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Convex Extensions of Convex Functions by Sequential Processes

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Abstract

The aim of this paper is to extend a real-valued convex function f into a real-valued convex function $\widehat{f}$, defined on a convex subset of the closure of the domain of f. When f is sequentially lower semi-continuous we study whether $\widehat{f}$ is sequentially lower semi-continuous. The extended function $\widehat{f}$ is constructed by a sequential process.

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1 Extensions by sequential processes

In this section we consider a topological space X, a nonempty subset $C \subset X$ and a function $f : C \to \mathbb{R}$. Our first objective is to construct an extension of f by a sequential process whenever f satisfies some lower semi-continuity properties.

In the sequel we consider a subset $\mathcal{S} \subset X^{\mathbb{N}}$ which contains all constant sequences; some of the notions introduced in this paper depend on the choice of $\mathcal{S}$. We define the subset

$$\widehat{C} := \widehat{C}(\mathcal{S}, f) = \left\{x \in X : \exists (x_n) \in \mathcal{S} \cap C^{\mathbb{N}},\ x_n \to x,\ \liminf f(x_n) < +\infty\right\}.$$

For each point $x \in X$ we define the subset

$$S(x, C, \mathcal{S}) = \left\{(x_n) \in X^{\mathbb{N}} : (x_n) \in \mathcal{S} \cap C^{\mathbb{N}},\ x_n \to x\right\}$$

and we set $\check{C} := \check{C}(\mathcal{S}) = \{x \in X : S(x, C, \mathcal{S}) \neq \emptyset\}$.

Obviously $C \subset \widehat{C} \subset \check{C}$; for each x in $\widehat{C}$ we define

$$\widehat{f}(x) = \inf_{(x_n) \in S(x,C,\mathcal{S})} \liminf f(x_n).$$

We first state some elementary properties of $\widehat{f}$.

Proposition 1 *$\widehat{f}$ is a function from $\widehat{C}$ to $[-\infty, +\infty[$ such that $\widehat{f}_{|C} \leq f$. Moreover $\inf_C f = \inf_C \widehat{f} = \inf_{\widehat{C}} \widehat{f}$; in particular, $\widehat{f}$ is a function from $\widehat{C}$ to $\mathbb{R}$ provided that f is lower bounded.*

Proof. By construction of $\widehat{C}$ and $\widehat{f}$ it is obvious that $\widehat{f}$ never takes the value $+\infty$ on $\widehat{C}$. By considering the constant sequences of C we get $\widehat{f}_{|C} \leq f$. In particular we get $\inf_C \widehat{f} \leq \inf_C f$. By the definition of $\widehat{f}$ we can conclude that $\widehat{f}(x) \geq \inf_C f$ for each $x \in \widehat{C}$. Thus $\inf_C f \leq \inf_{\widehat{C}} \widehat{f} \leq \inf_C \widehat{f}$ which finishes the proof. $\square$

We say that f is $\mathcal{S}-$*lower semi-continuous* ($\mathcal{S}-$l.s.c.) at x if for each $(x_n) \in S(x, C, \mathcal{S})$ we have $\liminf f(x_n) \geq f(x)$; f is said to be $\mathcal{S}-$lower semi-continuous on C when f is $\mathcal{S}-$l.s.c. at each point of C. When $\mathcal{S}$ is the set of all sequences in X, $\mathcal{S}-$lower semi-continuity coincides with *sequential lower semi-continuity* (s.l.s.c.).

We conclude directly that f is $\mathcal{S}-$l.s.c. at $x \in C$ iff $f(x) \leq \widehat{f}(x)$; f is $\mathcal{S}-$l.s.c. on C iff $f \leq \widehat{f}_{|C}$. We can assert:

Proposition 2 *If f is $\mathcal{S}-$l.s.c. on C then $\widehat{f}$ is a function from $\widehat{C}$ to $[-\infty, +\infty[$ such that $\widehat{f}_{|C} = f$.*

We say that a function g from D to $[-\infty, +\infty[$ is $\mathcal{S}-$*l.s.c. on D relatively to C* if (i) $C \subset D \subset \check{C}$, (ii) for each $x \in D$ and $(x_n) \in S(x, C, \mathcal{S})$ we have $\liminf g(x_n) \geq g(x)$.

