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On time-harmonic Maxwell's equations in Lipschitz and Multiply-connected domains of $\mathbb{R}^3$.

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Abstract

In this paper we deal with time-harmonic Maxwell's equations in Lipschitz and multiply connected bounded regions of $\mathbb{R}^3$. We prove the wellposedness of the current source problem by means of an appropriate compact operator.

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AMS Classification:

1 Preliminaries.

The harmonic magnetic field $\boldsymbol{H}$ in a cavity Ω of $\mathbb{R}^3$ is described by curl-curl system

$$\begin{aligned} \mathbf{curl}\,(\epsilon^{-1}\mathbf{curl}\,\boldsymbol{u}) - \omega^2\mu\boldsymbol{u} &= \mathbf{curl}\,(\epsilon^{-1}\boldsymbol{j}), \\ \mathrm{div}\,(\mu\boldsymbol{u}) &= 0. \end{aligned} \tag{1}$$

where $\boldsymbol{j}$ is the imposed source of electric current density. The parameters ϵ and μ refer to the premittivity and the permeability of the medium. For a perfect conducting boundary $\partial\Omega$, the magnetic field satisfies the boundary condition

$$\mu\boldsymbol{u}.\boldsymbol{n}|_{\partial\Omega} = 0. \tag{2}$$

Note that the electric field is given by $\boldsymbol{E} = (i\omega\epsilon)^{-1}(\mathbf{curl}\,\boldsymbol{u} - \boldsymbol{j})$. When the domain is smooth, the analysis of the time harmonic Maxwell's equations has been carried through successfully by means of the Maxwell operator (see, e. g., [7], [3]). However, when the

domain is non-smooth, namely if Ω contains inward edges and corners, the treatment of time-harmonic Maxwell's equations involves some serious complications. This is due mainly to the appearance of singularities near these corners and edges (see [2]).
The purpose of this paper is to treat the current source problem (1)+(2) in a non-smooth and multiply connected domains of $\mathbb{R}^3$. The approach we use for solving (1) is based on a formulation of this problem in terms of an adequate compact vector potential operator.

Let Ω be a bounded open set of $\mathbb{R}^3$ and denote by $\partial\Omega$ its boundary. We assume that Ω is Lipschitz-continuous and that its boundary $\partial\Omega$ is the union of $p+1$ connected components Γ_0,...,Γ_p where Γ_0 is the boundary of the only unbounded connected component of $\mathbb{R}^3/\Omega$. Note that $p = 0$ when $\partial\Omega$ is connected. We assume also that Ω is connected but not necessarily simply-connected. If Ω is multiply-connected, we suppose that there exists m smooth surfaces $\Sigma_1, ..., \Sigma_m$ ("cuts") such that

1. For any $i \in \{1, ..., m\}$, Σ_i is an open part of a smooth manifold $\mathcal{M}_i$.
2. For any $i \in \{1, ..., m\}$, the boundary of Σ_i is contained in $\partial\Omega$.
3. The intersection $\bar{\Sigma}_i \cap \bar{\Sigma}_j$ is empty if $i \neq j$.
4. The open set $\overset{\circ}{\Omega} = \Omega / \bigcup_{i=1}^{m} \Sigma_i$ is simply connected and pseudo-Lipschitz[1].

By convention, we set $m = 0$ when Ω is simply-connected. In the sequel we denote by $(.,.)$ the scalar product in $L^2(\Omega)$. For any $i \leq m$, $H^{1/2}(\Sigma_i)$ is the space of restrictions to Σ_i of the distributions belonging to $H^{\frac{1}{2}}(\mathcal{M}_i)$ and $H^{1/2}(\Sigma_i)'$ is its dual space.
Now, consider the spaces

$$\begin{aligned} H(\mathrm{div};\,\Omega) &= \{\boldsymbol{v} \in L^2(\Omega)^3 \mid \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\}, \\ H(\mathrm{curl};\,\Omega) &= \{\boldsymbol{v} \in L^2(\Omega)^3 \mid \mathbf{curl}\,\boldsymbol{v} \in L^2(\Omega)^3\}, \end{aligned}$$

equipped with the usual norms $\|\boldsymbol{v}\|_{H(\mathrm{div};\,\Omega)}$ and $\|\boldsymbol{v}\|_{H(\mathrm{curl};\,\Omega)}$. We recall the following properties of these spaces

