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Uniqueness of solution for the 2D Primitive Equations with friction condition on the bottom*

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Abstract

Uniqueness of solution for the Primitive Equations with Dirichlet conditions on the bottom is an open problem even in 2D domains. In this work we prove a result of additional regularity for a weak solution v for the Primitive Equations when we replace Dirichlet boundary conditions by friction conditions. This allows to obtain uniqueness of weak solution global in time, for such a system [3]. Indeed, we show weak regularity for the vertical derivative of the solution, $\partial_z v$ for all time. This is because this derivative verifies a linear pde of convection-diffusion type with convection velocity v, and the pressure belongs to a L^2-space in time with values in a weighted space.

Keywords: Boundary conditions of type Navier, 2D Primitive Equations, uniqueness
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1 Introduction and motivation.

Primitive Equations are one of the models used to forecast the fluid velocity and pressure in the ocean. Such equations are obtained from the dimensionless Navier-Stokes equations, letting the aspect ratio (quotient between vertical dimension and horizontal dimensions) go to zero. The first results about existence of solution (weak, in the sense of the Navier-Stokes equations) are proved for boundary conditions of Dirichlet type on the bottom of the domain and with wind traction on the surface, in the works by Lions-Temam-Wang,

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[5, 6], for domains with vertical walls and in the work of Azérad-Guillén, [1], for domains without vertical walls. However, uniqueness of solution remained as an open problem due to the necessity of a more regular solution. In the case of vertical sidewalls, the authors proved in [4] the existence of a more regular solution, global in time for small data or local in time for any data. In these cases, uniqueness of solution is guaranteed.

But, from a physical point of view, homogeneous Dirichlet boundary conditions (on the bottom) are only justified when considering a molecular viscosity fluid. In many geophysical fluids, the role of this viscosity is negligible, being more relevant the viscosity due to turbulent effects. It seems then logical to use Navier boundary conditions for the Primitive Equations. Moreover, they prevent the appearance of a boundary layer phenomena on the bottom.

The authors obtained the Primitive Equations model with Navier type boundary conditions from the Navier-Stokes equations in [2]. Here, we will focus on the uniqueness problem in the 2D case, see also [3]. We will present what we consider is the first result of uniqueness of weak solution for the 2D Primitive Equations.

2 The model.

The domain considered is defined by:

$$\Omega = \{(x,z) \in \mathbb{R}^2 / \, x \in S, \, -h(x) < z < 0\},$$

where S (ocean surface) is an open interval and $h : \bar{S} \to \mathbb{R}_+$ is a nonnegative continuous function defined on $\bar{S}$ that vanishes on ∂S. The boundary of the domain is $\partial\Omega = \bar{\Gamma}_b \cup \Gamma_s$, where the bottom is $\Gamma_b = \{(x,z) \in \mathbb{R}^2 : \, x \in S, \, z = -h(x)\}$ and the surface $\Gamma_s = \{(x,0) : x \in S\}$. Therefore, the fluid velocity (v, w) and the pressure p satisfy the following equations:

$$(PE) \left\{ \begin{aligned} \partial_t v + v\,\partial_x v + w\,\partial_z v - \nu_h \partial^2_{xx} v - \nu_v \partial^2_{zz} v + \partial_x p \; &= \; f \quad \text{in } (0,T) \times \Omega, \\ \partial_z p \; = \; 0, \quad w(t,x,z) = \int_z^0 \partial_x v(t,x,s) ds, \quad \langle v \rangle \; &= \; 0 \quad \text{in } (0,T) \times S, \\ \nu_v \partial_z v \; &= \; \alpha |v_{air}| (v_{air} - v) \quad \text{on } (0,T) \times \Gamma_s, \\ \nu_v \partial_z v \; &= \; \beta(x)\, v \quad \text{on } (0,T) \times \Gamma_b, \\ v|_{t=0} \; &= \; v_0 \quad \text{in } \Omega, \end{aligned} \right.$$

where $\langle v \rangle (t;x) = \displaystyle\int_{-h(x)}^0 v(t;x,z) dz$, v_{air} is the horizontal velocity of the wind at the surface, v_0 the horizontal initial velocity, (ν_h, ν_v) the anisotropic turbulent viscosity, $\alpha \in \mathbb{R}$

a positive constant and $\beta = \beta(x)$ a positive function defined on S.