Proposition 3 *Assume that f is $\mathcal{S}-$l.s.c. on C. Then $\widehat{f}$ is $\mathcal{S}-$l.s.c. on $\widehat{C}$ relatively to C. Moreover, if $g : D \supset C \to [-\infty, +\infty[$ is an extension of f which is $\mathcal{S}-$l.s.c. on D relatively to C then $g_{|D \cap \widehat{C}} \leq \widehat{f}_{|D \cap \widehat{C}}$.*

Proof. For each $x \in \widehat{C}$ and each $(x_n) \in S(x, C, \mathcal{S})$ we have $\liminf \widehat{f}(x_n) = \liminf f(x_n)$ since $\widehat{f}_{|C} = f$. By the definition of $\widehat{f}(x)$ we also get $\liminf f(x_n) \geq \widehat{f}(x)$. Let $g : D \supset C \to [-\infty, +\infty[$ be an extension of f which is $\mathcal{S}-$l.s.c. on D relatively to C and let $x \in D \cap \widehat{C}$. For each $(x_n) \in S(x, C, \mathcal{S})$, we have $g(x) \leq \liminf g(x_n) = \liminf f(x_n)$. By definition of $\widehat{f}(x)$ we conclude that $g(x) \leq \widehat{f}(x)$. $\square$

Corollary 4 *If f is $\mathcal{S}-$l.s.c. on C then $\widehat{f}$ is the greatest extension of f on $\widehat{C}$ with values in $[-\infty, +\infty[$ which is $\mathcal{S}-$l.s.c. on $\widehat{C}$ relatively to C.*

In particular, if f is s.l.s.c. on C then $\widehat{f}$ is s.l.s.c. on $\widehat{C}$ relatively to C. In some particular cases, for instance when X is a metric space, we can show that $\widehat{f}$ is s.l.s.c. on $\widehat{C}$ provided that f is s.l.s.c. on C. More generally:

Proposition 5 *If f is s.l.s.c. on C and if each point of $\widehat{C}$ admits a countable base of neighbourhoods for the induced topology on $\widehat{C}$ then $\widehat{f}$ is s.l.s.c. on $\widehat{C}$.*

PROOF. Let $x \in \widehat{C}$ such that $\widehat{f}(x) > -\infty$ and let (z_n) be a sequence of points of $\widehat{C}$ which converges to x; we can consider a subsequence (x_n) s.t. $\liminf \widehat{f}(z_n) = \lim \widehat{f}(x_n)$. Let $\epsilon > 0$ be given. For each integer n, we can consider a sequence $(x_{n,p})_p$ of points of C which converges to x_n such that $(f(x_{n,p}))_p$ converges to a real number between $\widehat{f}(x_n)$ and $\widehat{f}(x_n) + \epsilon/2$; without loss of generality we can assume that $f(x_{n,p}) \leq \widehat{f}(x_n) + \epsilon$ for all p. Consider now a decreasing countable base (V_n) of open neighborhoods of x (for the induced topology on $\widehat{C}$). We set $s(0) = r(0) = 0$; for each integer $n \geq 1$ there exists $s(n)$ s.t. $s(n) > s(n-1)$ and $x_m \in V_n$ whenever $m \geq s(n)$. Since $x_{s(n)}$ belongs to the open set V_n and $(x_{s(n),p})_p$ converges to $x_{s(n)}$ there exists $r(n)$ s.t. $r(n) > r(n-1)$ and $x_{s(n),p} \in V_n$ for all $p \geq r(n)$. In particular, $y_n := x_{s(n),r(n)} \in V_n$ for each integer n and thus (y_n) is a sequence of points of C which converges to x and therefore $\widehat{f}(x) \leq \liminf f(y_n)$. We have $f(y_n) \leq \widehat{f}(x_{s(n)}) + \epsilon$ for each integer n and thus $\liminf f(y_n) \leq \liminf \widehat{f}(x_{s(n)}) + \epsilon$. We conclude that $\widehat{f}(x) \leq \liminf \widehat{f}(z_n) + \epsilon$. We conclude that $\widehat{f}$ is s.l.s.c. at x. $\square$