1. Let $\boldsymbol{v} \in H(\mathrm{div};\,\Omega)$. Then, $\boldsymbol{v}$ has a normal component $\boldsymbol{v}.\boldsymbol{n}$ in $H^{-1/2}(\partial\Omega)$ and the following Green's formula holds

$$\forall \varphi \in H^1(\Omega), \quad (\boldsymbol{v}, \nabla\varphi) = -(\mathrm{div}\,\boldsymbol{v}, \varphi) + \langle \boldsymbol{v}.\boldsymbol{n}, \varphi\rangle_{\partial\Omega}. \tag{3}$$

Moreover, for any $i \in \{1, ..., m\}$, $\boldsymbol{v}$ has also a normal component $\boldsymbol{v}.\boldsymbol{n}$ in $H^{1/2}(\Sigma_i)'$ and (see [1], Lemma 3.10):

$$\forall \theta \in H^1(\overset{\circ}{\Omega}), \quad \int_{\overset{\circ}{\Omega}} \boldsymbol{v}.\nabla\theta d\boldsymbol{x} + \int_{\overset{\circ}{\Omega}} (\mathrm{div}\,\boldsymbol{v})\,\theta d\boldsymbol{x} = \sum_{i=1}^{m} \langle \boldsymbol{v}.\boldsymbol{n}, [\theta]_i\rangle_{\Sigma_i}, \tag{4}$$

where $[\theta]_i$ denotes the jump of θ through Σ_i.

[1]see [1] for the definition.

2. Similarly, if $\boldsymbol{v} \in H(\mathrm{curl};\Omega)$, then $\boldsymbol{v}$ has a tangential component $\boldsymbol{v} \times \boldsymbol{n}$ in $H^{-1/2}(\partial\Omega)^3$ and the following Green's formula holds

$$\forall \boldsymbol{w} \in H^1(\Omega)^3, \quad (\mathbf{curl}\,\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{w}) + \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\Omega}. \tag{5}$$

Observe that this formula remains valid if $\boldsymbol{w} \in H(\mathrm{curl};\Omega)$ and $\boldsymbol{v} \in H_0(\mathrm{curl};\Omega)$.

Consider also the following subspaces of $H(\mathrm{div};\Omega)$ and $H(\mathrm{curl};\Omega)$:

$$\begin{aligned} H_0(\mathrm{div};\Omega) &= \{\boldsymbol{v} \in H(\mathrm{div};\Omega) \mid \ \boldsymbol{v}.\boldsymbol{n} = 0 \text{ on } \Gamma\}, \\ H_0(\mathrm{curl};\Omega) &= \{\boldsymbol{v} \in H(\mathrm{curl};\Omega) \mid \ \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \Gamma\}. \end{aligned}$$

We introduce now the spaces

$$\begin{aligned} Y_T(\Omega) &= H_0(\mathrm{div};\Omega) \cap H(\mathrm{curl};\Omega), \\ Y_N(\Omega) &= H(\mathrm{div};\Omega) \cap H_0(\mathrm{curl};\Omega), \end{aligned}$$

equipped with the norm $\|\boldsymbol{v}\|_Y = (\|\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathrm{div}\,\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega}^2)^{1/2}$, and we set

$$\begin{aligned} G_T &= \{\boldsymbol{v} \in Y_T(\Omega) |\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0}\}, \\ G_N &= \{\boldsymbol{v} \in Y_N(\Omega) \mid \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0}\}. \end{aligned}$$

