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Insertion of a non cooperative elliptic system involving Schrödinger operators into a cooperative one

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Abstract

We study here a non cooperative system of n equations defined on $I\!R^N$ which we insert into a cooperative system of n+1 equations to obtain a "Maximum Principle."
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1 Introduction

We consider the following elliptic system on $I\!R^N$:

$$(1)\begin{cases} \text{for } 1 \le i \le n, \\ (1i) \quad L_{q_i}u_i := (-\Delta + q_i)u_i = \sum_{j=1}^n a_{ij}u_j + f_i \text{ in } I\!R^N \end{cases}$$

where:

(H1) for $1 \le i, j \le n$, $a_{ij} \in L^\infty(I\!R^N)$

(H2) for $1 \le i \le n$, q_i is a continuous potential defined on $I\!R^N$ such that:
$q_i(x) \ge 1, \quad \forall x \in I\!R^N$ and $q_i(x) \to +\infty$ when $|x| \to +\infty$

(H3) for $1 \le i \le n$, $f_i \in L^2(I\!R^N)$

Our paper is organized as follow:
- first, we recall some results on M-matrices and on cooperative systems
- in section 2, we adapt a method used by D.G. de Figueiredo and E. Mitidieri (see [8]) for insertion of a non cooperative system of two equations into a cooperative system of three equations to obtain a "Maximum Principle"
-in section 3, we obtain a "Maximum Principle" for a non cooperative system of 3 equations then of n equations

Definition 1.1 *([4]) A matrix $M = sI - B$ is called a non singular M-matrix if B is a positive matrix and $s > \rho(B) > 0$ the spectral radius of B.*

Proposition 1.1 *([4] th2.3 p.134) If M is a matrix with nonpositive off-diagonal coefficients, the conditions $(P0)$, $(P1)$ and $(P2)$ are equivalents where:*
$(P0)$ M is a non singular M-matrix
$(P1)$ all the principal minors of M are strictly positive
$(P2)$ M is semi-positive i.e.: $\exists X >> 0$ such that $MX >> 0$
$X >> 0$ signify $\forall i,\ X_i > 0$ if $X = (X_1, ..., X_n)$

Let $\mathcal{D}(I\!\!R^N)$ be the set of functions $\mathcal{C}^\infty$ on $I\!\!R^N$ with compact support and q be a continuous potential in $I\!\!R^N$ such that: $q \geq 1$ and $q(x) \to +\infty$ when $|x| \to +\infty$. The variational space is $V_q(I\!\!R^N)$: the completion of $\mathcal{D}(I\!\!R^N)$ for the norm $\|.\|_q$ where $\|u\|_q = [\int_{I\!\!R^N} |\nabla u|^2 + q|u|^2]^{\frac{1}{2}}$. $(V_q(I\!\!R^N), \|.\|_q)$ is an Hilbert space whose embedding into $L^2(I\!\!R^N)$ is dense. (see A.Abachti-Mchachti [1] prop.I.1.1)

Proposition 1.2 *(see [1] p25 to 27; [2] th1.1p4,p6,8,11; [3] p3,th3.2p45; [7] p488,489) $-\Delta + q$, considered as an operator in $L^2(I\!\!R^N)$, is positive, selfadjoint, with compact inverse. Its spectrum is discrete and consists in an infinite sequence of positive eigenvalues tending to $+\infty$. The smallest one, denoted by $\lambda(q)$, is a principal eigenvalue, positive and simple.*

For $a \in L^\infty(I\!\!R^N)$, let $a^* = \sup_{I\!\!R^N} a$ and $\lambda(q - a) = \inf\{\frac{\int_{I\!\!R^N}[|\nabla\phi|^2 + (q-a)\phi^2]}{\int_{I\!\!R^N}\phi^2}; \phi \in \mathcal{D}(I\!\!R^N);\ \phi \neq 0\}$.