We note that in the previous proposition we can conclude that $\widehat{f}$ is in fact l.s.c. on $\widehat{C}$. Using a classical result of functional analysis we get:

Corollary 6 *Let X be a normed linear space with a separable dual space and C be a nonempty bounded subset of X. If f is weakly l.s.c. on C then $\widehat{f}$ is weakly l.s.c. on $\widehat{C}$.*

A subset D of X is sequentially compact (resp., relatively sequentially compact) if each sequence of points of D has a subsequence which converges to a point of D (resp., of X). It is well known that a s.l.s.c. function has at least one minimizer on every nonempty sequentially compact subset. In particular, under the assumptions of the previous proposition, $\widehat{f}$ has at least a minimizer on $\widehat{C}$ whenever this set is sequentially compact. It is rather surprising that $\widehat{f}$ admits a minimizer on $\widehat{C}$ when this set is sequentially compact without knowing whether $\widehat{f}$ is s.l.s.c. on $\widehat{C}$ or not.

Proposition 7 *If f is s.l.s.c. on C and if C is relatively sequentially compact then $\widehat{f}$ is a function from $\widehat{C}$ to $[-\infty, +\infty[$ and $\inf_C f = \min_{\widehat{C}} \widehat{f} = \widehat{f}(\widehat{c})$ for some $\widehat{c}$ in $\widehat{C}$.*

PROOF. Consider a sequence $(x_n) \in C$ such that $(f(x_n))$ converges to $\inf_C f$. There exists a subsequence (c_n) which converges to a point $\widehat{c}$ of X. Let us show that $\widehat{c} \in \widehat{C}$: $\widehat{c}$ is limit of $(c_n) \subset C$ and $(f(c_n))$ is upper-bounded since it converges to $\inf\limits_C f \in [-\infty, +\infty[$. Moreover $\liminf f(c_n) \geq \widehat{f}(\widehat{c})$ since $\widehat{f}$ is s.l.s.c. on $\widehat{C}$ relatively to C and $\widehat{f}_{|C} = f$. We get $\widehat{f}(\widehat{c}) \leq \inf\limits_C f$. Since we also have $\inf\limits_C f \leq \inf\limits_{\widehat{C}} \widehat{f}$, the result follows. □

2 Convex extensions

Our objective is to construct a finite convex extension of a finite convex function f defined on C; we will show under weak assumptions that the function $\widehat{f}$ defined on $\widehat{C}$ is a solution of this problem.

In the sequel we assume that X is a topological vector space. We also assume that $\mathcal{S}$ is a subset of the set of sequences of X such that:

(A1) $\mathcal{S}$ contains the subset $\mathcal{S}_0$ of all sequences of X whose image is contained in some line segment,

(A2) if $(a_n) \in \mathcal{S}$ then each of its subsequences belongs to $\mathcal{S}$,

(A3) if $(a_n) \in \mathcal{S}$ and $b \in X$ then $(\theta a_n + (1-\theta)b) \in \mathcal{S}$ for each $\theta \in [0,1]$.

We say that $\mathcal{S}$ is stable by convex combinations when

(A4) if $(a_n) \in \mathcal{S}$ and $(b_n) \in \mathcal{S}$ then $(\theta a_n + (1-\theta)b_n) \in \mathcal{S}$ for each $\theta \in [0,1]$.

We first note that $\mathcal{S}_0$ satisfies assumptions (A1)-(A3) but not (A4). The set of all sequences of X satisfies (A1)-(A4). We keep in mind that the construction of $\widehat{f}$ and $\widehat{C}$ depends on the choice of $\mathcal{S}$.

