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Study and improvement of the condition number of the Electric Field Integral Equation

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Abstract

The discretization of the Electric Field Integral Equation (EFIE) leads to the solution of a dense complex non-hermitian linear system. It is well-known that the related linear system becomes ill-conditioned particularly when the frequency increases. As a consequence, its solution by a subspace Krylov iterative algorithm has a very slow convergence. In this work, we examine the possibility to speed up the convergence by constructing a Calderón-type implicit preconditioner.

Keywords: preconditioning, boundary element method, conjugate gradient

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1 Introduction

A typical difficulty arising in the numerical solution of electromagnetic scattering problems is related to the *unbounded* character of the domain. Among the most widely used approaches, a possible solution consists in equivalently rewritting the initial problem as an integral equation set on the surface of the scatterer. It is well-known that this technique yields schemes being both robust and accurate. However, the operator defining such an equation is a *non-local* pseudodifferential operator. From a discrete point of view, it generates a dense complex non-hermitian linear system of size $N \times N$, where N stands for the number of degrees of freedom arising in the considered problem. As a consequence, the application of a direct solver to compute the solution is both time and memory consuming and requires approximately $\mathcal{O}(N^3)$ operations. This is still more problematic at the high frequency regime since N is in the order of $\lambda/10$, where λ denotes the wavelength.

An alternative to this approach consists in applying a Krylov subspace iterative solver (CGS, GMRES, QMR,...) [4] to the equation for computing an approximate solution.

This results in a computational cost in the order of $\mathcal{O}(n^{\text{iter}}N^2)$, where n^{iter} designates the number of iterations needed for obtaining a satisfactory approximate solution. The complexity $\mathcal{O}(N^2)$ coming from the evaluation of Matrix/Vector products can be efficiently reduced by using for instance the Rokhlin's Multilevel Fast Multipole Method (FMM) [3]. Another way to reduce the CPU time is to prospect how to speed up the algorithm by diminishing n^{iter}. Integral equations become ill-conditioned specially at the high-frequency regime hence breaking down the convergence of the iterative solver. However, a significant improvement can be gained by efficiently preconditioning the linear system. Here, we focus on Calderón-type preconditioners. They are based on the construction of a continuous approximate inverse of the exact operator. to use Calderón's integral relations recently extended to the three-dimensional Here we choose to design an implicit inverse operator based on the construction of an approximate boundary-value problem by truncating the exterior domain by a local On-Surface Radiation Condition (OSRC) [1]. The main tool involved in the use of a Calderón-type preconditioner is the numerical approximation of the Dirichlet-Neumann operator. Here, this approximation is handled by a finite element solution of a boundary-value problem set on a shell around the scatterer surface. This first study is only prospective and hence is limited to the two-dimensional case.

The paper is organized as follows. In Section 2, we briefly recall how to derive the Electric Field Integral Equation (EFIE) for the scattering problem of a transverse polarized electromagnetic wave by a perfectly conducting body. In Section 3, we review some recent results of Chew and Warnick [2] giving a sharp estimate for the condition number of the EFIE for a strip in terms of the wavelength and the mesh size. In Section 4, we introduce the implicit sparse preconditioner. Then, we give some numerical results illustrating its robustness and efficiency.

2 The EFIE for a TM-polarized incident field

Let $\Omega^- \subset \mathbb{R}^2$ be a bounded domain with a $\mathcal{C}^\infty$ boundary $\Gamma = \partial\Omega$. Let us denote by $\Omega^+ = \mathbb{R}^2 \setminus \overline{\Omega^-}$ the associated exterior domain of propagation. Consider u^{inc} as an incident (usually plane) wave defined by a wavenumber k. We denote by $\lambda = 2\pi/k$ the wavelength. We now assume that the incident field u^{inc} generates a scattered wave u in the domain Ω^+ solution to the boundary-value problem for the Helmholtz equation

$$\begin{cases} \text{Find } u \in \mathcal{C}^\infty(\overline{\Omega^+}) \text{ such that} \\ \triangle u + k^2 u = 0, \text{ in } \Omega^+, \\ u = g, \text{ on } \Gamma, \\ \lim\limits_{|x|\to+\infty} |x|^{1/2}(\nabla u \cdot \dfrac{x}{|x|} - iku) = 0. \end{cases} \tag{1}$$