We say that System (1) is called cooperative if the hypothesis $(H1*)$: $for 1 \leq i, j \leq n,\ a_{ij} \in L^\infty(I\!\!R^N); a_{ij} \geq 0\ a.e$ for $i \neq j$, is satisfied.
We say that System (1) satisfies the Maximum Principle if: $\forall f_i \geq 0$, $1 \leq i \leq n$, each solution $u = (u_1, ..., u_n)$ of (1) is nonnegative.
For any matrix $A = (a_{ij})$ with bounded coefficients, let: $A^* = (a^*_{ij})$ and let $E = (e_{ij})$ be the matrix $n \times n$ defined by: $\forall 1 \leq i \leq n,\ e_{ii} = \lambda(q_i - a_{ii})$ and $\forall 1 \leq i, j \leq n,\ i \neq j \Rightarrow e_{ij} = -a^*_{ij}$. Let $F = (f_{ij})$ the matrix $n \times n$ be defined by: $\forall 1 \leq i \leq n,\ f_{ii} = \lambda(q_i - a_{ii})$ and $\forall 1 \leq i, j \leq n,\ i \neq j \Rightarrow f_{ij} = -\mid a_{ij} \mid^*$.

Recall the following theorems.

Theorem 1.1 *(see [5] Th 4.2.2) Assume that $(H1*), (H2), (H3)$ are satisfied. If E is a non singular M-matrix, then System (1) satisfies the Maximum Principle.*

Theorem 1.2 *(see [6] Th 3.1) Assume that $(H1), (H2), (H3)$ are satisfied. If F is a non singular M-matrix, then System (1) has a solution.*

2 Insertion of a non cooperative system of two equations into a cooperative system of three equations

We redefine System (1) for n=2 by

$$(1')\begin{cases}(-\Delta+q_1)u=au+bv+f \text{ in } I\!R^N\\(-\Delta+q_2)v=cu+dv+g \text{ in } I\!R^N\end{cases}$$

where $(H1)$ becomes $a,b,c,d\in L^\infty(I\!R^N)$. We follow here a method used in [8].

Theorem 2.1 *(see [5] Th 5.2.1) Consider System* $(1')$ *where* a,b,c,d *are reals and* $q_1=q_2=q,\ b<0,\ c>0,\ a>d,\ (a-d)^2+4bc\geq 0$. *Assume* $(H2)$ *and* $(H3)$ *satisfied. Let:* $\delta=(a-d)^2+4bc,\ r=\frac{-2bc}{a-d+\sqrt{\delta}},\ s=\frac{a+d-\sqrt{\delta}}{2},\ \gamma=-\frac{b}{r}$. *If* $\lambda(q)>a-r,\ \lambda(q)>d,\ \lambda(q)>s$, *then:* $f\geq 0,\ g\geq 0,\ f-\gamma g\geq 0\Rightarrow u\geq 0,\ v\geq 0$.

Proof of Theorem 2.1:

Let $w=u-\gamma v$, where (u,v) is a solution of $(1')$. Then (u,v,w) is a solution of the following cooperative system (2):

$$(2)\begin{cases}(-\Delta+q)u=(a-r)u+(b+r\gamma)v+rw+f \text{ in } I\!R^N\\(-\Delta+q)v=cu+dv+g \text{ in } I\!R^N\\(-\Delta+q)w=(a-c\gamma-s)u+(b-d\gamma+s\gamma)v+sw+f-\gamma g \text{ in } I\!R^N\end{cases}$$

Let:

$$B=\begin{pmatrix}\lambda(q)-a+r & 0 & -r\\ -c & \lambda(q)-d & 0\\ 0 & 0 & \lambda(q)-s\end{pmatrix}$$

Since B is a non singular M-matrix, System (2) satisfies the Maximum Principle.
So: $f\geq 0, g\geq 0, f-\gamma g\geq 0\Rightarrow u\geq 0, v\geq 0, w\geq 0$.

Theorem 2.2 *Consider System* $(1')$. *Assume that* $(H1),(H2),(H3)$ *are satisfied. Let* (u,v) *solution of* $(1')$. *Let:*

(H4) $b<0\ ;\ c>0$

(H5) $q_2=q_1+k$ *with* $k\in L^\infty(I\!R^N)\cap C(I\!R^N)$.