Lemma 1 ([4], [1]). *The spaces G_T and G_N are finite dimensional and dim $G_T = m$, dim $G_N = p$. Moreover, there exists a basis $(\boldsymbol{q}_i)_{i=1,\dots,m}$ (resp. $(\boldsymbol{f}_i)_{i=1,\dots,p}$) of G_T (resp. of G_N) such that:*

$$\forall i,\, j \in \{1,...,m\} \qquad \langle \boldsymbol{q}_i.\boldsymbol{n}, 1\rangle_{\Sigma_j} = \delta_{i,j}, \quad \forall i,\, j \in \{1,...,p\} \qquad \langle \boldsymbol{f}_i.\boldsymbol{n}, 1\rangle_{\Gamma_j} = \delta_{i,j}. \tag{6}$$

We shall denote by $\mathcal{P}_T$ (resp. $\mathcal{P}_N$) the orthogonal projection from $Y_T(\Omega)$ (resp. from $Y_N(\Omega)$) on G_T (resp. on G_N) with respect to inner product associated with the norm $\|.\|_Y$. It is worth noting that

$$\mathcal{P}_N \boldsymbol{v} = \sum_{i=1}^{m} \langle \boldsymbol{v}.\boldsymbol{n}, 1\rangle_{\Sigma_i} \boldsymbol{q}_i$$

for any $\boldsymbol{v} \in L^2(\Omega)^3$ such that $\mathrm{div}\,\boldsymbol{v} = 0$ (see [4], [1]).

Lemma 2 ([4], [1]). *The mapping*

$$\boldsymbol{v} \longrightarrow |\boldsymbol{v}|_{Y_T(\Omega)} = (\|\mathrm{div}\,\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega}^2 + \sum_{i=1}^{m} |\langle \boldsymbol{v}.\boldsymbol{n}, 1\rangle_{\Sigma_i}|^2)^{1/2},$$

is a norm on the space $Y_T(\Omega)$ equivalent to the norm $\|.\|_Y$. Similarly, the mapping $\boldsymbol{v} \longrightarrow |\boldsymbol{v}|_{Y_N(\Omega)} = (\|\mathrm{div}\,\boldsymbol{v}\|_{0,\Omega}^2 + \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega}^2 + \sum_{i=1}^{p} |\langle \boldsymbol{v}.\boldsymbol{n}, 1\rangle_{\Gamma_i}|^2)^{1/2}$, is a norm on the space $Y_N(\Omega)$ equivalent to the norm $\|.\|_Y$.

In the sequel, we set

$$\alpha_0 = \inf_{\boldsymbol{v} \in Y_T(\Omega),\ \boldsymbol{v} \neq \mathbf{0}} \frac{|\boldsymbol{v}|_{Y_T(\Omega)}}{\|\boldsymbol{v}\|_{0,\Omega}}. \tag{7}$$

Then, according to Lemma 2, we have $\alpha_0 > 0$.

1.1 Statement of the problem. The main result.

Let us consider the system: *given* $\boldsymbol{j} \in L^2(\Omega)^3$, *we look for* $\boldsymbol{u} \in Y_T(\Omega)$

$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} - k^2\boldsymbol{u} &= \mathbf{curl}\,\boldsymbol{j}, & (8)\\ \operatorname{div}\boldsymbol{u} &= 0, & (9)\\ \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n}|_{\partial\Omega} &= \boldsymbol{j}\times\boldsymbol{n}, & (10) \end{aligned}$$

where k is the wave number given by $k = \sqrt{\epsilon\mu}\,\omega$ with ϵ and μ supposed non-negative and constants. Observe that the boundary condition (10) is meaningfull if $\boldsymbol{j} \in H(\mathbf{curl}\,,\Omega)$ (thus $\mathbf{curl}\,\boldsymbol{u} \in H(\mathbf{curl}\,,\Omega)$). If $\boldsymbol{j}$ belongs only to $L^2(\Omega)^3$, we interpret the problem (8)-(10) in a weaker form; a vector field $\boldsymbol{u}$ in $Y_T(\Omega)$ is called a *generalized* or *a weak* solution of (8)-(10) if it satisfies

$$(\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})+\gamma(\operatorname{div}\boldsymbol{u},\operatorname{div}\boldsymbol{v})+\delta(\mathcal{P}_T\boldsymbol{u},\mathcal{P}_T\boldsymbol{v})-k^2(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{j},\mathbf{curl}\,\boldsymbol{v}),\ \forall\boldsymbol{v}\in Y_T(\Omega),\quad (11)$$

where γ and δ are two nonnegative real constants. The following proposition state the relationship between the weak problem (11) and the continuous problem (8):

