Monografías del Seminario Matemático García de Galdeano. **27:** 81–88, (2003).

Mathematical Modelling of Sedimentary Basin Formation

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Abstract

New stratigraphic modellings (sedimentary basins formation), developed by the Institut Français du Pétrole, lead to mathematical questions difficult to answer. Such models describe erosion-sedimentation processes and take into account a limited weathering via non standard unilateral problems. Various theoretical results and research procedures are presented for solving the monolithologic column case.
Keywords: Stratigraphic modelling, variational inequalities, inverse problem, limited weathering.
AMS Classification: 35K20, 35K85, 74F10

1 Introduction

The geological problem proposed by R. Masson (I.F.P.) deals with the erosion-sedimentation phenomenon in sedimentary basins [4]. The model is based on three main assumptions, given below:

Claim 1: The model is weather-limited. It means that $-\partial_t h$, the erosion speed if one denotes by h the height of sediments, has to be smaller than a given function E, a.e. in the base of the basin Ω (see Fig.1), at every moment.

Claim 2: Non standard unilateral constraints of instantly control type are considered on outflow boundary, say Γ_s (see condition (4)).

Claim 3: In order to reconcile these two claims with a conservative formulation, a third hypothesis is needed. Therefore, one assumes that the flow of matter is proportional to ∇h and one states the following equation for the mass balance of the sediment:

$$\partial_t h - div(\lambda \nabla h) = 0 \quad in\ ]0, T[\times\Omega,$$

where λ is a suitable multiplier, *a priori* in $[0, 1]$.

2 The mathematical model

Let us consider Ξ a sedimentary basin, $\Omega \subset R^N$ $(N = 1, 2)$ its base and $Q =]0, T[\times\Omega$. Let us denote by Γ_e and Γ_s two parts of the boundary such that $\partial\Omega = \overline{\Gamma}_e \cup \overline{\Gamma}_s$ and by h the sediment's h

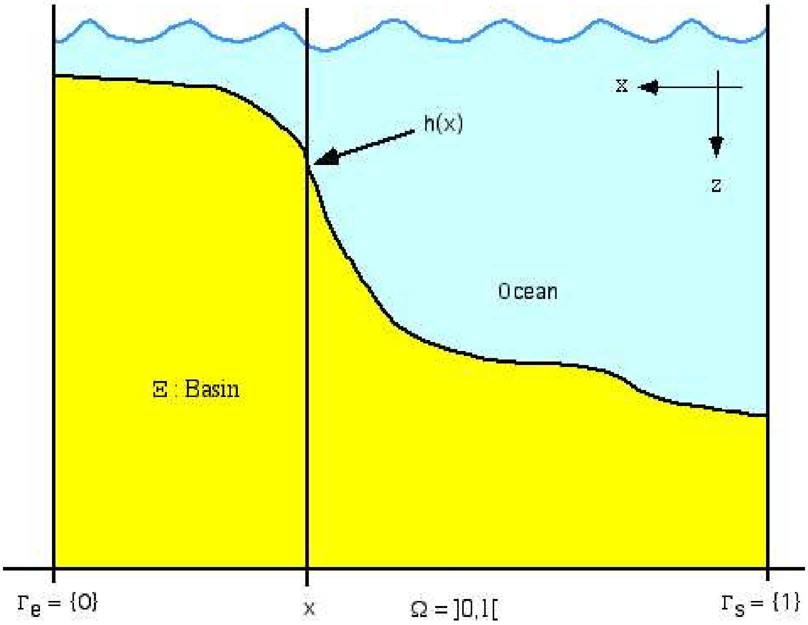


Figure 1: Sedimentary basin

Then, the mathematical modelling of the problem writes as follows:

$$\partial_t h - div(\lambda\nabla h) = 0 \quad in\ Q. \tag{1}$$

$$\partial_t h \geq -E \quad in\ Q, \quad where \quad E(t,x) \geq 0, \tag{2}$$

$$-\lambda\partial_n h = f_e \quad on\ ]0, T[\times\Gamma_e, \tag{3}$$

$$\left.\begin{array}{c} \lambda\partial_n h + f_s \geq 0\ ; \quad \partial_t h + E \geq 0\ ; \\ (\lambda\partial_n h + f_s)(\partial_t h + E) = 0 \end{array}\right] \quad on\ ]0, T[\times\Gamma_s, \tag{4}$$