Remark 2.1 *The model for Primitive Equations with Navier conditions deduced in [2] was formed by* $(PE)_1$, $\partial_z p = 0$ *and* $\partial_x v + \partial_z w = 0$ *in* $(0,T)\times\Omega$, $\nu_v \partial_z v = \alpha(v_{air} - v)$ *and* $w = 0$ *on* $(0,T)\times\Gamma_s$, $\nu_v\partial_z v = \beta v$ *and* $(v,w)\cdot\mathbf{n} = 0$ *on* $(0,T)\times\Gamma_b$ *and* $v|_{t=0} = v_0$ *in* Ω. *The equation* $\partial_x v + \partial_z w = 0$ *and boundary conditions for* w *imply that* $w(t;x,z) = \int_z^0 \partial_x v(t;x,s)ds$ *and* $\partial_x\langle v\rangle = 0$. *Finally, as* $\langle v\rangle$ *is a 1-dimensional function, the hypothesis* $\langle v\rangle = 0$ *on* $(0,T)\times\partial S$ *implies that* $\langle v\rangle = 0$ *on* $(0,T)\times S$.

3 Definitions and previous results.

For the velocity v, we introduce the following spaces:

$$\mathcal{V} = \{\varphi \in C_s^\infty(\overline{\Omega}) : \langle\varphi\rangle = 0 \text{ in } S,\}$$

where $C_s^\infty(\overline{\Omega})$ is the space of C^∞-functions that vanish in a neighbourghood of $\partial\Gamma_s$. We will denote by H and V its closures in the $L^2(\Omega)$ and $H^1(\Omega)-$ norms respectively.

Definition 3.1 (Weak solution) *We say that* v *is a weak solution for* (PE) *in* $(0,T)$ *if:*

$$v \in L^\infty(0,T;H)\cap L^2(0,T;V),$$

satisfies the variational formulation: $\forall\varphi \in C^1([0,T];\mathcal{V})$ *with* $\varphi(T) = 0$,

$$\left\{\begin{array}{rl} - & \displaystyle\int_0^T\int_\Omega (\partial_t\varphi + v\partial_x\varphi + w\partial_z\varphi)\,v + \int_0^T\int_\Omega(\nu_h\partial_x v\partial_x\varphi + \nu_v\partial_z v\partial_z\varphi) \\ & \displaystyle\quad + \int_0^T\int_S \delta(x) v|_{\Gamma_b}\varphi|_{\Gamma_b} + \int_0^T\int_S \alpha|v_{air}|(v|_{\Gamma_s} - v_{air})\varphi|_{\Gamma_s} \\ = & \displaystyle\int_\Omega v_0\varphi(0) + \int_0^T\int_\Omega f\,\varphi + \nu_h\int_0^T\int_S v|_{\Gamma_b}\partial_x[\varphi|_{\Gamma_b}h'(x)], \end{array}\right.$$

with $w = \int_z^0 \partial_x v$ *and satisfies the following energy inequality*

$$\left\{\begin{array}{l} \displaystyle\frac{1}{2}\|v(t)\|^2_{L^2(\Omega)} + \nu_h\int_0^t \|\partial_x v(s)\|^2_{L^2(\Omega)} + \nu_v\int_0^t\|\partial_z v(s)\|^2_{L^2(\Omega)} \\ \displaystyle\quad + \int_0^t\int_S\gamma(x)|v|_{\Gamma_b}|^2 + \frac{1}{2}\int_0^t\int_S\alpha|v_{air}||v|_{\Gamma_s}|^2 \le \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t\int_S\alpha|v_{air}|^3 \end{array}\right.$$

with $\delta(x) = \beta(x)\left(1 + \dfrac{\nu_h}{\nu_v}|h'(x)|^2\right)$ *and* $\gamma(x) = \delta(x) - \dfrac{\nu_h}{2}h''(x)$.

Remark 3.1 *In order to ensure that the system is dissipative (necessary property from a physical point of view), we assume that* $\gamma(x)\ge 0$.