Proposition 1. *Let* $\boldsymbol{j} \in L^2(\Omega)^3$ *and suppose that* $k > 0$ *and that* γ *and* δ *are such that:* $\gamma > 0$, $\delta > 0$ *and*

$$\frac{k^2}{\gamma} \notin EV(\Delta^{neu}),\ \frac{k^2}{\delta} \neq 1, \quad (12)$$

where $EV(\Delta^{neu})$ *is the set of eigenvalues of the Laplace operator with an homogenous Neumann condition. Then, any solution of (11) satisfies (8) and (9) in the sense of distributions. Moreover, if* $\boldsymbol{j}$ *belongs to* $H(\mathbf{curl}\,;\Omega)$*, then the problems (11) and (8)-(10) are equivalent.*

When the wave number k is smaller than the parameter α_0 defined by (7), the existence and the uniqueness of solutions of (11) stem immediately from Lax-Milgram theorem. Here, we treat the problem (11) when k is not necessarily small. We state the following

Theorem 1. *Assume that* $\boldsymbol{j} \in L^2(\Omega)^3$ *and that (12) is fullfilled. Then, there exists a countable sequence of real values* $\{\alpha_i, i \in \mathbb{N}\}$*, tending to* $+\infty$ *such that*

1. *If* $k \notin \{\alpha_i, i \in \mathbb{N}\}$ *then (11) admits one and only one solution* $\boldsymbol{u} \in Y_T(\Omega)$.

2. *If* $k = \alpha_m$ *for some* $m \in \mathbb{N}$*, then the homogeneous problem (when* $\boldsymbol{j} = \mathbf{0}$*) admits a finite dimensional space* E_m *of solutions, and (11) is solvable in* $Y_T(\Omega)$ *iff*

$$(\boldsymbol{j},\mathbf{curl}\,\boldsymbol{\varphi}) = 0,\ \ \forall\boldsymbol{\varphi}\in E_m. \quad (13)$$

If this condition is fulfilled, the solution of (11) is unique up to elements of E_m.

We state also the following regularity results when the domain has a smooth boundary and when it is a parallelepiped (as involved by pseudo-spectral and spectral methods). Note that the general case of a polygonal domain contains some technical complications, due to the appearance of the singularities, and which are beyond the scope of this paper (see, e. g., [2]) (observe that the inclusion $Y_T(\Omega) \subset H^1(\Omega)^3$ does not hold in general).

Corollary 1. *Assume that Ω is of class $\mathcal{C}^{m,1}$ with $m \geq 2$ and let $\boldsymbol{j} \in L^2(\Omega)^3$ such that*

$$\mathbf{curl}\,\boldsymbol{j} \in H^{m-2}(\Omega)^3,\ \boldsymbol{j} \times \boldsymbol{n} \in H^{m-3/2}(\partial\Omega)^3.$$

Then, the solution $\boldsymbol{u}$ of (11) belongs to $H^m(\Omega)^3$.

Corollary 2. *Assume that Ω is a rectangular parallelepiped of $I\!R^3$. Suppose that $\boldsymbol{j} \in H(\mathbf{curl}\,;\,\Omega)$ and satifies $\boldsymbol{j} \times \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$. Then, the solution of the problem (11) belongs to $H^2(\Omega)^3$.*

Proof of Theorem 1.

The proof of Theorem 1 is composed of four steps. In step 1 we introduce and study a new operator. Step 2 deals with its adjoint operator. In the third step we rewrite the problem in a Fredholm form. The Fredholm's alternative is finally applied in step 4.

Step 1. An operator.

