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Why do the failure rates decrease?

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Abstract

It is quite plausible that any device or system reliability shows an increasing failure rate (IFR) as age, use or both may cause it to wear out over time. Nevertheless, explaining a decreasing failure rate, which means an improvement as time goes by, is far from intuitive. Mixtures of lifetime distributions turn out to be the most widespread explanation for this “positive ageing”. In this work, the proportional hazard rate (PHR) is revisited and related to an exponential mixture. We also analyse the effect of mixing on the patterns of dependence on time that the mixture exhibits.

Keywords: Failure Rate, Mixtures, Proportional Hazard Model

AMS Classification: 60K10, 62N05

1 Introduction

When assessing reliability of systems, an increasing failure rate seems to be very reasonably as age, use or both may cause the system to wear out over time. The reasons of decreasing failure rate, that is, a system improving as time goes by are less intuitive. Mixtures of lifetime distributions turn out to be the most widespread explanation for this “positive ageing”.

The more paradoxical case corresponds to the exponential distribution that shows a constant failure rate while exponential mixtures belong to the decreasing failure rate (DFR) class. Proschan, found that a mixture of exponential distributions was the appropriate choice to model the failures in the air-conditioning systems of planes [9]. Such mixing was the reason of the decreasing failure rate that the aggregated data exhibited. The decreasing failure rate (DFR) as well as the decreasing failure rate average (DFRA) classes, are closed under mixtures [2]. A similar result does not hold for the increasing failure rate (IFR) class, however, many articles focus on mixtures of IFR distributions that reverse this property over time and exhibit a decreasing failure rate as time elapses [8].

In studying the reliability of systems, operating conditions are far from constant over time. Moreover, the occurrence of changes in the operational environment is unpredictable in both the timing and the size of its effects and may be the reason why heterogeneous populations and, hence, mixtures arise in practice. Mixtures also emerge, for instance, when studying the reliability of a component exposed to many types of failure or when it can proceed from different suppliers.

In what follows, X denotes the random time to failure of the system with $f(x)$ and $F(x)$ being the corresponding density and distribution functions. The following measures are usually considered so as to evaluate the reliability of the system.

- The reliability function

$$\begin{cases} R(x) = P(X > x) = 1 - F(x), \quad x > 0 \\ \\ R(0) = 1 \end{cases}$$

- The failure rate

$$r(x) = \lim_{\Delta x \to 0} \frac{P\left(x < X < x + \Delta x | X > x\right)}{\Delta x} = \frac{f(x)}{R(x)}$$

- The failure rate average

$$H(x) = \frac{\int_0^x r(u)du}{x} = -\frac{\ln R(x)}{x}$$

The reliability function, $R(x)$, is the probability that the system fails after time x. The failure rate at time x, $r(x)$, represents the probability of an imminent failure given that the system had survived up x. Whenever the functioning over an interval, $[0, x]$, should be assured, the failure rate average will be preferred to evaluate the system reliability.

The effect of mixtures of distributions on its ageing characteristics (reliability function, failure rate, failure rate average, and mean residual life) has been widely studied ([3], [5], [6]). Therefore, the consequences of mixing should be taken into account when assessing system reliability. The Cox model [4], is the most cited mixture model and has been also studied by Gupta and Gupta [7]. In this article we consider the analysis of the Cox model deriving properties of its ageing characteristics. In particular new simplified expressions for them are obtained. In addition, the behaviour of both the failure and the failure rate average and its corresponding derivatives is described.

2 The proportional hazard model

One of the most frequently used mixture models is the Cox's proportional hazard [4]. In the Cox model, Z is a strictly non-negative random variable and the conditional failure rate, given $Z = z$, is

$$r(x|z) = zr(x) \tag{2.1}$$

with $r(x)$ being the failure rate of a non-negative random variable.

The random variable Z introduces the environmental heterogeneity and $X^\star$ represents the lifetime of a system under changing conditions. The aging characteristics depend on Z: the conditional distribution of $X^\star$, given $Z = z$, has reliability function $R(x|z)$ or, equivalently, failure rate $r(x|z)$ and failure rate average $H(x|z)$.

Definition 1 *The proportional hazard model mixture denoted, $X^\star$, consists of mixing over Z as specified in equation (2.1) a collection of positive random variables.*

Gupta and Gupta [7] study aging characteristics of the foregoing mixture. They also provide examples such that $r(x)$ is the failure rate of a Weibull distribution.