where the Dirichlet datum is $g = -u^{inc}$ and the Sommerfeld radiation condition at infinity is imposed to select the physical outgoing wave. This problem models the scattering of a TM time-harmonic wave by a perfect conductor infinite cylinder. System (1) has first to be reduced to an equivalent integral equation set on Γ. Several approaches can be used for such a reduction when Γ is a closed curved. Usually, the Combined Field Integral Equation (CFIE), being well-conditioned, is the most used formulation. However, when Γ is an open curve, only the Electric Field Integral Equation (EFIE) can be applied

$$Vp(x) = g(x), \;\; x \in \Gamma, \tag{2}$$

where V is the single-layer potential defined by

$$Vp(x) = \int_\Gamma G(x,y)p(y)d\Gamma(y), \;\; x \in \Gamma.$$

Here, G is the free-space Green's function in $\mathbb{R}^2$ given by

$$G(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|), \;\; x \neq y,$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. The unknown density p is related to the field u by the jump relation $p := [\partial_{\mathbf{n}} u]_\Gamma$, with $[\partial_{\mathbf{n}} u]_\Gamma = \partial_{\mathbf{n}} u^-_{|\Gamma} - \partial_{\mathbf{n}} u^+_{|\Gamma}$, $\partial_{\mathbf{n}} u^\pm_{|\Gamma}$ denoting the boundary-value of $\partial_{\mathbf{n}} u$ on Γ from its value within $\Omega^\pm$. Vector $\mathbf{n} = (n_1, n_2)$ is the outwardly directed unit normal to Ω^-. To numerically solve (2), we make use of a $\mathbb{P}_1$ boundary element method. The density of mesh nodes must be sufficient (generally in the order of $\lambda/10$) to correctly reproduce the oscillations of the field and the singularities of the scatterer.

3 Asymptotic study of the conditioning of the EFIE

The discretization of the EFIE leads to the computation of the solution to a linear system defined by a dense non-hermitian complex-valued matrix $[V]$. Since the aim is to use a Krylov iterative solver for high-frequency problems, we have to investigate which parameters damage the condition number K of $[V]$. To this end, we recall some recent results of Chew and Warnick [2] related to the particular case of the strip in the limit of small wavelengths.

We consider here a screen $\Gamma = [-d/2, d/2]$. We designate by h the uniform mesh size of Γ arising in the discretization of the EFIE by a boundary element method. Let us respectively denote by $D = d/\lambda$ and $n_\lambda = \lambda/h$ the dimensionless length of the strip in wavelengths and the number of nodes by wavelength. The pseudodifferential operator V has the property of being both non-hermitian and non-normal. Rather than splitting V into a static part and a compact perturbation like in the derivation of usual error

estimates, Chew and Warnick [2] choose the following decomposition: $V = \mathcal{H} + \mathcal{R}$, where $\mathcal{H}$ is the normal part of V and $\mathcal{R} = V - \mathcal{H}$. Next, based on this decomposition and using an asymptotic analysis relatively to $D \to \infty$, they prove that $\mathcal{H}$ gives a correct representation of the behaviour of the spectrum of V and that $\mathcal{R}$ remains bounded with respect to D.

It is well-known that, for an infinite plane, modes of the form $e^{i\beta x}$ are the eigenfunctions of V for a frequency β in the spectral decomposition. It results that, according to [2], the high spectral frequencies ($|\beta| > 1$) give rise to eigenvalues which behave like $\lambda(4\pi)^{-1}(\beta^2 - 1)^{-1/2}$ and lie on the real positive axis. They physically correspond to evanescent rays. For low spectral frequencies ($|\beta| < 1$), the eigenvalues are in the order of $i\lambda(4\pi)^{-1}(1-\beta^2)^{-1/2}$ and are located on the axis of the purely imaginary numbers. They are linked to the propagative rays. Finally, the interaction between the two endpoints of the finite domain Γ generates a strong coupling between the evanescent and propagative modes through the surface waves. They are described by eigenvalues in the form $\sqrt{2}\lambda(1+i)D^{1/2}/(6\pi)$, with $|\beta| \simeq 1$. the discrete version all the aspects of the spectrum