Recall that $f : C \subset X \to \mathbb{R}$ is a quasi-convex function iff C is a convex subset and for each $(x, y, \theta) \in C \times C \times [0,1]$ we have $f(\theta x + (1-\theta)y) \leq \max(f(x), f(y))$.

We first give a sufficient condition for $\widehat{C}$ to be a convex subset. We say that $D \subset X$ is *convex relatively to* $C \subset X$ if $C \subset D$ and for each $(x, y, \theta) \in D \times C \times [0,1]$ we have $\theta x + (1-\theta)y \in D$.

Proposition 8 *If C is a convex subset and f is a quasi-convex function then the subset $\widehat{C}$ is convex relatively to C. If moreover $\mathcal{S}$ is stable by convex combinations then $\widehat{C}$ is a convex subset.*

Proof. Given $(x, y, \theta) \in \widehat{C} \times \widehat{C} \times [0,1]$ we can consider sequences (x_n) and (y_n) of C belonging to $\mathcal{S}$ and converging respectively to x and y in such a way that (by (A2)) $(f(x_n))$ and $(f(y_n))$ are upper bounded. If assumption (A4) is not satisfied we will assume that in fact $y \in C$ and we will take $y_n = y$. By convexity of C the points $z_n := \theta x_n + (1-\theta)y_n$ are in C and by quasi-convexity of f we have $f(z_n) \leq \max(f(x_n), f(y_n))$. The point $z := \theta x + (1-\theta)y$ is limit of the sequence (z_n) of points of C which belongs to $\mathcal{S}$ (by (A3) or (A4)) and $(f(z_n))$ is upper bounded: $\theta x + (1-\theta)y$ belongs to $\widehat{C}$. □

We will show the convexity of f is preserved by $\widehat{f}$ when (A4) holds. We say that a function $f : D \subset X \to \mathbb{R}$ is *convex relatively to* $C \subset X$ if D is convex relatively to C and if for each $(x, y, \theta) \in D \times C \times [0, 1]$ we have $f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$.

Proposition 9 *Let f be a convex function from C to $\mathbb{R}$. If f is a $\mathcal{S}-$l.s.c. convex function then $\widehat{f}$ is convex relatively to C. If $\mathcal{S}$ is stable by convex combinations then $\widehat{f}$ is a convex function.*

Proof. Let $(x, \theta) \in \widehat{C} \times [0, 1]$ and let $y \in C$ or, when (A4) holds, $y \in \widehat{C}$; the point $z := \theta x + (1 - \theta)y$ is then in $\widehat{C}$. By definition of $\widehat{f}$ (and possible extraction), for any $L \in \left]\widehat{f}(x), +\infty\right[$ and $M \in \left]\widehat{f}(y), +\infty\right[$, there exist two sequences (x_n) and (y_n) of C belonging to $\mathcal{S}$ which converge respectively to x and y and such that $(f(x_n))$ and $(f(y_n))$ converge respectively to some points of $[-\infty, +\infty[$ denoted by λ and μ which satisfy $\widehat{f}(x) \leq \lambda \leq L$ and $\widehat{f}(y) \leq \mu \leq M$. We will take $y_n = y \in C$ and $\mu = f(y)$ when (A4) does not hold; indeed, $f(y) = \widehat{f}(y)$ by Proposition 2. Each point $z_n := \theta x_n + (1 - \theta)y_n$ is in C and satisfies $f(z_n) \leq \theta f(x_n) + (1 - \theta)f(y_n)$ by convexity of f; obviously (z_n) belongs to $\mathcal{S}$ and converges to z and $(f(z_n))$ is upper bounded. We get $\widehat{f}(z) \leq \liminf f(z_n) \leq \theta\lambda + (1 - \theta)\mu \leq \theta L + (1 - \theta)M$. We conclude that $\widehat{f}(z) \leq \theta\widehat{f}(x) + (1 - \theta)\widehat{f}(y)$. □

In fact, convexity of f also implies that $\widehat{f}$ has real values whenever $\widehat{f}$ is an extension of f and C satisfies a regularity property defined below.