Remark 3.2 *Notice that the boundary condition on the bottom is not standard because $\partial_z v$ is not the Neumann condition respect to the laplacian operator. This fact produces the term $\nu_h \int_0^T \int_S v|_{\Gamma_b} \partial_x[\varphi|_{\Gamma_b} h'(x)]$ in the variational formulation. In other words, giving a weak solution v, we can get an associate pressure p through the De Rham Lemma (as a Lagrange multiplier) in such a way that (v, w, p) verify the differential problem (PE) in the distribution sense (see [3] for more details). In particular, the following mixed variational formulation can be obtained: $\forall \varphi \in C^1([0,T]; C_s^\infty(\overline{\Omega}))$, with $\varphi(T) = 0$, there exists a function ψ smooth enough, satisfying $(\varphi, \psi) \cdot \mathbf{n}|_{\partial\Omega} = 0$ such that:*

$$\begin{aligned} &-\int_0^T \int_\Omega (\partial_t \varphi + v\partial_x \varphi + w\partial_z \varphi)\, v + \int_0^T \int_\Omega (\nu_h \partial_x v \partial_x \varphi + \nu_v \partial_z v \partial_z \varphi) \\ &+ \int_0^T \int_S \delta(x) v|_{\Gamma_b} \varphi|_{\Gamma_b} + \int_0^T \int_S \alpha |v_{air}| \, (v|_{\Gamma_s} - v_{air})\, \varphi|_{\Gamma_s} \\ &= \int_\Omega v_0 \varphi(0) + \int_0^T \int_\Omega f\varphi + \nu_h \int_0^T \int_S v|_{\Gamma_s} \partial_x \left(\varphi|_{\Gamma_b} h'\right) + \int_0^T \int_\Omega p\, \nabla \cdot (\varphi, \psi). \end{aligned} \tag{1}$$

Theorem 3.2 *(See [2] for a proof of this result.) Suppose that $h \in H^2(S)$ with $|h'| > 0$ on ∂S, $\beta \in L^\infty(S)$, $f \in L^2(0,T; L^2(\Omega))$, $v_{air} \in L^3(0,T; L^3(S))$, $v_0 \in H$ and $\gamma(x) \geq 0$ on S. Then, there exists a weak solution v for (PE) in $(0,T)$.*

Definition 3.3 (Weak-vorticity solution) *We will say that v is a weak-vorticity solution of (PE) in $(0,T)$ if it is a weak solution that also satisfies the additional regularity:*

$$\partial_z v \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)).$$

Remark 3.3 *$\partial_z v$ can be seen as the vorticity associated to the Primitive Equations. Indeed, if we consider the vorticity for the 2D Navier-Stokes equations, $\omega_{NS} = \partial_z v_{NS} - \partial_x w_{NS}$, letting the aspect ratio go to zero we arrive at $\partial_z v$.*

4 Main result.

Theorem 4.1 (Uniqueness of weak solution) *Under the hypothesis of Theorem 3.2, if we also consider that $\beta \in H_0^1(S)$, $v_{air} \in L^\infty(0,T; H_0^1(S))$, $\partial_t v_{air} \in L^2(0,T; L^1(S))$, $\partial_z f \in L^2(0,T; H^{-1}(\Omega))$, $\partial_z v_0 \in L^2(\Omega)$ and the depth function h verifies $|h'|/h \leq c/dist(x, \partial S)$, then there exists a unique weak solution for (PE). Moreover, this solution is a weak-vorticity solution.*

Outline of the proof: Here, we will explain the main ideas that we have followed to prove Theorem 4.1. For a complete proof of this result see [3].

Following the method of P. L. Lions, [7], to prove uniqueness of weak solution for the Navier-Stokes equations we observed that additional regularity is necessary for one of the two solutions compared. Applying the argument to (PE), we observed that this regularity should be $\partial_z v \in L^4(0,T;L^4(\Omega))$. In order to obtain more regularity for $\partial_z v$, we search for the problem verified by $\partial_z v$. First, we formally derive $(PE)_1$ respect to z, obtaining that $\partial_z v$ satisfies in $\mathcal{D}'((0,T)\times\Omega)$:

$$\partial_t(\partial_z v) + v\,\partial_x(\partial_z v) + w\,\partial_z(\partial_z v) - \nu_h \partial^2_{xx}(\partial_z v) - \nu_v \partial^2_{zz}(\partial_z v) = \partial_z f.$$

Knowing v and w, the previous equation is linear and parabolic, because the pressure p has disappeared, so we could expect weak regularity for $\partial_z v$. To this end, we need to study a homogeneous system, so we consider the auxiliary function $\psi = \nu_v \partial_z v - \phi\, v - e$ with

$$\phi(t;x,z) = -\alpha\left(1 + \frac{z}{h(x)}\right)|v_{air}(t;x)| - \frac{z}{h(x)}\beta(x)$$

and

$$e(t;x,z) = \alpha |v_{air}(t;x)| v_{air}(t;x)\left(1 + \frac{z}{h(x)}\right)$$

auxiliary functions such that $\psi|_{\partial\Omega} = 0$. Then, ψ verifies the problem:

$$(P)\left\{\begin{array}{rcll} \partial_t\psi + v\,\partial_x\psi + w\,\partial_z\psi - \nu_v\partial^2_{xx}\psi - \nu_v\partial^2_{zz}\psi &=& F & \text{in } (0,T)\times\Omega, \\ \psi &=& 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi|_{t=0} &=& \nu_v\partial_z v_0 - \phi|_{t=0} v_0 - e|_{t=0} & \text{in } \Omega, \end{array}\right.$$

where $F = G(\phi, v, w, e, f) + \phi\,\partial_x p$.