From this analysis, we can see that the largest eigenvalue $\overline{\Lambda}_{max}$ of $[V]$ corresponds to the surface modes

$$\overline{\Lambda}_{max} \simeq \frac{\sqrt{2}\lambda}{6\pi}(1+i)D^{1/2}.$$

For a fixed λ and $D \to \infty$, it behaves like the square root of the size of the scatterer expressed in wavelengths units. Hence, it constitutes a specific parameter of the problem itself independent of the discretization scheme, and therefore cannot be reduced without the use of an appropriate preconditioner. Moreover, the smallest eigenvalue of $[V]$ is related to the evanescent modes being truncated by the approximation scheme

$$\overline{\Lambda}_{min} \simeq \frac{2\lambda}{\pi^3 n_\lambda}.$$

A direct consequence is that the condition number K of $[V]$ can be approximated by

$$K = \left(\frac{\pi}{2}\right)^2 \frac{2}{3} n_\lambda D^{1/2}.$$

increasing with n_λ. As a conclusion, the condition number mainly depends on two essential terms

- the number of nodes by wavelength n_λ directly related to the discretization process and linked to the highest evanescent rays being correctly approximated by the numerical scheme,
- ratio $D =$ (size of the scatterer)/(wavelength) which represents the global length of the path covered by the creeping rays.

behaviour of the condition number $K(\overline{V})$ with respect to $\Gamma = [-1, 1]$).

4 A preconditioned algorithm for the EFIE

4.1 Construction of a preconditioner for the EFIE

As mentioned in Section 2, the problem to be solved is the following

$$\begin{cases} \text{Find the density} p \text{ such that} \\ Vp(x) = g(x), \;\; x \in \Gamma. \end{cases} \tag{3}$$

Following the discussion in the previous section, this system has to be suitably preconditioned before it can be solved by a Krylov iterative procedure. Of course, a perfect candidate to this objective is the inverse of operator V given by

$$p = \mathcal{B}g.$$

In a more realistic way, it is enough to construct an approximation of this inverse much more less expensive. Moreover, from the previous analysis developed in Section 3, its effectiveness is closely related to its capability to reproduce the characteristics of the problem incorporating both the infinite plane approximation and the creeping rays.

To fix the notations, we consider the case of the strip $\Gamma = [-1, 1]$. Let us define the thin layer Ω_l by

$$\Omega_l = [-(1 + hl), (1 + hl)] \times [-hl, hl], \;\; l \in \mathbb{N}^*.$$

The integer l allows a control of the size of Ω_l. In the sequel, we use the notation $\mathbf{n}_l$ for the outwardly directed unit normal vector to Ω_l at the fictive boundary Σ_l.

If the Dirichlet-Neumann operator Λ^+ is set on the exterior boundary Σ_l of Ω_l, the boundary-value problem

$$\begin{cases} \text{Find } w \text{ such that} \\ \triangle w + k^2 w = 0, \text{ in } \Omega_l, \\ w_{|\Gamma} = g, \;\; \text{on } \Gamma, \\ \partial_{\mathbf{n}_l} w + \Lambda^+ w = 0, \text{ on } \Sigma_l, \end{cases} \tag{4}$$

has a unique solution w exactly giving the solution u in the domain Ω_l. In some meaning, it is the exact inverse of V. To simplify the solution of problem (4), we rather use a microlocalization of Λ^+ in the high-frequency regime. Following [1], this leads to local non-reflecting boundary operators $\mathcal{P}^m(s, \partial_s)$ of order m ($m \in \mathbb{N}^*/2$) on Γ, variable s denoting the anticlockwise directed curvilinear abscissa along Γ. The boundary condition on Σ_l is called an On-Surface Radiation Condition (OSRC) of order m

$$\partial_{\mathbf{n}_l} w + \mathcal{P}^m(s, \partial_s) w = 0, \text{ on } \Sigma_l,$$

and w is sought as the solution to

$$\begin{cases} \text{Find } w \text{ such that} \\ \triangle w + k^2 w = 0, \text{ in } \Omega_l, \\ w_{|\Gamma} = g, \text{ on } \Gamma, \\ \partial_{\mathbf{n}_l} w + \mathcal{P}^m(s, \partial_s) w = 0, \text{ on } \Sigma_l. \end{cases} \tag{5}$$

This gives rise to an approximation of $u_{|\Omega_l}$. Here we consider as OSRCs [1]

- the Sommerfeld OSRC of order 1/2, that is, the usual radiation condition,

$$\partial_{\mathbf{n}_l} w - ikw = 0, \text{ on } \Sigma_l,$$

- and the second-order symetric Bayliss-Turkel-like OSRC

$$\left(\partial_- ik + \frac{\kappa}{2} - \frac{\kappa^2}{8(\kappa - ik)}\right) w - \partial_s \left(\frac{1}{2(\kappa - ik)} \partial_s\right) w = 0, \text{ on } \Sigma_l.$$

Here function $\kappa(s)$ stands for the curvature at point $s \in \Sigma_l$.