The subset of *linearly accessible points from* C is defined by $lina(C) := \{x \in X : \exists c \in C, c \neq x,]x, c[\subset C\}$; we set $lin(C) := C \cup lina(C)$.

We introduce $\widetilde{C} = \widetilde{C}(\mathcal{S}) := \left\{x \in X : \exists (x_n) \in \mathcal{S} \cap C^{\mathbb{N}}, x_n \to x\right\}$. We say that the convex subset C is *$\mathcal{S}-$semi-regular* if $\widetilde{C} = lin(C)$. Let us note that we always have $\widetilde{C} \subset cl(C)$ (the closure of C) and, by (A1), $lin(C) \subset \widetilde{C}$.

Proposition 10 *If f is a $\mathcal{S}-$l.s.c. convex function from C to $\mathbb{R}$ and if C is $\mathcal{S}-$semi-regular then $\widehat{f}$ is a function from $\widehat{C}$ to $\mathbb{R}$.*

Proof. We have to show that $\widehat{f}$ takes finite values on $\widehat{C}$. We can assume that C is not reduced to a point. Let $x \in C$; since $x \in lin(C)$ we can consider a point $a_x \in C \setminus \{x\}$ such that $]a_x, x[\subset C$. Then $b_x := 2^{-1}x + 2^{-1}a_x \in C$ and, by Proposition 9, $\widehat{f}(b_x) \leq 2^{-1}\widehat{f}(x) + 2^{-1}\widehat{f}(a_x)$. By Proposition 2 we get $\widehat{f}(x) \geq 2f(b_x) - f(a_x) > -\infty$. □

In many classical cases, the property of $\mathcal{S}-$semi-regularity is satisfied.

Proposition 11 *Let C be a nonempty convex subset of X.*

(i) C is $\mathcal{S}_0-$semi-regular.

(ii) If X is a finite dimensional space then C is $\mathcal{S}-$semi-regular.

(iii) If $int(C)$ is nonempty then C is $\mathcal{S}-$semi-regular and $\widetilde{C} = cl(C)$.

(iv) If $lin(C)$ is closed then C is $\mathcal{S}-$semi-regular and $\widetilde{C} = cl(C)$.

Proof. (i) Let $x \in \widetilde{C} \backslash C$ (if nonempty); x is limit of a sequence of points of C given by $x_n = (1 - t_n)x + t_n a$ where $a \in X$ and $t_n \downarrow 0$ in $[0, 1]$. We remark that $]x, x_0] = \bigcup_n]x_{n+1}, x_n]$; by convexity $[x_{n+1}, x_n] \subset C$ and thus $]x, x_0] \subset C$: $x \in lina(C)$. We have shown that $\widetilde{C} \subset lin(C)$ which finishes the proof.

(ii) Without loss of generality we can assume that the affine hull of C is X; the result follows from (iii).

(iii) It is well known that $lin(C) = cl(C)$ (see [2, p. 59]); we use (iv).

(iv) Since $C \subset lin(C) \subset \widetilde{C} \subset cl(C) \subset cl(lin(C))$ the conclusion is obvious. □

In infinite dimensions, C can be a $\mathcal{S}-$semi-regular convex subset with an empty interior. Consider the space $X = L^2(]0, 1[)$ and $C := \{f \in X : f(x) > 0$ a.e. $x \in]0, 1[\}$; we have $lin(C) = \widetilde{C} = D$ where $D := \{f \in X : f(x) \geq 0$ a.e. $x \in]0, 1[\}$ and we have $int(C) = int(D) = \emptyset$.