Definition 2 *The conditional distribution of an exponential mixture, $X^\star_{exp}$, given $Z = z$, has the following reliability function*

$$R(x|z) = e^{-zx}$$

The failure rate and the failure rate average of the exponential mixture are denoted, respectively, $r^\star_{exp}$, and $H^\star_{exp}$.

Next lemma shows the monotonicity properties that the ageing characteristics of $X^\star_{exp}$ exhibit.

Lemma 1 *The failure rate, $r^\star_{exp}(x)$, and the failure rate average, $H^\star_{exp}(x)$, of the exponential mixture verify*

(i) $r^\star_{exp}(x)$ is non increasing and $r^\star_{exp}(0) = E(Z)$.

(ii) $H^\star_{exp}(x)$ is non increasing and $H^\star_{exp}(0) = E(Z)$.

The proof can be found in [1].

Exponential mixtures play an essential role in the derivation of ageing characteristics of the proportional hazard model mixtures, difficult to obtain otherwise.

The relations between the failure rate, $r^\star(x)$, and failure rate average, $H^\star(x)$, of the mixture and the corresponding to an exponential mixture are examined in the next proposition.

Proposition 1 *For any proportional hazard mixture, the following properties hold*

(i)

$$r^\star(x) = r(x) r^\star_{exp}(\Lambda(x))$$

(ii)

$$H^\star(x) = H(x) H^\star_{exp}(\Lambda(x))$$

where $\Lambda(x) = \int_0^x r(u)du = -\ln R(x)$, *and* $H(x) = \dfrac{\Lambda(x)}{x}$

Proof

(i) The conditional reliability function corresponding to the failure rate given in (2.1) is

$$R(x|z) = e^{-z\Lambda(x)}$$

hence

$$r^\star(x) = \frac{E(r(x|Z)R(x|Z))}{E(R(x|Z))} = r(x)\frac{E(Ze^{-Z\Lambda(x)})}{E(e^{-Z\Lambda(x)})}$$

and (i) holds.

(ii) From part (i) in Proposition 1, it follows that

$$\Lambda^\star(x) = \int_0^x r^\star(u)du = \int_0^x r(u) r^\star_{exp}(\Lambda(u))du = \int_0^{\Lambda(x)} r^\star_{exp}(u)du = \Lambda^\star_{exp}(\Lambda(x))$$

The last inequality is obtained by setting $\Lambda(u) = x$. In addition

$$H^\star(x) = \frac{\Lambda^\star(x)}{x} = \frac{\Lambda^\star_{exp}(\Lambda(x))}{\Lambda(x)}\frac{\Lambda(x)}{x}$$

so, (ii) holds. This proposition simplifies the computations needed to obtain the aging characteristics of the mixture. ■

Next theorem provides the relation between the aging characteristics of the mixture and the distributions in the mixture.

Theorem 1 *The mixture given in Definition 1 satisfies the following properties*

(i) If $r^\star(x)$ *is non-decreasing, then* $r(x)$ *is also non-decreasing.*

(ii) If $H^\star(x)$ *is non-decreasing, then* $H(x)$ *is also non-decreasing.*

(iii) If Z *is not a constant and* $r(x) > 0$ *almost everywhere,* $r^\star(x)$ *and* $r(x)$ *cross at most at one point. The crossing point* t_0 *is the unique root of the following equation*

$$r^\star_{exp}(\Lambda(t_0)) = \frac{E\left(Ze^{-Z\Lambda(t_0)}\right)}{E\left(e^{-Z\Lambda(t_0)}\right)} = 1$$

(iv) If Z *is not a constant and* $r(x) > 0$ *almost everywhere,* $H^\star(x)$ *and* $H(x)$ *cross at most at one point. The crossing point,* t_1*, is the unique root of the equation below*

$$H^\star_{exp}(\Lambda(t_1)) = \frac{-\ln E\left(e^{-Z\Lambda(t_1)}\right)}{\Lambda(t_1)} = 1$$

(v) The following inequalities are true

$$r^\star(x) \le E(Z)r(x) \quad \textit{and} \quad H^\star(x) \le E(Z)H(x)$$

(vi) If $r(x)$ and $r^\star_{exp}(x)$ are differentiable and $r(x)$ is an increasing function, then

$$\frac{dr^\star(x)}{dx} \le E(Z)\frac{dr(x)}{dx}$$

(vii) If $H(x)$ and $H^\star_{exp}(x)$ are differentiable and $H(x)$ is an increasing function, then

$$\frac{dH^\star(x)}{dx} \le E(Z)\frac{dH(x)}{dx}$$

(viii) If $r(x)$ is non increasing, then $X^\star$ is IFR. In case of $H(x)$ being non increasing, $X^\star$ belongs to the DFRA class.