At this point, we have 2 problems: getting an additional regularity for the pressure p to obtain weak regularity for ψ, and identifying $\psi + \phi\, v + e$ with $\nu_v \partial_z v$. Once these problems are solved, then $\partial_z v \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ and in particular belongs to $L^4(0,T;L^4(\Omega))$, so we will be able to conclude weak uniqueness for (PE).

5 Additional regularity for the pressure.

Thanks to $\partial_z p = 0$, we can identify p with a function p_s only defined on S, $p_s(x) = p(x,z)$, through the relation:

$$\int_\Omega p(x,z)\varphi(x,z)dx\,dz \;=\; \int_S p_s(x)\langle\varphi\rangle(x)dx \quad \forall \varphi \in L^2(\Omega).$$

Theorem 5.1 *Assume the hypothesis for the data of Theorem* 4.1*. If* (v,p) *is a weak solution of* (PE)*, we have:*

$$\sqrt{h}\,\partial_x p_s \in L^2(0,T;H^{-1}(S)).$$

Outline of the proof: For the equations of Navier-Stokes type, the pressure regularity is normally obtained from the regularity of the remaining terms of the equation. The term

$\partial_t v$ prevents a L^2-regularity in time for the pressure. The fact that $\langle v \rangle = 0$ on $(0,T)\times S$ implies that $\partial_t \langle v \rangle = 0$ on $(0,T)\times S$, so integrating $(PE)_1$ in z we try to improve the regularity for the pressure. In a rigorous form, this vertical integration corresponds to take test functions independent from z in the mixed variational formulation (1).

On the other hand, as the pressure p is independent from z, its integration on z only adds a factor $h(x)$ multiplying p. Moreover, for (φ, ψ) any test functions in (1),

$$\int_\Omega p \, \nabla \cdot (\varphi, \psi) \, d\Omega = \int_S p_s \, \partial_x \langle \varphi \rangle \, dx.$$

Then, we choose $\varphi = \zeta / \sqrt{h}$ with $\zeta \in C_0^1([0,T]; C_0^\infty(S))$ as a test function (in particular, this space is dense in $L^2(0,T; H_0^1(S))$). Concretely, we have to give sense to the term

$$\int_0^T \int_S p_s(t;x) \partial_x(\sqrt{h}\, \zeta)(t;x) dx\, dt.$$

To this aim, we prove that the others terms from the mixed variational formulation are well-defined and bounded in function of the $L^2(0,T; H_0^1(S))$-norm of ζ. Additional regularity required for the data, hypothesis $|h'|/h \leq c/dist(x, \partial S)$ jointly with Hardy inequalities and the fact that $\partial_t \langle v \rangle = 0$ let finish the proof. ■

6 Identification of $\psi + \phi\, v + e$ with $\nu_v \partial_z v$.

Using a Galerkin method, the additional regularity for p let us obtain weak regularity for ψ, so $\psi \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$. To get $\partial_z v \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$, we prove that $\psi + \phi\, v + e = \nu_v \partial_z v$.

The first idea to get this result was to use the uniqueness of weak solution for problem (P), but the problem was that we could not assure the weak regularity for $\partial_z v$ (only $\partial_z v \in L^2(0,T; L^2(\Omega))$). Consequently, we looked for a new method to our purpose: We call $a = \psi + \phi\, v + e$ and define $\widetilde{v} \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ such that $\nu_v \partial_z \widetilde{v} = a$ in Ω and $\langle \widetilde{v} \rangle = 0$ on S. In fact, we can choose:

$$\widetilde{v}(x,z) = -\frac{1}{\nu_v} \int_z^0 a(x,s) ds + \frac{1}{\nu_v} \frac{1}{h(x)} \left(\int_{-h(x)}^0 \left(\int_z^0 a(x,s) ds \right) dz \right).$$

The idea is to obtain uniqueness for both velocities v and $\widetilde{v}$, and then $\partial_z v = \partial_z \widetilde{v} \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$.