We now introduce a stronger form of the $\mathcal{S}-$semi-regularity condition in order to transmit some properties of f to $\widehat{f}$. We will say that a point c of the convex subset C is a *regular point of* C if $\theta c + (1 - \theta)x \in C$ for any $(\theta, x) \in]0, 1[\times lin(C)$; we denote $reg(C)$ the possibly empty subset of regular points of C. We say that the convex subset C is $\mathcal{S}-$*regular* if C is $\mathcal{S}-$semi-regular (i.e. $\widetilde{C} = lin(C)$) and if there exists a point $c \in C$ such that $\theta c + (1 - \theta)x \in C$ for any $(\theta, x) \in]0, 1[\times lin(C)$ (i.e. $reg(C) \neq \emptyset$).

Let us give an important case for which $reg(C) \neq \emptyset$.

Proposition 12 *Let C be a convex subset of X. If $icor(C)$ is nonempty then C has regular points; moreover $icor(C) \subset reg(C)$.*

Proof. We can assume that $0 \in C$ and $icor(C)$ is the core of C considered as a subset of $span(C)$. We have $\theta\, icor(C) + (1 - \theta)lin(C) \subset icor(C)$ for each $\theta \in]0, 1[$ by [2, p. 10] and thus $icor(C) \subset reg(C)$. □

In general , a convex subset may have regular points but an empty intrinsic core. For example let $C := \{f \in L^2(]0, 1[) : f(x) > 0$ a.e. $x \in]0, 1[\}$. Then $reg(C) = C$, C is $\mathcal{S}-$regular but $icor(C) = \emptyset$ (see Proposition 13 (iv)).

The property of $\mathcal{S}-$regularity is satisfied in many classical situations.

Proposition 13 *Let C be a nonempty convex subset of X.*

(i) If $icor(C)$ is nonempty then C is $\mathcal{S}_0-$regular.

(ii) If X is a finite dimensional space then C is $\mathcal{S}-$regular.

(iii) If $int(C)$ is nonempty then C is $\mathcal{S}-$regular.

(iv) If $lin(C)$ is closed and $icor(C)$ is nonempty then C is $\mathcal{S}-$regular.

Proof. It follows by Propositions 11 and 12 (for (ii) see [2, p. 9]). □

In many cases, the function $\widehat{f}$ can be determined via an elementary calculus.

Proposition 14 *If f is a $\mathcal{S}-$l.s.c. convex function from C to $\mathbb{R}$ and if C is $\mathcal{S}-$regular then for each $(a, x) \in reg(C) \times \widehat{C}$ and (t_n) converging in $]0, 1[$ to 0 then*

$$\widehat{f}(x) = \lim_n f((1 - t_n)x + t_n a).$$

PROOF. The sequence (z_n) defined by $z_n = (1-t_n)x + t_n a$ where $(a, x, t_n) \in reg(C) \times \widehat{C} \times]0, 1[$ is in $\mathcal{S}$ by (A1). Moreover (z_n) is a sequence of points of C; indeed, $(a, x, t_n) \in reg(C) \times lin(C) \times]0, 1[$ since $x \in \widehat{C} \subset \widetilde{C}$ and $\widetilde{C} = lin(C)$. The function $\widehat{f}$ is convex on $\widehat{C}$ relatively to C (Proposition 9) and is an extension of f (Proposition 2); it follows that $f(z_n) = \widehat{f}(z_n) \leq (1 - t_n)\widehat{f}(x) + t_n\widehat{f}(a)$ and finally $\limsup f(z_n) \leq \widehat{f}(x)$. We also have $\liminf f(z_n) \geq \widehat{f}(x)$ since $(z_n) \in S(x, C, \mathcal{S})$ and $\widehat{f}$ is $\mathcal{S}-$l.s.c. on $\widehat{C}$ relatively to C (Proposition 3). □

Proposition 15 *If f is a $\mathcal{S}-$l.s.c. convex function from C to $\mathbb{R}$ and if C is $\mathcal{S}-$regular then $\widehat{f}$ is a $\mathcal{S}-$l.s.c. convex function from $\widehat{C}$ to $\mathbb{R}$.*