Consider the closed subspace of $H(\mathrm{div}\,;\Omega)$

$$X = \{\boldsymbol{v} \in L^2(\Omega)^3 \mid \ \mathrm{div}\,\boldsymbol{v} = 0 \text{ and } \langle \boldsymbol{v}.\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0,\ \ 1 \leq i \leq p\}. \tag{14}$$

For any vector function $\boldsymbol{w}$ in X consider the problem: *Find $\boldsymbol{z} \in Y_T(\Omega)$ such that*

$$\mathbf{curl}\,\boldsymbol{z} = \boldsymbol{w},\ \ \mathrm{div}\,\boldsymbol{z} = 0,\ \ \forall i \in \{1, ..., m\}\ \ \ \langle \boldsymbol{z}.\boldsymbol{n}, 1\rangle_{\Sigma_i} = 0. \tag{15}$$

Lemma 3 ([1]). *The problem (15) has a unique solution $\boldsymbol{z} \in Y_T(\Omega)$ and there exists a constant C, depending only on Ω such that*

$$\|\boldsymbol{z}\|_{Y_T(\Omega)} \leq C(\Omega)\|\boldsymbol{w}\|_{0,\Omega}. \tag{16}$$

In the sequel, we shall denote by $\boldsymbol{\mathcal{K}}$ the linear and continuous operator from X into X defined by

$$\boldsymbol{\mathcal{K}}:\ \ \boldsymbol{w} \in X \mapsto \boldsymbol{z} \in X \text{ solution of (15)},$$

Lemma 4. *$\boldsymbol{\mathcal{K}}$ is a compact operator.*

Proof of Lemma 4 – For proving the compactness of $\boldsymbol{\mathcal{K}}$, the following lemma turns to be useful. The reader can consult [5] (Theorem 3.1) for the proof.

Lemma 5. *A function $\boldsymbol{w}$ in $L^2(\Omega)^3$ belongs to X if and only if there exists a vector function $\boldsymbol{\varphi}$ in $H^1(\Omega)^3$ satisfying $\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{\varphi}$. Moreover, there exists a constant C depending only on Ω such that for any $\boldsymbol{\varphi} \in X$, the corresponding vector function $\boldsymbol{v}$ can be chosen such that*

$$\|\boldsymbol{\varphi}\|_{H^1(\Omega)^3} \leq C\|\boldsymbol{w}\|_{0,\Omega}.$$

Now, let $\boldsymbol{w}_n$ be a sequence in X such that $\|\boldsymbol{w}_n\|_{0,\Omega} \leq C_1$, where C_1 is a constant not depending on n. Then, by virtue of Lemma 5, there exists a sequence $\boldsymbol{\varphi}_n$ in $H^1(\Omega)^3$ such that: $\forall n$, $\mathbf{curl}\,\boldsymbol{\varphi}_n = \boldsymbol{w}_n$, $\|\boldsymbol{\varphi}_n\|_{1,\Omega} \leq C$. Thus, there exists a subsequence still denoted by $\boldsymbol{\varphi}_n$ which converges strongly in $L^2(\Omega)^3$.
Now, for any n, let s_n be the unique solution in $H^1(\Omega)/I\!R$ of the Neumann problem

$$\forall \Psi \in H^1(\Omega)/I\!R, \quad \int_\Omega \nabla s_n . \nabla \Psi d\boldsymbol{x} = \int_\Omega \boldsymbol{\varphi}_n . \nabla \Psi d\boldsymbol{x}$$

and set $\boldsymbol{\varphi}_n^* = \tilde{\boldsymbol{\varphi}}_n - \mathcal{P}_T \tilde{\boldsymbol{\varphi}}_n$, where $\tilde{\boldsymbol{\varphi}}_n = \boldsymbol{\varphi}_n - \nabla \mathbf{s}_n$. The sequence $\tilde{\boldsymbol{\varphi}}_n$ belongs to $Y_T(\Omega)$. Moreover, it is quite obvious that $(s_n)_n$ converges in $H^1(\Omega)^3/I\!R$. Thus, $\tilde{\boldsymbol{\varphi}}_n$ converges in $L^2(\Omega)^3$ to an element $\tilde{\boldsymbol{\varphi}}$ of $Y_T(\Omega)$. Moreover, $\mathcal{P}_T \tilde{\boldsymbol{\varphi}}_n$ converges also to $\mathcal{P}_T \tilde{\boldsymbol{\varphi}}$ since

$$\|\mathcal{P}_T \tilde{\boldsymbol{\varphi}}_n\|_{0,\Omega} \leq \|\tilde{\boldsymbol{\varphi}}_n\|_{0,\Omega}.$$