$$h(0, .) = h_0\ in\ \Omega, \tag{5}$$

$$\lambda \in [a, 1], \quad a \geq 0. \tag{6}$$

Finally, let us denote by Λ_{ad} the admissible set:

$$\Lambda_{ad} = \{\lambda \in L^\infty(]0, T[\times\Omega)\ /\ \exists\ h,\ (\lambda, h)\ satisfying\ (1)\ to\ (6)\}.$$

Let us remark that if $f_e = 0,\ f_s \geq 0$ with $E = 0$ and $h = h_0 = c$, any λ is solution. Therefore, one needs some more assumptions in order to characterize λ. Two models are

proposed:
in the first one, it is assumed that λ has to be maximal in Λ_{ad}, *i.e.*

$$\exists\ \overline{\lambda} \in \Lambda_{ad},\ \forall \lambda \in \Lambda_{ad},\ \lambda \leq \overline{\lambda} \tag{7}$$

alternatively, in the second one, non standard unilateral constraints are imposed:

$$(1-\lambda)(\partial_t h + E) = 0 \ in\ Q. \tag{8}$$

3 Some concrete examples for an heuristic approach

• Let us consider that $E = 0$, $f = 0$ and assume that $h_0 = c > 0$.

In this case, one can remark that for any bounded measurable λ satisfying (6), (λ, h_0) is a solution of (1) to (6).
On the one hand, (8) is satisfied, but the solution is not unique.
On the other hand, $\lambda \equiv 1$ is the unique solution of (7).
So, one concludes that (8) is not well-posed.

• Let us consider the one-dimensional case $\Omega =]0,1[$ and assume that $E = 0$, $f = 0$ and that $h_0 = \sum_{i=1}^{n-1} \alpha_i \mathbb{I}_{[a_{i-1},a_i[} + \alpha_n \mathbb{I}_{[a_{n-1},a_n]}$ is a non negative step function.

One remarks that for any continuous λ satisfying (6) and such that $\lambda(a_i) = 0$ ($i = 1,..,n-1$), (λ, h_0) is a solution of (1) to (6).
One the one hand, a solution of (8) exists, but it is not unique.
On the other hand, it is obvious that if one has a solution (λ, h) of (7), then $\lambda = 1$.
Therefore, the continuity of the derivation operator in $\mathcal{D}'(\Omega)$ leads to: $\Delta h_0 \geq 0$ in $\mathcal{D}'(\Omega)$.
Since this last assertion is impossible, (7) has no solution.
Let us point out that, under the assumption $E = c_0 > 0$, the existence of a solution of (8) is an open problem.

• We are now intersted by the travelling waves which are solutions of the one-dimension model (1) to (6).

1) Let us consider that $\Omega =]0, x_1[$, $\Gamma_e = \{0\}$, $\Gamma_s = \{x_1\}$ and assume that $E(\xi) = E\mathbb{I}_{[0,\xi_0[} + E^*\mathbb{I}_{[\xi_0,+\infty[}$ where $\xi = \mu x + t$ ($\mu > 0$), $0 < \xi_0 < \xi_1$ and $E^* > E \geq 0$.