PROOF. Let $(x, y, \theta) \in \widehat{C} \times \widehat{C} \times [0, 1]$ and a be given in the nonempty subset $reg(C)$; we denote $z(\theta) := \theta x + (1 - \theta)y$. Since C is $\mathcal{S}-$regular, $x_n := (1 - n^{-1})x + n^{-1}a$ and $y_n := (1-n^{-1})y + n^{-1}a$ belong to C (for $n \geq 1$). We set $z_n(\theta) := (1-n^{-1})z(\theta) + n^{-1}a$ (for $n \geq 1$); the sequence $(z_n(\theta))$ is in $\mathcal{S}$. We have $z_n(\theta) = \theta x_n + (1 - \theta)y_n$ and thus $(z_n(\theta))$ is a sequence of points of C which converges to $z(\theta)$ ((x_n) and (y_n) converge respectively to x and y). By convexity of f we can write $f(z_n(\theta)) = f(\theta x_n + (1 - \theta)y_n) \leq \theta f(x_n) + (1 - \theta)f(y_n)$. By Proposition 14, we have $\widehat{f}(x) = \lim_n f((1 - n^{-1})x + n^{-1}a) = \lim_n f(x_n)$ and $\widehat{f}(y) = \lim_n f(y_n)$. We deduce $\liminf f(z_n(\theta)) \leq \theta\widehat{f}(x) + (1 - \theta)\widehat{f}(y) < +\infty$; we can assert that $z(\theta)$ belongs to $\widehat{C}$ and finally $\widehat{C}$ is a convex subset. By Proposition 14, $\widehat{f}(z(\theta)) = \lim_n f((1-n^{-1})z(\theta)+n^{-1}a) = \lim_n f(z_n(\theta))$; thus $\widehat{f}(z(\theta)) \leq \theta\widehat{f}(x)+(1-\theta)\widehat{f}(y)$ and we conclude $\widehat{f}$ is a convex function on $\widehat{C}$. By Proposition 10, $\widehat{f}$ has real values on $\widehat{C}$.

To show that $\widehat{f}$ is a $\mathcal{S}-$l.s.c. function on $\widehat{C}$, consider (x_n) in $\mathcal{S} \cap \widehat{C}^{\mathbb{N}}$ which converges to a point $x \in \widehat{C}$ and such that $\liminf \widehat{f}(x_n) < +\infty$. Let a be a regular point of C and $\theta \in]0, 1[$; we define (y_n) by $y_n := (1 - \theta)x_n + \theta a$. We see that (y_n) is in $\mathcal{S} \cap C^{\mathbb{N}}$ and converges to the point $y := (1 - \theta)x + \theta a$. We have $y \in C$ and $(y_n) \in S(y, C, \mathcal{S})$; since f is $\mathcal{S}-$l.s.c. we get $\liminf f(y_n) \geq f(y)$. By convexity of $\widehat{f}$, $f(y_n) = \widehat{f}(y_n) \leq (1-\theta)\widehat{f}(x_n) + \theta\widehat{f}(a)$. We obtain $f((1-\theta)x + \theta a) \leq (1-\theta) \liminf \widehat{f}(x_n) + \theta f(a)$. We first deduce $\liminf \widehat{f}(x_n) > -\infty$ since $f(y)$ is real. We conclude $\liminf f((1 - n^{-1})x + n^{-1}a) \leq \liminf \widehat{f}(x_n)$. By Proposition 14, $\lim f((1 - n^{-1})x + n^{-1}a) = \widehat{f}(x)$ and thus $\liminf \widehat{f}(x_n) \geq \widehat{f}(x)$. □

We first study the convex extensions of a convex function. We say that f is *algebraically lower semi-continuous* (a.l.s.c.) if f is $\mathcal{S}_0-$l.s.c. We say that C is *algebraically regular* if C is $\mathcal{S}_0-$regular, i.e. $reg(C) \neq \emptyset$. In particular, convex subsets with nonempty intrinsic core (relative algebraic interior) are algebraically regular.