(H6) $a+k-d>0;\ \delta=(a+k-d)^2+4bc>0$

(H7) $m=\sup_{I\!R^N}(\frac{-a-k+d-\sqrt{\delta}}{2c})\leq M=\inf_{I\!R^N}(\frac{-a-k+d+\sqrt{\delta}}{2c})$.

Let $\alpha\in]m,M[$ $(\alpha=m$ *if* $m=M)$. *Note that* $\alpha<0$.

(H8) $\lambda(q_1-a)>0;\ \lambda(q_2-d)>0;\ \lambda(q_1-a)\lambda(q_2-d)>(-b)^*c^*>0$

(H9) $\lambda(q_1 - a) > (a + k - d)^* > 0$

Assume that Hypothesis $(H4)$ *to* $(H9)$ *are satisfied.*
Then: $(f \geq 0,\ g \geq 0,\ f + \alpha g \geq 0 \Rightarrow u \geq 0,\ v \geq 0.)$

Proof of Theorem 2.2: Denote $k_1 = -\frac{1}{2}q_1 + \frac{1}{2}q_2$, $r = \frac{b}{\alpha} \in L^\infty(I\!R^N)$ and $s = a + \alpha c + k_1 \in L^\infty(I\!R^N)$. Let $w = u + \alpha v$ and $q_3 = \frac{1}{2}q_1 + \frac{1}{2}q_2$. We have:

$$(2) \begin{cases} (-\Delta + q_1)u = (a - r)u + (b - \alpha r)v + rw + f \text{ in } I\!R^N \\ (-\Delta + q_2)v = cu + dv + g \text{ in } I\!R^N \\ (-\Delta + q_3)w = (a + \alpha c - s + k_1)u + (b + \alpha d - s\alpha + \alpha(k_1 - k))v \\ \qquad\qquad\qquad\qquad + sw + f + \alpha g \text{ in } I\!R^N \end{cases}$$

By $(H4)$ and $(H6)$ we show that System (2) is cooperative. Let:

$$L = \begin{pmatrix} \lambda(q_1 - a + r) & 0 & -r^* \\ -c^* & \lambda(q_2 - d) & 0 \\ 0 & -[b + \alpha d - s\alpha + \alpha(k_1 - k)]^* & \lambda(q_3 - s) \end{pmatrix}$$

We use $(H8)$ and $(H9)$ to prove that L is a non singular M-matrix. Applying Theorem 1.1, we deduce that: $(f \geq 0,\ g \geq 0,\ f + \alpha g \geq 0 \Rightarrow u \geq 0,\ v \geq 0.)$

3 Insertion of a non cooperative system of n equations into a cooperative system of $n + 1$ equations

First consider in this section the non cooperative System (1) for $n = 3$ and $q = q_1 = q_2 = q_3$. We study two cases: first, we study a particular case when one off-diagonal coefficient is equal to 0, another one is negative and all the others are positive; then we study another case when all the off-diagonal coefficients are constants not equal to 0.

Theorem 3.1 *Let:*

(H10) $a_{21} = 0;\ a_{12} \in I\!R^{*-};\ a_{13} \in I\!R^{*+}$.

(H11) $\forall (i,j) \notin \{(1,2);(2,1);(1,3)\},\ \ a_{ij} \in L^\infty(I\!R^N);\ a_{31} \not\equiv 0$ *and*
$\forall (i,j) \notin \{(1,2);(2,1);(1,3)\} i \neq j \Rightarrow a_{ij} \geq 0$

(H12) $\frac{a_{13}a_{32} + a_{22}a_{12}}{a_{12}} = \frac{a_{13}a_{33} + a_{23}a_{12}}{a_{13}} = \gamma \in L^\infty(I\!R^N)$

(H13) $\exists r \in I\!R^{*+},\ a_{11} > \gamma + r = s \in L^\infty(I\!R^N)$

(H14) *F a non singular M-matrix (see section 1 for the definition of F)*

(H15) $(\lambda(q - a_{11}))^2 > r(a_{11} - s)^* + (a_{13}a_{31})^*$

Assume that the hypothesis $(H2),(H3)$ and $(H10)$ to $(H15)$ are satisfied.
Then: $(f_1 \geq 0,\ f_2 \geq 0,\ f_3 \geq 0,\ f_1 + \frac{a_{12}}{r} f_2 + \frac{a_{13}}{r} f_3 \geq 0 \Rightarrow u_1 \geq 0,\ u_2 \geq 0,\ u_3 \geq 0.)$