Starting from the variational formulation for ψ, taking $\chi = \displaystyle\int_z^0 \eta(x,s) ds$ as test functions, where $\eta \in \mathcal{D}(\Omega)$ with $\langle \eta \rangle = 0$ and taking into account that

$$\nu_v \partial_z \widetilde{v} = \alpha |v_{air}| (v_{air} - v) \text{ on } \Gamma_s \quad \text{and} \quad \nu_v \partial_z \widetilde{v} = \beta v \text{ on } \Gamma_b,$$

we can easily deduce that $\widetilde{v}$ verifies the following variational formulation $(\widetilde{FV})$: $\forall \eta \in C^1([0,T];\mathcal{V})$,

$$\begin{cases} \displaystyle\int_0^t \langle \partial_t \widetilde{v}, \eta\rangle_\Omega + \int_0^t \int_\Omega (v\,\partial_x \widetilde{v} + w\,\partial_z \widetilde{v})\,\eta \\ \displaystyle + \int_0^t \int_\Omega (\nu_h \partial_x \widetilde{v}\,\partial_x \eta + \nu_v \partial_z \widetilde{v}\,\partial_z \eta) + \int_0^t \int_S \alpha |v_{air}|\,(v|_{\Gamma_s} - v_{air})\,\eta|_{\Gamma_s} \\ \displaystyle + \int_0^t \int_S \delta(x) v|_{\Gamma_b}\,\eta|_{\Gamma_b} = \int_0^t \int_\Omega f\,\eta \\ \displaystyle + \int_0^t \int_\Omega \left\{ v\,\partial_x \widetilde{v} + \int_z^0 \partial_x (v \partial_z \widetilde{v})(x,s)ds \right\} \eta + \nu_v \int_0^t \int_S v|_{\Gamma_b}\,\partial_x \left[\eta|_{\Gamma_b}\, h'(x)\right]. \end{cases}$$

On the other hand, we know that v satisfies the following variational formulation (FV): $\forall \varphi \in C^1([0,T];\mathcal{V})$,

$$\begin{cases} \displaystyle\langle v(t), \varphi(t)\rangle_\Omega - \int_0^t \int_\Omega (\partial_t \varphi + v \partial_x \varphi + w \partial_z \varphi)\, v \\ \displaystyle + \int_0^t \int_\Omega (\nu_h \partial_x v \partial_x \varphi + \nu_v \partial_z v \partial_z \varphi) \\ \displaystyle + \int_0^t \int_S \alpha |v_{air}|\,(v|_{\Gamma_s} - v_{air})\,\varphi|_{\Gamma_s} + \int_0^t \int_S \delta(x) v|_{\Gamma_b} \varphi|_{\Gamma_b} \\ \displaystyle = \int_\Omega v_0 \varphi(0) + \nu_h \int_0^t \int_S v|_{\Gamma_b} \partial_x [\varphi|_{\Gamma_b} h'(x)] + \int_0^t \int_\Omega f \varphi, \end{cases}$$

Taking into account the weak regularity for $\widetilde{v}$ and $\partial_z \widetilde{v}$ and arguing by density, we can take $\widetilde{v}$ as a test function in (FV) and v as a test function in $(\widetilde{FV})$. Subtracting both expressions to the energy equality of $\widetilde{v}$ and the energy inequality of v, we arrive at ([3]): a. e. $t \in (0,T)$,

$$\begin{aligned} &\frac{1}{2}\|v(t) - \widetilde{v}(t)\|^2_{L^2(\Omega)} + \int_0^t \Big(\nu_h \|\partial_x (v - \widetilde{v})(s)\|^2_{L^2(\Omega)} + \nu_v \|\partial_z (v - \widetilde{v})(s)\|^2_{L^2(\Omega)} \Big) ds \\ &\le \int_0^t \int_\Omega \left\{ v\,\partial_x \widetilde{v} + \int_z^0 \partial_x (v\,\partial_z \widetilde{v})(x,s) ds \right\} (\widetilde{v} - v) d\Omega ds \qquad (2) \\ &+ \frac{\nu_h}{2} \int_0^t \int_S |\widetilde{v}|_{\Gamma_b} - v|_{\Gamma_b}|^2 h''(x) dx ds \equiv I + J. \end{aligned}$$