Theorem 16 *If f is an a.l.s.c. convex function from C to $\mathbb{R}$ and C is algebraically regular then $\widehat{f}$ is a convex extension from $\widehat{C}$ to $\mathbb{R}$. Moreover, if g is a function from $D \subset lin(C)$ to $\mathbb{R}$ which is a convex extension of f then $D \subset \widehat{C}$ and $\widehat{f}_{|D} \leq g$.*

Proof. From the previous results $\widehat{f}$ is a convex function from $\widehat{C}$ to $\mathbb{R}$ and $\widehat{f}_{|C} = f$. Let g be a function from $D \subset lin(C)$ to $\mathbb{R}$ which is a convex extension of f. Let $x \in D$; since C is algebraically regular we can choose $a \in reg(C)$. Then $x_n := (1 - n^{-1})x + n^{-1}a \in C$ for each $n \geq 1$. The sequence (x_n) converges to x and is in $\mathcal{S}_0$. The sequence $(f(x_n))$ is upper bounded : $f(x_n) = g(x_n) \leq (1 - n^{-1})g(x) + n^{-1}g(a)$. We conclude that $x \in \widehat{C}$. We also have $\widehat{f}(x) \leq \liminf f(x_n) \leq \liminf[(1 - n^{-1})g(x) + n^{-1}g(a)] \leq g(x)$. □

When the convex subset C has a nonempty core (algebraic interior) and f is an a.l.s.c. convex function from C to $\mathbb{R}$, it is proved in [1] that the smallest convex extension Ef of f exists on the whole space X but takes values in $\mathbb{R} \cup \{+\infty\}$. We easily show $Ef_{|\widehat{C}} = \widehat{f}$ and $Ef_{|lin(C)\setminus\widehat{C}} = +\infty$.

We now study the $\mathcal{S}-$l.s.c. convex extensions of a $\mathcal{S}-$l.s.c. convex function.

Theorem 17 *If f is a $\mathcal{S}-$l.s.c. convex function from C to $\mathbb{R}$ and C is $\mathcal{S}-$regular then $\widehat{f}$ is a $\mathcal{S}-$l.s.c. convex extension from $\widehat{C}$ to $\mathbb{R}$. Moreover, if g is a function from $D \subset lin(C)$ to $\mathbb{R}$ which is an a.l.s.c. convex extension of f then $D \subset \widehat{C}$ and $\widehat{f}_{|D} = g$; thus g is $\mathcal{S}-$l.s.c.*

Proof. The first part is consequence of Propositions 2 and 15. Consider $g : D \subset lin(C) \to \mathbb{R}$ an a.l.s.c. convex extension of f. We know from the previous theorem that $D \subset \widehat{C}$ and $\widehat{f}_{|D} \leq g$. From Proposition 3 we have $g \leq \widehat{f}_{|D}$ and we conclude. □

We say that C is *sequentially regular* if C is $\mathcal{S}_\infty-$regular where $\mathcal{S}_\infty := X^{\mathbb{N}}$. In particular, convex subsets with nonempty interior, convex subsets of finite dimensional spaces are sequentially regular. We note that if $\mathcal{S} = \mathcal{S}_\infty$ we have $\widetilde{C} = \{x \in X : \exists (x_n) \in C^{\mathbb{N}}, x_n \to x\}$; the convex subset C is sequentially regular iff each limit of a convergent sequence of points of C is in $lin(C)$ and C has regular points.

Corollary 18 *If f is a s.l.s.c. convex function from C to $\mathbb{R}$ and C is sequentially regular then $\widehat{f}$ is a s.l.s.c. convex extension from $\widehat{C}$ to $\mathbb{R}$. Moreover, if g is a function from $D \subset lin(C)$ to $\mathbb{R}$ which is a s.l.s.c. convex extension of f then $D \subset \widehat{C}$ and $\widehat{f}_{|D} = g$.*

References

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