Proof

The key of the proof rests on Lemma 1 and Proposition 1.

(i) Part (i) in Lemma 1 along with $\Lambda(x)$ being non-decreasing lead to $r^\star_{exp}(\Lambda(x))$ being also non-increasing. The representation of $r^\star(x)$ given in part (i) of Proposition 1 allows to express $r(x)$ as the product of two positive and non-decreasing functions, hence, (i) holds.

(ii) An analogous use of conditions (ii) in both Lemma 1 and Proposition 1 leads to the result.

(iii) $\frac{r^\star(x)}{r(x)} = r^\star_{exp}(\Lambda(x))$ is a non-increasing function. Moreover, $r^\star_{exp}(x)$ is strictly decreasing except in case of Z was a constant: differentiating in the probabilistic representation of $r^\star_{exp}(x)$ previously used in part (i) of Proposition 1, along with the Cauchy-Schwartz inequality, we have that

$$\frac{dr^\star_{exp}(x)}{dx} = \frac{-E(Z^2e^{-Zx})E(e^{-Zx}) + E^2(Ze^{-Zx})}{E^2(e^{-Zx})} \le 0$$

The expression above equals zero if and only if Z is a constant. In addition, $\Lambda(x)$ is strictly increasing if $r(x) > 0$ almost everywhere, implying that $r^\star_{exp}(\Lambda(x))$ is strictly decreasing as well. Therefore $r^\star(x)$ and $r(x)$ cross at most at one point, t_0, satisfying $r^\star_{exp}(\Lambda(t_0)) = 1$.

(iv) By differentiation in the following probabilistic representation of $H^\star_{exp}(x)$

$$H^\star_{exp}(x) = -\frac{\ln E\left(e^{-Zx}\right)}{x}$$

along with Jensen's inequality applied to the convex function $y \ln y$, it follows that

$$\frac{dH^\star_{exp}(x)}{dx} = \frac{E(Ze^{-Zx})x + E(e^{-Zx})\ln E(e^{-Zx})}{E(e^{-Zx})x^2} \le 0$$

The expression above equals zero just in case of Z being a constant.

Hence, it is also deduced that $H^\star_{exp}(\Lambda(x))$ is strictly decreasing, therefore $\frac{H^\star(x)}{H(x)} = H^\star_{exp}(\Lambda(x))$ is strictly decreasing and the result in (iv) yields.

(v) $\Lambda(x)$ is nonnegative and satisfies $\Lambda(0) = 0$. In addition, $r^\star_{exp}(\Lambda(x))$ is non-increasing and, from (i) in Lemma 1, it has a maximum at 0, and $r^\star_{exp}(0) = E[Z]$. $H^\star_{exp}(\Lambda(x))$ is also non-increasing and verifies $H^\star_{exp}(0) = E[Z]$. The result is obtained from (i) and (ii) in Proposition 1.

(vi) By differentiation of (i) in Proposition 1, we arrive at

$$\frac{dr^\star(x)}{dx} = \frac{dr(x)}{dx} r^\star_{exp}(\Lambda(x)) + r^2(x)\frac{dr^\star_{exp}(u)}{du}|_{u=\Lambda(x)}$$

Part (i) in Lemma 1 shows that $r^\star_{exp}(x)$ is non-increasing, hence

$$\frac{dr^\star(x)}{dx} \leq \frac{dr(x)}{dx} r^\star_{exp}(\Lambda(x)) \tag{2.2}$$

From (i) in Lemma 1, it is deduced that $r^\star_{exp}(x) \leq E[Z]$, and the result follows.

(vii) Next equation is obtained by differentiation of (ii) in Proposition 1

$$\frac{dH^\star(x)}{dx} = \frac{dH(x)}{dx} H^\star_{exp}(\Lambda(x)) + H(x)r(x)\frac{dH^\star_{exp}(u)}{du}|_{u=\Lambda(x)}$$

$H^\star_{exp}(x)$ is non-increasing (part (ii) in Lemma 1), hence

$$\frac{dH^\star(x)}{dx} \leq \frac{dH(x)}{dx} H^\star_{exp}(\Lambda(x))$$

Moreover, $H^\star_{exp}(x) \leq E[Z]$, and the result holds.