Therefore, plugging $h(t,x) = h(\xi)$ and $\lambda(t,x) = \lambda(\xi)$ into (1) to (6) with $(\xi_1 - \xi_0)/\mu^2 + E/E^* \leq 1$ leads to:

	for ξ in $]0,\xi_0[$	for ξ in $]\xi_0,\xi_1[$
$h(t,x) =$	$\mu^2 E\, e^{-\frac{\xi_0}{\mu^2}}[1 - e^{\frac{\xi}{\mu^2}}] + h_0(0)$	$E^*(\xi_0 - \xi) + h_0(0) - \mu^2 E[1 - e^{-\frac{\xi_0}{\mu^2}}]$
$\lambda(t,x) =$	1	$\frac{\xi - \xi_0}{\mu^2} + \frac{E}{E^*}$

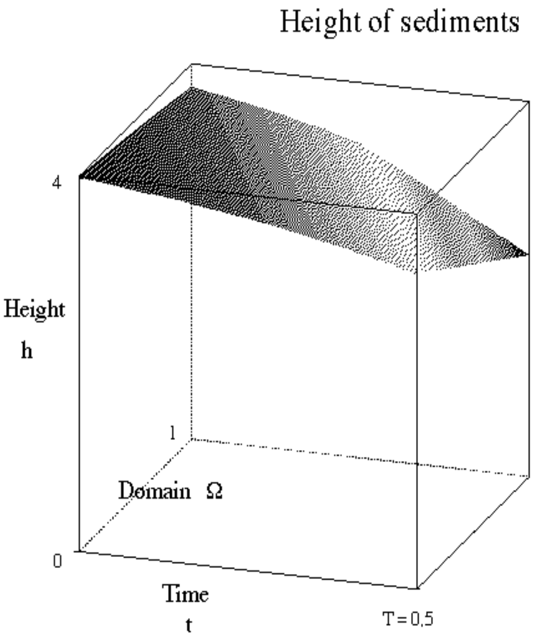
2) The same explicit construction with $\xi = x + \mu t$, leads to

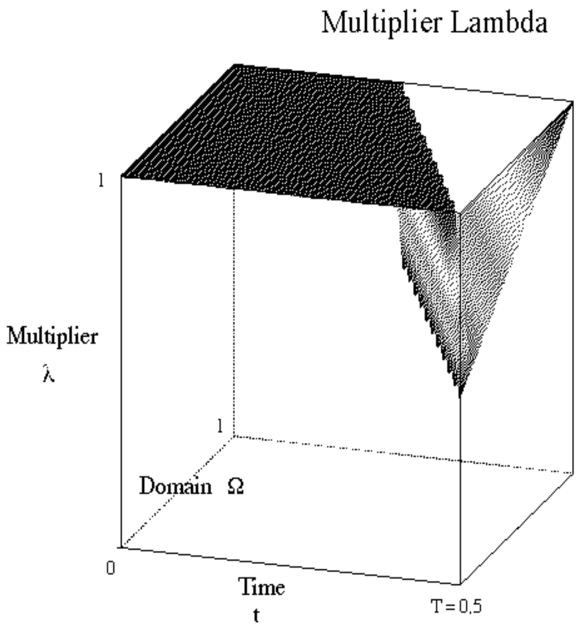
	for ξ in $]0, \xi_0[$	for ξ in $]\xi_0, \xi_1[$
$h(t,x) =$	$\dfrac{E}{\mu^2}\, e^{-\mu\xi_0}[1 - e^{\mu\xi}] + h_0(0)$	$\dfrac{E^*}{\mu}(\xi_0 - \xi) + h_0(0) - \dfrac{E}{\mu^2}[1 - e^{-\mu\xi_0}]$
$\lambda(t,x) =$	1	$\mu(\xi - \xi_0) + \dfrac{E}{E^*}$

where $\xi_1 = \min\{\xi_0 + \dfrac{\mu}{E^*}(h_0(0) - \dfrac{E}{\mu^2}, \xi_0 + \dfrac{E^* - E}{\mu E^*}\}$.