Proof of Theorem 3.1:

a) Remark: The existence of a solution for System (1) is due to the hypothesis $(H2)$, $(H3)$ and $(H14)$.

b) Let $u_4 = u_1 + \frac{a_{12}}{r} u_2 + \frac{a_{13}}{r} u_3$. By $(H12)$ and since $s = r + \gamma$, (u_1, u_2, u_3, u_4) is solution of the following System (2).

$$(2) \begin{cases} (-\Delta + q)u_1 = (a_{11} - r)u_1 + ru_4 + f_1 \text{ in } I\!R^N \\ (-\Delta + q)u_2 = a_{22}u_2 + a_{23}u_3 + f_2 \text{ in } I\!R^N \\ (-\Delta + q)u_3 = a_{31}u_1 + a_{32}u_2 + a_{33}u_3 + f_3 \text{ in } I\!R^N \\ (-\Delta + q)u_4 = \frac{(a_{11}-s)r + a_{13}a_{31}}{r} u_1 + su_4 + f_1 + \frac{a_{12}}{r} f_2 + \frac{a_{13}}{r} f_3 \text{ in } I\!R^N \end{cases}$$

c) We show by $(H10)$ and $(H13)$ that System (2) is cooperative.

d) Let

$$D = \begin{pmatrix} \lambda(q - a_{11} + r) & 0 & 0 & -r \\ 0 & \lambda(q - a_{22}) & -a_{23}^* & 0 \\ -a_{31}^* & -a_{32}^* & \lambda(q - a_{33}) & 0 \\ -(\frac{(a_{11}-s)r + a_{13}a_{31}}{r})^* & 0 & 0 & \lambda(q - s) \end{pmatrix}$$

We verify that D is a non singular M-matrix.Indeed, by $(H14)$ and $(H15)$, we prove that all the principal minors of D are positive.

Hence System (2) satisfies the Maximum Principle and:

$$f_1 \geq 0,\ f_2 \geq 0,\ f_3 \geq 0,\ f_1 + \frac{a_{12}}{r} f_2 + \frac{a_{13}}{r} f_3 \geq 0 \Rightarrow u_1 \geq 0,\ u_2 \geq 0,\ u_3 \geq 0.$$

Remark: If $a_{12} = -a_{13} \in I\!R^{*-}$, then $(H12)$ becomes $a_{22} - a_{32} = a_{33} - a_{23}$.

Theorem 3.2 *Let:*

(H16) $\forall i \neq j,\ a_{ij} \in I\!R^*$ *and* $\exists i_0 \neq j_0,\ a_{i_0 j_0} > 0$.

(H17) $a_{13}a_{23} > 0,\ a_{21}a_{31} > 0,\ a_{12}a_{32} > 0$.
If $i \neq j$ *and* $a_{ij} > 0$, *let* $s_{ij} < 0$ *such that* $a_{ij} + s_{ij} > 0$.
If $i \neq j$ *and* $a_{ij} < 0$, *let* $s_{ij} > 0$ *such that* $a_{ij} + s_{ij} > 0$.

(H18) $s_{12}s_{31}s_{23} = s_{21}s_{13}s_{32}$.

By $(H17)$ and $(H18)$ we can choose $a_{14} > 0, a_{24} > 0, a_{34} > 0$ and $\alpha_1, \alpha_2, \alpha_3$ reals such that: $s_{21} + a_{24}\alpha_1 = s_{31} + a_{34}\alpha_1 = s_{12} + a_{14}\alpha_2 = s_{32} + a_{34}\alpha_2 = s_{13} + a_{14}\alpha_3 = s_{23} + a_{24}\alpha_3 = 0.$
Let:

(H19) $a_{11} = -\frac{\alpha_2 a_{21} + \alpha_3 a_{31}}{\alpha_1}, \quad a_{22} = -\frac{\alpha_1 a_{12} + \alpha_3 a_{32}}{\alpha_2}, \quad a_{33} = -\frac{\alpha_1 a_{13} + \alpha_2 a_{23}}{\alpha_3}.$

(H20)

$$M = \begin{pmatrix} \lambda(q - a_{11} + a_{14}\alpha_1) & -(a_{12} + s_{12}) & -(a_{13} + s_{13}) \\ -(a_{21} + s_{21}) & \lambda(q - a_{22} + a_{24}\alpha_2) & -(a_{23} + s_{23}) \\ -(a_{31} + s_{31}) & -(a_{32} + s_{32}) & \lambda(q - a_{33} + a_{34}\alpha_3) \end{pmatrix}$$

a non singular M-matrix.

(H21) *F a non singular M-matrix.*

Assume that the hypothesis $(H2)$, $(H3)$ and $(H16)$ to $(H21)$ are satisfied.
Then: $(f_1 \geq 0,\ f_2 \geq 0,\ f_3 \geq 0,\ \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \geq 0 \Rightarrow u_1 \geq 0,\ u_2 \geq 0,\ u_3 \geq 0.)$

Proof of Theorem 3.2:

a) Remarks: The existence of a solution for System (1) is due to the hypothesis $(H2)$, $(H3)$ and $(H21)$. Note that: $\forall i,\ \alpha_i$ has the same sign than $a_{ji}\ \forall j \neq i$. By $(H16)$, there exists at least one $\alpha_i > 0$.

b) Let $u_4 = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$. By $(H19)$, (u_1, u_2, u_3, u_4) is solution of the following cooperative System (S).

$$(S)\begin{cases} (-\Delta + q + \alpha_1 a_{14})u_1 = a_{11}u_1 + (a_{12} + s_{12})u_2 + (a_{13} + s_{13})u_3 + a_{14}u_4 + f_1 \text{ in } I\!R^N \\ (-\Delta + q + \alpha_2 a_{24})u_2 = (a_{21} + s_{21})u_1 + a_{22}u_2 + (a_{23} + s_{23})u_3 + a_{24}u_4 + f_2 \text{ in } I\!R^N \\ (-\Delta + q + \alpha_3 a_{34})u_3 = (a_{31} + s_{31})u_1 + (a_{32} + s_{32})u_2 + a_{33}u_3 + a_{34}u_4 + f_3 \text{ in } I\!R^N \\ (-\Delta + q)u_4 = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \text{ in } I\!R^N \end{cases}$$

c) Let

$$O = \begin{pmatrix} \lambda(q - a_{11} + a_{14}\alpha_1) & -(a_{12} + s_{12}) & -(a_{13} + s_{13}) & -a_{14} \\ -(a_{21} + s_{21}) & \lambda(q - a_{22} + a_{24}\alpha_2) & -(a_{23} + s_{23}) & -a_{24} \\ -(a_{31} + s_{31}) & -(a_{32} + s_{32}) & \lambda(q - a_{33} + a_{34}\alpha_3) & -a_{34} \\ 0 & 0 & 0 & \lambda(q) \end{pmatrix}$$

By $(H20)$, O is a non singular M-matrix.

Hence System (S) satisfies the Maximum Principle and:

$$f_1 \geq 0,\ f_2 \geq 0,\ f_3 \geq 0,\ \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \geq 0 \Rightarrow u_1 \geq 0,\ u_2 \geq 0,\ u_3 \geq 0.$$

We give some examples.