We conclude by observing that $\tilde{\boldsymbol{\varphi}}_n^* = \boldsymbol{\mathcal{K}} \boldsymbol{w}_n$. $\diamond$

Step 2. The adjoint operator.

We need the following lemma

Lemma 6 ([1], [4]). *A field $\boldsymbol{v}$ in $H(\mathrm{div}\,;\Omega)$ satisfies*

$$\mathrm{div}\,\boldsymbol{v} = 0, \ \boldsymbol{v}.\boldsymbol{n} = 0 \ \textit{on } \partial\Omega, \ \langle \boldsymbol{v}.\boldsymbol{n}, 1\rangle_{\Sigma_i} = 0, \ i = 1, ..., m,$$

if and only if there exists a unique vector potential $\boldsymbol{\Phi} \in Y_N(\Omega)$ such that

$$\mathbf{curl}\,\Phi = \boldsymbol{v}, \ \mathrm{div}\,\boldsymbol{\Phi} = 0, \ \langle \boldsymbol{\Phi}.\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0, \ i = 1, ..., p. \tag{17}$$

In particular, this lemma implies that any vector field $\boldsymbol{w}$ in $L^2(\Omega)^3$ admits a unique decomposition into the form

$$\boldsymbol{w} = \overset{\circ}{\nabla} q + \mathbf{curl}\,\boldsymbol{\Phi}, \tag{18}$$

where $\boldsymbol{\Phi}$ belongs to $Y_N(\Omega)$ and verifies $\mathrm{div}\,\boldsymbol{\Phi} = 0$, $\langle \boldsymbol{\Phi}.\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$, $0 \leq i \leq p$, while q belongs to the space $\Theta = \{s \in H^1(\overset{\circ}{\Omega}) \mid [s]_{\Sigma_i} = \textit{constant}, \ 1 \leq i \leq m\}$, and is the unique solution in $\Theta/I\!R$ of the quasi-Neumann problem

$$\forall p \in \Theta, \ \int_{\overset{\circ}{\Omega}} \nabla s . \nabla p \, d\boldsymbol{x} = \int_\Omega \boldsymbol{w} . \overset{\circ}{\nabla} p d\boldsymbol{x},$$

where $\overset{\circ}{\nabla} p$ denotes the extension in $L^2(\Omega)^3$ of the gradient ∇p considered in the sense of distributions in $\mathcal{D}'(\overset{\circ}{\Omega})$. Moreover, the decomposition (18) is unique in $(\Theta/\mathbb{R}) \times Y_N(\Omega)$. The operator $\boldsymbol{\mathcal{K}}^*$ is defined as follows

$$\boldsymbol{\mathcal{K}}^* : \boldsymbol{w} \in L^2(\Omega)^3 \mapsto \boldsymbol{\Phi} \in X,$$

where $\boldsymbol{\Phi}$ is the unique fonction in the decomposition (18). $\boldsymbol{\mathcal{K}}^*$ is a continuous operator from $L^2(\Omega)$ into X. The following lemma gives the relationship between $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{K}}^*$:

Lemma 7. *The restriction of $\boldsymbol{\mathcal{K}}^*$ to X is the adjoint operator of $\boldsymbol{\mathcal{K}}$.*

STEP 3. A NEW FORMULATION OF THE PROBLEM

Let us now rewrite the problem (11) in terms of the operator $\boldsymbol{\mathcal{K}}$.