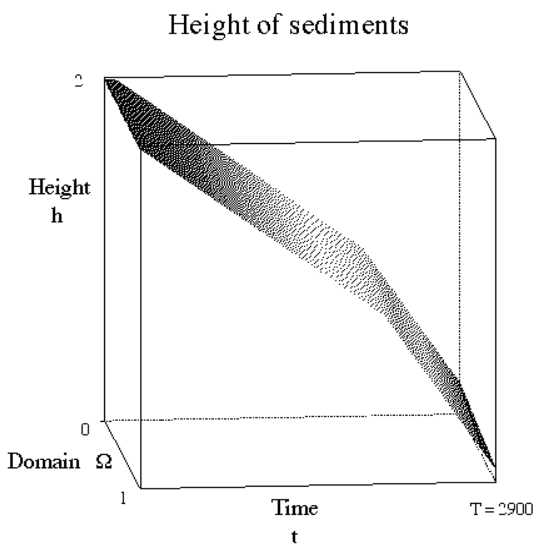
3) Let us now give some graphics of the exact solutions proposed by formula 1) or 2) when $\Omega =]0,1[\,,\ \Gamma_e = \{x = 0\}$ and $\Gamma_s = \{x = 1\}$.

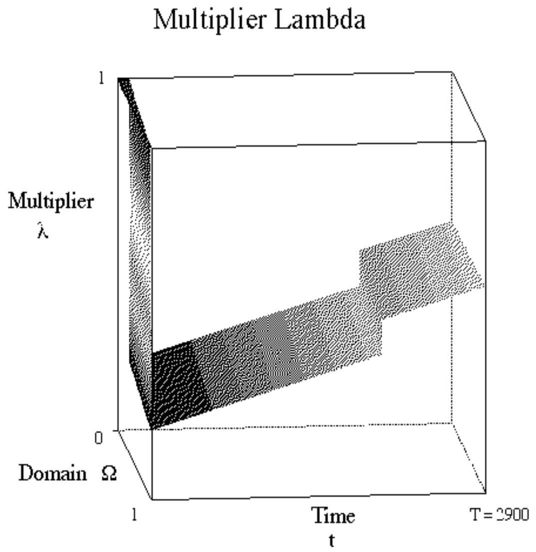
i) Assume that $h_0 = h_0(0) = 4,\ E = \mu = 1,\ E^* = 2,\ \xi = x + t,\ \xi_0 = 1,\ \xi_1 = 1{,}5$ *and* $E(x,t) = \chi_{\{0 \le x+t < 1\}} + 2\chi_{\{1 \le x+t \le 1{,}5\}}$ Then, for $T = 0{,}5$, one has the following figures:



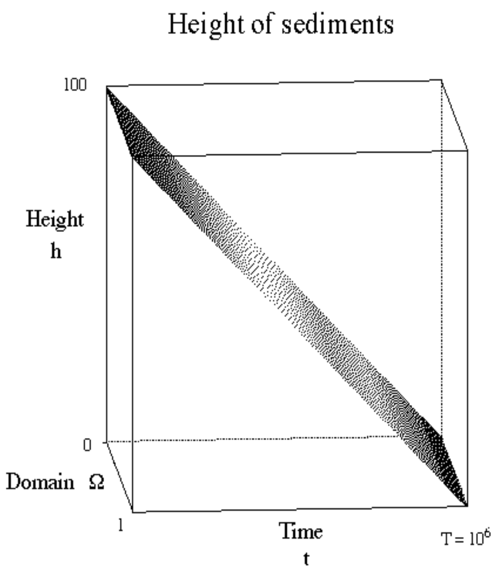


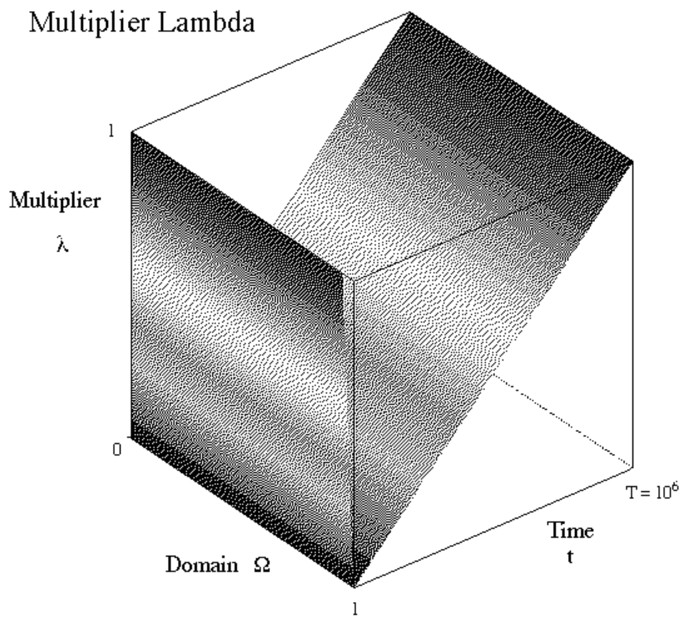
ii) Assume that $h_0 = h_0(0) = 2,\ E = 0{,}0001,\ \mu = 0{,}01,\ E^* = 0{,}0005,\ E^{**} = 0{,}001,\ \xi = x + \frac{1}{100}t,\ \xi_0 = 1,\ \xi_1 = 21,\ \xi_2 = 30,$ *and* $E(x,t) = E\,\chi_{\left\{0 \le x + \frac{t}{100} < 1\right\}} + E^*\,\chi_{\left\{1 \le x + \frac{t}{100} < 21\right\}} + E^{**}\,\chi_{\left\{21 \le x + \frac{t}{100} < 30\right\}}$. Then, for $T = 2900$, one has the following figures:





iii) Assume that $h_0 = h_0(0) = 100,\ E = 0,\ \mu = 0,001,\ E^* = 0,0001,\ \xi = x + \frac{1}{1000}t,$ $\xi_0 = 1,\ \xi_1 = 1001$ and $E(x,t) = 0,0001\,\chi_{\{1 \leq x + \frac{1}{1000}t \leq 1001\}}$. Then, for $T = 10^6$, one has the following figures:





4 Toward the definition of a strong solution

As mentioned previously, the global problem is an ill-posed problem. Nevertheless, thanks to the work of G. Duvaut & J. L. Lions [3] concerning the "thermical enslavement", some aspects are already studied and some mathematical tools are available.

Let us first consider, for a fixed smooth λ, the non degenerated and non weather-limited case.

Definition 1 *A strong solution of [1 and of 3 to 6] is a couple (λ, h) of $L^\infty(]0,T[\times\Omega) \times L^2(0,T;H^1(\Omega))$ such that:*

$$\partial_t h \in \{u \in L^2(0,T;H^1(\Omega)),\ u + E \geq 0 \textit{ on } \Gamma_s\}$$

$$0 \leq \lambda \leq 1\ ; \quad h(0) = h_0 \quad \textit{a.e. in } \Omega$$

and $\forall v \in H^1(\Omega), \quad t$ *a.e in* $]0,T[$,

$$\int\limits_\Omega \partial_t h\,(v - \partial_t h)\,dx + \int\limits_\Omega \lambda\,\nabla h\,\nabla(v - \partial_t h)\,dx + \int\limits_\Gamma f\,(v - \partial_t h)\,d\sigma + \int\limits_{\Gamma_s} \chi_{\mathbb{R}^+}(v + E)\,d\sigma \geq 0$$

where $\chi_{\mathbb{R}^+}(x) = 0$ *if* $x \geq 0$ *and* $+\infty$ *if* $x < 0$.

Proposition 1 *Assume that $a > 0$ in (6), that λ, E and f_s are regular functions and that*
$\exists g \in L^2(\Omega),\ g \geq 0,$ *with*

$$\forall v \in H^1(\Omega), \quad \int_\Omega (g - E(0))v\,dx + \int_\Omega \lambda(0)\,\nabla h_0\,\nabla v\,dx + \int_\Gamma f(0)\,v\,d\sigma = 0.$$

Then there exists a unique h such that (λ, h) is a strong solution to $[1-3-4-5-6]$.

