h\|_{-1}} \geq \beta \|(\mathsf{D}^{-1} + \xi_h)u_h\|_0 - Ch|\widehat{u}_h(0)| \geq (\beta' - Ch)\|u_h\|_{-1}.$$

This proves the result for h small enough.

Suppose now that V is a general integral operator with logarithmic kernel. Define $a(s) := A(s,s) \neq 0$ for all s. The operator $\mathsf{W}u := \mathsf{V}(u/a)$ satisfies the hypothesis of the first part of the proof and $\mathsf{V}u_h = \mathsf{W}(au_h)$. Since the elements of S_h are linear combinations of Dirac deltas, we have that $au_h \in S_h$ for all $u_h \in S_h$. Hence,

$$\sup_{r_h \in T_h} \frac{|\langle \mathsf{V} u_h, r_h \rangle_h|}{\|r_h\|_{-1}} = \sup_{r_h \in T_h} \frac{|\langle \mathsf{W}(au_h), r_h \rangle_h|}{\|r_h\|_{-1}} \geq \beta \|au_h\|_{-1} \geq \beta' \|u_h\|_{-1}$$

and the result is proven. $\square$

The case $\epsilon = 1/2$ must be excluded since the matrix of the numerical method can be singular. For instance, this happens when $V(s,t) := \log(\sin^2(\pi(s-t)))$ and N is even.

Remark 4 Condition (7) can be shown to be equivalent to the existence of $C > 0$ such that

$$\|u_h\|_{-1} \leq C\|u\|_0,$$

being u the exact solution, u_h the numerical one, and C independent of h and u (see [4] for similar arguments). $\square$

3 Asymptotic behaviour of the error

Let us introduce the discrete operator

$$Q_h u := h \sum_{j=0}^{N-1} u(x_j)\delta_{x_j}.$$

This operator enjoys of good approximation properties in weak Sobolev norms (see [3] Lemma 3): for all $t > 1/2$

$$\|Q_h u - u\|_{-t} \leq Ch^t \|u\|_t, \qquad \forall u \in H^t.$$

Proposition 5 *There exists a sequence of differential operators* $\{\mathsf{G}_k\}$ *(*G_k *of order* $2k$*) and a sequence of periodic functions* C_k*, both independent of* h *and* ϵ*, such that for all for all* M*,* $u \in H^{M+1}$ *and* $r_h \in S_h$

$$\langle \mathsf{V}(Q_h u - u), r_h\rangle_h = \sum_{k=1}^{M} h^{2k-1} C_k(\epsilon)\langle \mathsf{G}_k u, r_h\rangle_h + \mathcal{O}(h^{M+1})\|u\|_{M+1}\|r_h\|_{-1}.$$

Moreover, if $\epsilon = 1/6$ *then* $C_1(1/6) = 0$.

Proof. It is a simple consequence of [3] Theorem 7. Since $C_1 = \log(2\sin^2(\pi\cdot))$, the choice $\epsilon = 1/6$ implies that the first term of expansion vanishes and the first power of h appearing is 3. $\square$

In the remainder of the paper we restrict ourselves to the case $\epsilon = 1/6$, since this value of the parameter will define the method of highest order. Let $\mathsf{V}u = f$ and $C_h u \in S_h$ be the solution of (2). By Remark 4, $C_h : H^0 \to S_h$ is uniformly bounded in h.

Theorem 6 *There exits a sequence of pseudodifferential operators* $\{\mathsf{D}_k\}$ *(with* D_k *of order* k*) such that for all* $u \in H^{M+1}$ *(and all* M*)*

$$\max_{j=0,\ldots,N-1} \Big| u(x_j) - u_j - h^3 \mathsf{D}_3 u(x_j) - \sum_{k=5}^{M} h^k \mathsf{D}_k u(x_j)\Big| \leq Ch^{M+1}\|u\|_{M+3}.$$

Proof. Notice that for all $r_h \in T_h$

$$\Big|\Big\langle \mathsf{V}(Q_h u - C_h u - \sum_{k=1}^{M} h^{2k-1} C_h \mathsf{V}^{-1}\mathsf{G}_k u), r_h\Big\rangle_h\Big| \leq Ch^{M+1}\|u\|_{M+1}\|r_h\|_{-1}.$$

Applying Theorem 3 it follows readly that

$$\Big\|Q_h u - C_h u - h^3 Q_h \mathsf{D}_3 u - \sum_{k=5}^{M} h^k Q_h \mathsf{D}_k u\Big\|_{-1} \leq C h^{M+1} \|u\|_{M+1},$$

being $\mathsf{D}_k : H^s \to H^{s-k}$ a new sequence of pseudodifferential operators. From [2] Lemma 9, we have the following estimate

$$h \sum_{j=0}^{N-1} |u_j| \leq C \Big\| \sum_{j=0}^{N-1} u_j \delta_{x_j} \Big\|_{-1} = \frac{1}{h} \|u_h\|_{-1}.$$

Therefore, using the definition of Q_h, that the order of D_k is k, and the Sobolev imbedding theorem, we have

$$\begin{aligned} \Big|u(x_j) - u_j - h^3 \mathsf{D}_3 u(x_j) - \sum_{k=5}^{M} h^k \mathsf{D}_k u(x_j)\Big| &\leq C h^{M+1} (\|u\|_{M+3} + \|\mathsf{D}_{M+1} u\|_\infty + \|\mathsf{D}_{M+2} u\|_\infty) \\ &\leq C' h^{M+1} \|u\|_{M+3} \end{aligned}$$

uniformly in j. □

The expansion of the error allows the use of Richardson extrapolation to improve the accuracy of the solution. In this way, if we denote by

$$(u_0^N, u_1^N, \ldots, u_{N-1}^N) \in \mathbb{C}^N$$

to the numerical solution with N unknowns, we have that one step of Richardson extrapolation

$$v_j^N := \frac{8u_{2j}^{2N} - u_j^N}{7}, \qquad j = 0, \ldots, N-1$$

defines a new solution that satisfies (assuming sufficient smoothing requirements over the exact solution)

$$\max_{j=0,\ldots,N-1} |v_j^N - u(j/N)| \leq C h^5.$$

More steps of Richardson extrapolation can be taken, increasing in one the order of accuracy with each step.

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