Proposition 2. *Let $\boldsymbol{j} \in L^2(\Omega)^3$ and let $\theta \in H_0^1(\Omega)$ be solution of the Dirichlet problem*

$$\Delta\theta = \operatorname{div} \boldsymbol{j} \in H^{-1}(\Omega), \ \theta = 0 \ \text{on } \Gamma.$$

We set $\boldsymbol{j}_1 = \boldsymbol{j} - \nabla\theta \in H(\operatorname{div}; \Omega)$, $\boldsymbol{j}^* = \boldsymbol{j}_1 - \mathcal{P}_N \boldsymbol{j}_1$. Then, $\boldsymbol{u}$ is solution of (11) iff $\hat{\mathbf{u}} = \boldsymbol{u} - \boldsymbol{\mathcal{K}}\boldsymbol{j}^*$ belongs to X and is solution of the problem

$$\hat{\mathbf{u}} - k^2 \boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^* \hat{\mathbf{u}} = k^2 \boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^* \boldsymbol{\mathcal{K}} \boldsymbol{j}^*. \tag{19}$$

Proof of Proposition 2– Firstly, observe that if we set $\boldsymbol{\ell} = \boldsymbol{j} - \boldsymbol{j}^* = \nabla\theta + \mathcal{P}_T \boldsymbol{j}_1$, then $\boldsymbol{\ell} \in H(\mathbf{curl}; \Omega)$ and $\mathbf{curl}\, \boldsymbol{\ell} = \mathbf{0}$, $\boldsymbol{\ell} \times \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$.

1. Let $\boldsymbol{u}$ solution of (11). Then, it stems from Proposition 1 that $\boldsymbol{u}$ satisfies (8) and (9) and $\mathcal{P}_T \boldsymbol{u} = \mathbf{0}$. We set $\hat{\mathbf{u}} = \boldsymbol{u} - \boldsymbol{\mathcal{K}}\boldsymbol{j}^*$. It follows immediately that $\hat{\mathbf{u}}$ belongs to $X \cap Y_T(\Omega)$ and
$$\mathbf{curl}\,\mathbf{curl}\,\hat{\mathbf{u}} - k^2 \hat{\mathbf{u}} = k^2 \boldsymbol{\mathcal{K}}\boldsymbol{j}^*, \ \ \mathcal{P}_T \hat{\mathbf{u}} = \mathbf{0}. \tag{20}$$
Thus, $\mathbf{curl}\,\hat{\mathbf{u}}$ belongs to $H(\mathbf{curl}; \Omega)$. Furthermore, (11) yields
$$(\mathbf{curl}\,\hat{\mathbf{u}}, \mathbf{curl}\,\boldsymbol{v}) - k^2(\hat{\mathbf{u}}, \boldsymbol{v}) = k^2(\boldsymbol{\mathcal{K}}\boldsymbol{j}^*, \boldsymbol{v}) + (\boldsymbol{\ell}, \mathbf{curl}\,\boldsymbol{v}), \ \forall \boldsymbol{v} \in Y_T(\Omega).$$
Choosing $\boldsymbol{v} \in H^1(\Omega)^3$ gives $\langle \mathbf{curl}\,\hat{\mathbf{u}} \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\Omega} = 0$. Thus $\mathbf{curl}\,\hat{\mathbf{u}} \times \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$. It follows that $\mathbf{curl}\,\hat{\mathbf{u}} = \boldsymbol{\mathcal{K}}^*(k^2 \hat{\mathbf{u}} + k^2 \boldsymbol{\mathcal{K}}\boldsymbol{j}^*)$. Moreover, $\hat{\mathbf{u}} = k^2 \boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*(\hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^*)$.