We give in the sequel only the steps of the proof.
First step. Thanks to a classical Galerkin method, based on a fixed point argument (see G. Duvaut & J. L. Lions [3] and J. L. Lions et al [5, 6, 8, 7] for example), one has:
for any positive ε and η, there exists a unique h^ε_η in $L^2(0,T;H^1(\Omega))$ such that

$$\partial_t h^\varepsilon_\eta \in L^2(0,T;H^1(\Omega))\ , \quad \partial^2_t h^\varepsilon_\eta \in L^2(0,T;L^2(\Omega))$$

and

$$h^\varepsilon_\eta(0) = h_0, \quad \partial_t h^\varepsilon_\eta(0) = g - E(0),$$

satisfying for any v in $H^1(\Omega)$, $\quad t$ a.e in $]0,T[$

$$\varepsilon \textstyle\int_\Omega \partial^2_t h^\varepsilon_\eta\,v\,dx + \int_\Omega \partial_t h^\varepsilon_\eta\,v\,dx + \int_\Omega \lambda\,\nabla h^\varepsilon_\eta\,\nabla v\,dx$$

$$+ \textstyle\int_\Gamma f\,v\,d\sigma - \int_{\Gamma_s} \beta_\eta(\partial_t h^\varepsilon_\eta + E)v\,d\sigma = 0.$$

where $\beta_\eta(x) = \frac{1}{\eta}[x(x^2+2)\mathbb{I}_{[-1,0]} - x\mathbb{I}_{]-\infty,-1[}]$.

Second step. Using *a priori* estimates (mainly obtained in the Galerkin scheme of the above equation and conserved thanks to the lower-semi-continuity of the norms for the weak convergence) and sub-sequences extractions, one gets:
there exists a unique h_η in $L^2(0,T;H^1(\Omega))$ with

$$\partial_t h_\eta \in L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \partial_t^2 h_\eta \in L^2(0,T;H^1(\Omega)')$$

such that $\quad h_\eta(0) = h_0 \quad$ and $\forall v \in H^1(\Omega), \quad t$ a.e in $]0,T[$

$$\int_\Omega \partial_t h_\eta \, v \, dx + \int_\Omega \lambda \, \nabla h_\eta \, \nabla v \, dx + \int_\Gamma f \, v \, d\sigma - \int_{\Gamma_s} \beta_\eta(\partial_t h_\eta + E) v \, d\sigma = 0.$$

Third step. In order to prove the existence of a solution, one has to pass to the limit in the above equation when η tends toward 0. Once again, *a priori* estimates are needed and the limit is obtained thanks to a technique proposed by G. Duvaut & J. L. Lions in [3]. The main difference is the use of a weight (depending on a and $||\partial_t \lambda||_\infty$) in the time integration since the bilinear form is time depending.

Last step. The uniqueness is obvious by monotonicity arguments.

Furthermore, thanks to the classical maximum principle, one has the following existence-uniqueness result of the strong solution of the non-degenerated weather-limited problem:

Proposition 2 *Assume moreover that λ, E and f fulfil*

$$\begin{aligned} &div(\partial_t \lambda \nabla h) + \partial_t E - div(\lambda(x,t)\nabla E) \geq 0 \quad in\ ]0,T[\times\Omega, \\ &\partial_t f - \partial_t \lambda \nabla h.\overrightarrow{n} + \lambda \nabla E.\overrightarrow{n} \geq 0 \quad on\ ]0,T[\times\Gamma_e, \end{aligned} \tag{9}$$

then h satisfies the weather-limited condition: $\partial_t h + E \geq 0 \quad in\ ]0,T[\times\Omega$.

Let us note that such a condition is satisfied if one assumes that $E(t,x) = E_0$, $f(t,x) = f(x)$ and $\lambda(t,x) = \lambda(x)$ (see S.N. Antontsev et al [1]).

It is important to remark here the first order hyperbolic aspect of problem ("travelling waves" examples.

5 Final remarks

First of all, we cann conclude that the problem of the existence and the uniqueness of the solution of (7) or (8) is still open. In fact, without any further assumption on the physical data, theses problems are ill-posed.

Concerning the forthcoming work, let us point out that the real industrial problem does not consist in finding (λ, h), but in solving the inverse problem: knowing h (the topography), is it possible to obtain a unique *sui generis* λ, solution of (7) or (8) ?

One may find in [2] some details of the proofs.

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