$(viii)$ In case of $r(x)$ being non increasing, part (i) in Proposition 1 and (i) of Lemma 1 lead $r^\star(x)$ to be the product of two positive and non increasing functions. In the same way, if $H(x)$ is non-increasing, with (ii) in Proposition 1 and (ii) in Lemma 1, it is deduced that $X^\star$ belongs to the DFRA class.$\blacksquare$

It is well known that if a random variable X is IFR (DFR), then it also belongs to the IFRA (DFRA) classes. However, as far as we know, there is no similar property when $r(x)$ is no monotonic. This paper also provides an original result useful in this case. Next, two forms of no monotonic behaviour are defined.

Definition 3 *A nonnegative function $g(x)$ is said to be of the type B (U) if there exists a point, x_0, such that $\frac{dg(x)}{dx} < 0 (> 0)$, for $x < x_0$, $\left.\frac{dg(x)}{dx}\right|_{x=x_0} = 0$ and $\frac{dg(x)}{dx} > 0 (< 0)$ for $x > x_0$.*

The following proposition that relates the monotonicity of $r(x)$ and $H(x)$ is stated without proof. It can be found in [1].

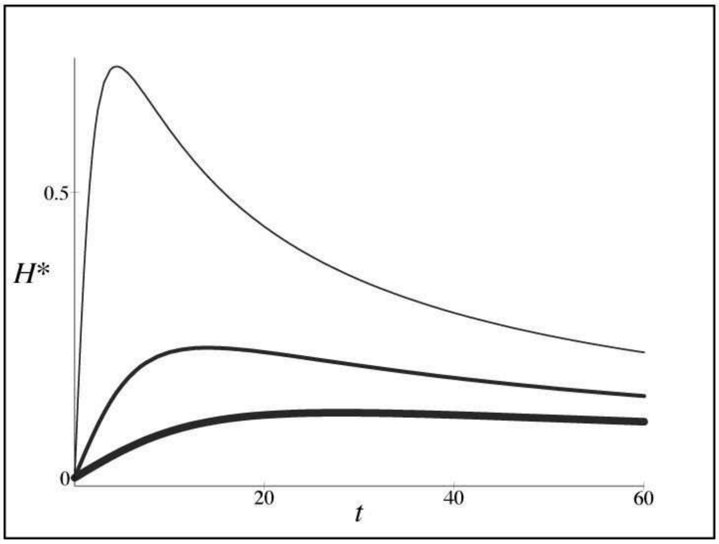

Figure 1: Graph of $H^{\star}(t)$ for an IFR Weibull mixture ($\beta = 2$).

Proposition 2 *Consider the following limit*

$$a = \lim_{x \to \infty} xr(x) - \Lambda(x)$$

If the failure rate, $r(x)$, is of the type B (U), then the failure rate average, $H(x)$, is of the type B (U) if $a > 0$ (< 0). In case of $a \leq 0$ (≥ 0), then $H(x)$ is non-increasing (non-decreasing).

The following example aims at illustrating the foregoing results.

Example 1 *Z follows a gamma distribution and $r(x)$ corresponds to an IFR Weibull distribution.*

For a Weibull distribution and $\beta > 0$:

$$\begin{aligned} r(x) &= \beta x^{\beta-1} \\ H(x) &= x^{\beta-1} \\ \Lambda(x) &= x^{\beta} \end{aligned}$$

In addition, the density function of Z is given by

$$f(z) = \frac{a^s e^{-az} z^{s-1}}{\Gamma(s)}, \qquad z > 0, \qquad a > 0, \qquad s > 0$$

From (i) and (ii) in Proposition 1

$$r^{\star}(x) = \frac{s\beta x^{\beta-1}}{a + x^{\beta}}, \quad H^{\star}(x) = \frac{s \ln\left(1 + \frac{x^{\beta}}{a}\right)}{x}$$

If $\beta > 1$, $r^\star(x)$ is of the type U with a maximum at $x_0 = (a(\beta-1))^{\frac{1}{\beta}}$.

Moreover

$$\lim_{x\to\infty} xr^\star - \Lambda^\star(x) = -\infty = a$$

From Proposition 2, in case $\beta > 1$, it follows that $H^\star(x)$ is of the U-type. In Figure 1 three Weibull mixtures for $\beta = 2$ and a =5, 50, and 200 are depicted, respectively, by progressively thicker lines. The expected value of the gamma distribution decreases with a. The greater the mean of the mixing distribution, the sooner the IFRA property is reversed.

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