Notice that if $\widetilde{v} = v$, then $I = 0$ and $J = 0$. Integrating by parts respect to z, we rewrite I as:

$$\begin{aligned} I &= \int_0^t \int_\Omega \{\partial_z \widetilde{v}\, \partial_x (v-\widetilde{v}) - \partial_x \widetilde{v}\, \partial_z (v-\widetilde{v})\} \left(\int_z^0 (v-\widetilde{v})(x,s) ds \right) d\Omega ds \\ &\leq \frac{\min\{\nu_h, \nu_v\}}{4} \int_0^t \|v-\widetilde{v}\|^2_{H^1(\Omega)} ds \\ &+ C(\nu_h,\nu_v) \int_0^t \left(\|\partial_x \widetilde{v}\|^2_{L^2(\Omega)} + \|\partial_x(\partial_z \widetilde{v})\|^{4/3}_{L^2(\Omega)} \right) \|v-\widetilde{v}\|^2_{L^2(\Omega)} ds. \end{aligned}$$

We bound J using the Trace and Interpolation Theory in $H^s(\Omega)$-spaces with $s \in \mathbb{R}$:

$$\begin{aligned} J &\leq C \int_0^t \|h''\|_{L^2(S)} \|(v-\widetilde{v})|_{\Gamma_b}\|^2_{L^4(S)} ds \\ &\leq C \int_0^t \|h''\|_{L^2(S)} \|v-\widetilde{v}\|^2_{H^{3/4}(\Omega)} ds \\ &\leq C \int_0^t \|h''\|_{L^2(S)} \|v-\widetilde{v}\|^{1/2}_{L^2(\Omega)} \|v-\widetilde{v}\|^{3/2}_{H^1(\Omega)} ds \\ &\leq \frac{\min\{\nu_h, \nu_v\}}{4} \int_0^t \|v-\widetilde{v}\|^2_{H^1(\Omega)} ds + C(\nu_h,\nu_v) \int_0^t \|h''\|^4_{L^2(S)} \|v-\widetilde{v}\|^2_{L^2(\Omega)} ds \end{aligned}$$

Then, (2) becomes:

$$\begin{aligned} &\|v(t)-\widetilde{v}(t)\|^2_{L^2(\Omega)} + \int_0^t \left(\nu_h \|\partial_x (v-\widetilde{v})(s)\|^2_{L^2(\Omega)} + \nu_v \|\partial_z (v-\widetilde{v})(s)\|^2_{L^2(\Omega)} \right) ds \\ &\leq C(\nu_h,\nu_v) \int_0^t \Big(\|\partial_z \widetilde{v}\|_{L^2(\Omega)} \|\partial_z \widetilde{v}\|_{H^1(\Omega)} + \|\partial_x \widetilde{v}\|^2_{L^2(\Omega)} \\ &\qquad + \|\partial_x(\partial_z \widetilde{v})\|^{4/3}_{L^2(\Omega)} + \|h''\|^4_{L^2(S)} \Big) \|v-\widetilde{v}\|^2_{L^2(\Omega)} ds. \end{aligned}$$

Since $\partial_z \widetilde{v}$ has weak regularity, we can use the Gronwall Lemma and deduce that $\widetilde{v} = v$. ∎

References

[1] P. Azérad & F. Guillén-González. Mathematical justification of the hydrostatic approximation in the Primitive Equations of Geophysical fluid dynamics. To appear in *Siam J. Math. Anal.* , Vol. 33, No. 4, 847-859.

[2] D. Bresch, F. Guillén-González, N. Masmoudi & M. A. Rodríguez-Bellido. Asymptotic derivation of a Navier condition for the Primitive Equations. Submitted to *Asymptotic Analysis*.

[3] D. Bresch, F. Guillén-González, N. Masmoudi & M. A. Rodríguez-Bellido. On the uniqueness of weak solutions of the two-dimensional Primitive Equations. Accepted for publication in *Diff. Int. Eq.*

[4] F. Guillén-González, N. Masmoudi & M. A. Rodríguez-Bellido. Anisotropic estimates and strong solutions of the Primitive Equations. *Diff. Int. Eq.*, **14**, 11, (2001), 1381-1408.

[5] J. L. Lions, R. Temam & S. Wang. New formulation of the primitive equations of the atmosphere and applications. *Nonlinearity*, **5**, (1992), 237-288.

[6] J. L. Lions, R. Temam & S. Wang. On the equations of the large scale Ocean. *Nonlinearity*, **5**, (1992), 1007-1053.

[7] P. L. Lions. Mathematical topics in fluid mechanics, Vol. 1: Incompressible models. *The Clarendon Press Oxford University Press*, New York, 1996.