Considering the boundary-value problem (5), we can now describe how to construct a sparse preconditioner $\tilde{\mathcal{B}}$ by using two successive variational formulations.

Step 1: Computation of w on Ω_l.

Let us begin by computing the approximate solution w on Ω_l. To this end, we consider a smooth enough test-function φ such that $\varphi_{|\Gamma} \equiv 0$. Using the Green's formula, we obtain

$$-\int_{\Omega_l} (\triangle w + k^2 w)\varphi d\Omega_l = -\int_{\partial\Omega_l} \partial_{\mathbf{n}} w \varphi d\partial\Omega_l + \int_{\Omega_l} (\nabla w \cdot \nabla \varphi - k^2 w\varphi) d\Omega_l.$$

Since we have $\partial\Omega_l = \Sigma_l \cup \Gamma$ and $\varphi_{|\Gamma} \equiv 0$, this last relation yields

$$\begin{aligned} -\int_{\Omega_l} (\triangle w + k^2 w)\varphi d\Omega_l &= -\int_{\Sigma_l} \partial_{\mathbf{n}_l} w \varphi d\Sigma_l + \int_{\Omega_l} (\nabla w \cdot \nabla \varphi - k^2 w\varphi) d\Omega_l \\ &= \int_{\Sigma_l} \mathcal{P}^m(s, \partial s) w \varphi d\Sigma_l + \int_{\Omega_l} (\nabla w \cdot \nabla \varphi - k^2 w\varphi) d\Omega_l = 0. \end{aligned}$$

The term involving the OSRC can be symetrized from the form itself of an OSRC. This direct computation allows us to determine the unknown field w not only on the whole computational domain Ω_l but also on the fictive boundary Σ_l. From a discrete point of view, the approximate solution w is computed by the use of a $\mathbb{P}_1$ Galerkin finite element method. This implies that we have to solve a linear system defined by a sparse symetric and complex-valued matrix.

Step 2: Computation of the density $[\partial_{\mathbf{n}} w]_\Gamma$.

To determine the density $[\partial_{\mathbf{n}} w]_\Gamma$, we now choose a test-function φ such that $\varphi_{|\Gamma} \neq 0$. The Green's formula gives the weak formulation

$$-\int_{\Omega_l} (\triangle w + k^2 w)\varphi d\Omega_l = -\int_{\Gamma} \partial_{\mathbf{n}} w \varphi d\Gamma - \int_{\Sigma_l} \partial_{\mathbf{n}_l} w \varphi d\Sigma_l + \int_{\Omega_l} (\nabla w \cdot \nabla \varphi - k^2 w\varphi) d\Omega_l.$$

A more explicit version of this last equation is given by

$$\int_\Gamma \partial_{\mathbf{n}} w\varphi d\Gamma = \int_{\Sigma_l} \mathcal{P}^m(s,\partial s) w\varphi d\Sigma_l + \int_{\Omega_l} (\nabla w \cdot \nabla\varphi - k^2 w\varphi) d\Omega_l.$$

However, we have the following equalities

$$\int_\Gamma \partial_{\mathbf{n}} w\varphi d\Gamma = \int_\Gamma (\partial_{\mathbf{n}} w)^- \varphi d\Gamma - \int_\Gamma (\partial_{\mathbf{n}} w)^+ \varphi d\Gamma = \int_\Gamma [\partial_{\mathbf{n}} w]_\Gamma \varphi d\Gamma.$$

As a consequence, a possible construction of a preconditioner $\tilde{\mathcal{B}}$ is given by using the approximate solution w computed in the first step

$$\int_\Gamma \tilde{\mathcal{B}} g\varphi d\Gamma = \int_{\Omega_l} (\nabla w \cdot \nabla\varphi - k^2 w\varphi) d\Omega_l + \int_{\Sigma_l} \mathcal{P}^m(s,\partial_s) w\varphi d\Sigma_l.$$

Since the construction of $\tilde{\mathcal{B}}$ requires the solution of a linear system, then this preconditioner is implicit.