Example 1 Let

$$(1)\begin{cases}(-\Delta+q)u_1=\frac{1}{2}u_1-u_2+2u_3+f_1 \text{ in } I\!R^N\\(-\Delta+q)u_2=-u_1+\frac{1}{2}u_2+2u_3+f_2 \text{ in } I\!R^N\\(-\Delta+q)u_3=-u_1-u_2+8u_3+f_3 \text{ in } I\!R^N\end{cases}$$

We can choose $s_{13}=s_{23}=-1,\ s_{21}=s_{31}=s_{12}=s_{32}=2,\ a_{14}=a_{24}=a_{34}=1,\ \alpha_1=\alpha_2=-2,\ \alpha_3=1.$

Example 2 Let $a\in I\!R^{*+}$ and

$$(1)\begin{cases}(-\Delta+q)u_1=-au_2+au_3+f_1 \text{ in } I\!R^N\\(-\Delta+q)u_2=-au_1+au_3+f_2 \text{ in } I\!R^N\\(-\Delta+q)u_3=-au_1-au_2+2au_3+f_3 \text{ in } I\!R^N\end{cases}$$

We can choose $s_{13}=s_{23}=-s_{21}=-s_{31}=-s_{12}=-s_{32}\in I\!R^{*-},\ a_{14}=a_{24}=a_{34}\in I\!R^{*+},\ \alpha_1=\alpha_2=-\alpha_3\in I\!R^{*-}.$

Example 3 Let α,β,γ reals, $a\in I\!R^{*+},\ b\in I\!R^{*-},\ c\in I\!R^{*-}$ and

$$(1)\begin{cases}(-\Delta+q)u_1=\alpha u_1+cu_2+au_3+f_1 \text{ in } I\!R^N\\(-\Delta+q)u_2=bu_1+\beta u_2+au_3+f_2 \text{ in } I\!R^N\\(-\Delta+q)u_3=bu_1+cu_2+\gamma u_3+f_3 \text{ in } I\!R^N\end{cases}$$

We can choose $s_{13}=s_{23}=s\in I\!R^{*-},\ s_{21}=s_{31}=s'\in I\!R^{*+},\ s_{12}=s_{32}=s"\in I\!R^{*+}.$
Then : $\exists\alpha_1\in I\!R^{*-},\ \exists\alpha_2\in I\!R^{*-},\ \exists\alpha_3\in I\!R^{*+}$ such that: $\frac{s'}{\alpha_1}=\frac{s"}{\alpha_2}=\frac{s}{\alpha_3}$.
$(H19)$ becomes: $\alpha=-b\frac{\alpha_3+\alpha_2}{\alpha_1},\ \beta=-c\frac{\alpha_3+\alpha_1}{\alpha_2},\ \gamma=-a\frac{\alpha_2+\alpha_1}{\alpha_3}$.

We conclude by giving a generalization for a system of n equations.

Theorem 3.3 *Let:* $\forall i, q_i=q$ *and*

(H22) $\forall i,j,\ a_{ij}\in I\!R$ *and* $\exists j_0,\ \forall i,\ i\neq j_0\Rightarrow a_{ij_0}>0$
Let $\forall j,\ \alpha_j=\min_i(a_{ij})$.

(H23) $\forall j,\ \alpha_j(a_{jj}-1)\geq-\sum_{i=1,i\neq j}^n\alpha_i a_{ij}$

(H24) $\forall i,\ \lambda(q)>1+\sum_{j=1}^n(a_{ij}-\alpha_j)$ *and* $\lambda(q)>\sum_{j=1}^n(\sum_{i=1}^n\alpha_i a_{ij}-\alpha_j)+1$

Assume that the hypothesis $(H2),(H3)$ *and* $(H22)$ *to* $(H24)$ *are satisfied.*
Then: $\forall i,\ f_i\geq 0$ *and* $\sum_i\alpha_i f_i\geq 0\Rightarrow\forall i,\ u_i\geq 0.$

Proof of Theorem 3.3: Let $u_{n+1}=\sum_{i=1}^n\alpha_i u_i$. We have: $\forall i=1,...,n,\ (-\Delta+q)u_i=\sum_{j=1}^n(a_{ij}-\alpha_j)u_j+u_{n+1}+f_i$ and $(-\Delta+q)u_{n+1}=\sum_{j=1}^n(\sum_{i=1}^n\alpha_i a_{ij}-\alpha_j)u_j+u_{n+1}+\sum_i\alpha_i f_i$. Let B be the matrix associated to the above system and ${}^tX=(1,...1)$. We have $BX>>0$ so B is a non singular M-matrix. Applying Theorem 1.1, we obtain the result of the Theorem 3.3.

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