2. Conversly, let $\hat{\mathbf{u}}$ be solution of (19). Then,
$$\mathbf{curl}\,\hat{\mathbf{u}} = k^2 \mathbf{curl}\,(\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*(\hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^*)) = k^2 \boldsymbol{\mathcal{K}}^*(\hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^*).$$
Thus,
$$\begin{aligned}(\mathbf{curl}\,\hat{\mathbf{u}}, \mathbf{curl}\,\boldsymbol{v}) &= k^2(\boldsymbol{\mathcal{K}}^*(\hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^*), \mathbf{curl}\,\boldsymbol{v}) = k^2(\mathbf{curl}\,(\boldsymbol{\mathcal{K}}^*(\hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^*)), \boldsymbol{v}) \\ &= k^2(\hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^*, \boldsymbol{v}) = k^2(\boldsymbol{u}, \boldsymbol{v}),\end{aligned}$$
since $\mathbf{curl}\,(\boldsymbol{\mathcal{K}}^*(\hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^*)) = \hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^*$. Hence, $\boldsymbol{u} = \hat{\mathbf{u}} + \boldsymbol{\mathcal{K}}\boldsymbol{j}^* \in Y_T(\Omega)$ satisfies $\operatorname{div} \boldsymbol{u} = 0$, $\mathcal{P}_T \boldsymbol{u} = \mathbf{0}$, and $(\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{v}) - k^2(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{j}^*, \mathbf{curl}\,\boldsymbol{v}) = (\boldsymbol{j}, \mathbf{curl}\,\boldsymbol{v})$. Thus, $\boldsymbol{u}$ is solution of (11) which is the desired result. $\diamond$

Step 4. Fredholm alternative.

Consider the operator $T = \boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*$. Then, T is obviously self-adjoint and is compact by virtue of Lemma 4. Let $s_1^2 \geq s_2^2 \geq ... \geq s_n^2 \geq ...$ be the real countable sequence of its eigenvalues. The numbers $s_1, s_2, ..., s_n, ...$ are indeed the *s-values* (or *singular values*) of the operator $\boldsymbol{\mathcal{K}}$ (namely, the eigenvalues of $(\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*)^{\frac{1}{2}}$). These numbers are in general different from the eigenvalues of $\boldsymbol{\mathcal{K}}$ since it is not a normal operator. The reader can consult [6] for more details about that question.

Now, applying the Fredholm alternative to the inhomogeneous problem (19) yields

- If $\dfrac{1}{k} \notin \{s_1, s_2, ...\}$, then the (19) admits one and only one solution.
- If $\dfrac{1}{k} = s_m$ for some $m \in \{1, 2, ..\}$, then (19) is solvable iff the right hand side verifies

$$(\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*\boldsymbol{\mathcal{K}}\boldsymbol{j}^*, \boldsymbol{\varphi}) = 0, \tag{21}$$

for any $\boldsymbol{\varphi}$ satisfying $\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*\boldsymbol{\varphi} = s_k^2\boldsymbol{\varphi}$. If this solvability condition is fulfilled, then (19) has a unique solution up to eigenfunctions of T corresponding to the eigenvalue s_m^2. Let us rewrite this solvability condition (21) differently. We have

$$\begin{aligned} 0 &= (\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*\boldsymbol{\mathcal{K}}\boldsymbol{j}^*, \boldsymbol{\varphi}) = (\boldsymbol{j}^*, \boldsymbol{\mathcal{K}}^*\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*\boldsymbol{\varphi}) = s_m^2(\boldsymbol{j}^*, \boldsymbol{\varphi}) = s_m^2(\boldsymbol{j} - \boldsymbol{\ell}, \boldsymbol{\mathcal{K}}^*\boldsymbol{\varphi}) \\ &= s_m^4(\boldsymbol{j} - \boldsymbol{\ell}, \mathbf{curl}\,\boldsymbol{\varphi}) = s_m^4(\boldsymbol{j}, \mathbf{curl}\,\boldsymbol{\varphi}). \end{aligned}$$

since $s_m^2\mathbf{curl}\,\boldsymbol{\varphi} = \mathbf{curl}\,(\boldsymbol{\mathcal{K}}\boldsymbol{\mathcal{K}}^*\boldsymbol{\varphi}) = \boldsymbol{\mathcal{K}}^*\boldsymbol{\varphi}$ and $\mathbf{curl}\,\boldsymbol{\ell} = \mathbf{0}$. This ends the proof of Theorem 1. $\diamond$

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