4.2 Numerical results

To test the efficiency of $\tilde{\mathcal{B}}$ used as a preconditioner, we consider the Conjuguate Gradient Squared (CGS) algorithm [4] as Krylov iterative solver. The implementation of the preconditioner is simply made by replacing each Matrix/Vector product $\mathbf{y} = \overline{V}\mathbf{x}$ involving in the initial algorithm by the two successive steps

$$\begin{cases} \tilde{\mathbf{x}} &= \tilde{\mathcal{B}}\mathbf{x}, \\ \mathbf{y} &= \overline{V}\tilde{\mathbf{x}}. \end{cases}$$

We choose a stopping criterion on the relative error

$$\frac{||\overline{V}\mathbf{x}_j - \overline{\mathbf{g}}||_2}{||\overline{\mathbf{g}}||_2} \leq \varepsilon,$$

where $\overline{\mathbf{g}}$ is the vectorial representation of the boundary element approximation of g involving in the EFIE formulation, vector $\mathbf{x}_j$ is the approximation of p at the j-th iteration of the CGS algorithm and ε is an *a priori* fixed tolerance. The two plots in Fig. 2 depict the number of Matrix/Vector products versus the relative norm of the residue for the preconditioned and not preconditioned CGS algorithms, ε being fixed to 10^{-6}. We report the results of two tests corresponding to $k = 20$ and $k = 40$. Only one layer of finite elements has been used to surround Γ.

Note that the number of iterations has been approximately divided by a factor 5. OSRCs of orders 1/2 and 2 have been tested and have yielded the same rate of convergence. Here, the choice of the layer does not take into account the curvature effect. We think that a high order OSRC would be more effective for surfaces where the curvature has more significant effect on the scattering. Moreover, we have observed that it is independent of

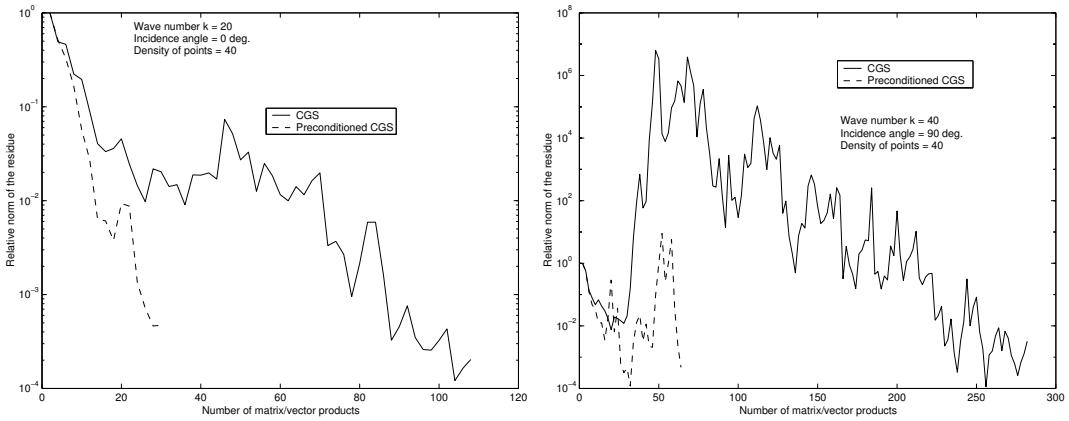

Figure 1: Evolution of the norm of the relative residual as a function of the number of Matrix/Vector products for the preconditioned and not preconditioned CGS algorithms.

the angle of incidence. Therefore, this preconditioner is robust relatively to the frequency and the angle of incidence.

As a conclusion, we have constructed and tested a new robust implicit preconditioner for the EFIE which is based on the resolution of an approximate local boundary-value problem. It applies to open surfaces as e.g. for the case of the strip. Its efficiency is linked to the fact that it takes into account the computation of the creeping waves and the evanescent rays, modifying hence suitably the spectrum of the EFIE. Our perspectives are now to make some more intensive calculations for other curved open scatterers and to extend it to the TE-polarization case. Moreover, it has to be incorporated in a FMM to really prove its efficiency.

References

[1] X. Antoine, H. Barucq and A. Bendali, *Bayliss-Turkel-like radiation conditions on surfaces of arbitrary shape*, J. Math. Anal. Appl. **229** (1999), pp.184-211.

[2] W. C. Chew and K. F. Warnick, *On the spectrum of the electric field integral equation and the convergence of the moment method*, Int. J. Numer. Meth. Engng. **51** (2001), pp.475-489.

[3] V. Rokhlin, *Rapid solution of integral equations of scattering theory in two-dimensions*, J. Comput. Phys. **186** (1990), pp.414-439.

[4] Y. Saad, *Iterative Methods for Sparse Linear Systems*, PWS Pub. Co., Boston, 1996.