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Study of the scalar Oseen equation

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Abstract

This paper is devoted to the scalar Oseen equation, a linearized form of the Navier-Stokes equations. Because of the various decay properties in various directions of $\mathbb{R}^N$, the problem is set in Sobolev spaces with anisotropic weights. In a first step, some weighted Hardy-type inequalities are obtained, which yield some norm equivalences. In a second step, we establish existence results.

Keywords: Oseen equations, anisotropic weights, Hardy inequality, Sobolev spaces, Exterior domains

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1 Introduction.

Let Ω be an exterior domain of $\mathbb{R}^N$, $N \geq 2$. We consider the following system:

$$\begin{cases} -\nu\Delta u + \rho u_0.\nabla u + \nabla P = f \quad \text{in} \quad \Omega \\ \text{div } u = 0 \quad \text{in} \quad \Omega \\ u = u_* \quad \text{on} \quad \partial\Omega \\ \lim\limits_{|x|\to\infty} u(x) = u_\infty. \end{cases} \tag{1}$$

C. W. Oseen [7] obtained (1) by linearising the Navier-stokes equations, describing the flow of a viscous and incompressible fluid past several obstacles, around a nonzero constant solution u_0. Thus, the result offers a better approximation than that of Stokes. The viscosity ν, the density ρ, the external force f, and the boundary values u_* on $\partial\Omega$ are given. The unknown velocity field u is assumed to converge to a constant vector u_∞, and the scalar P denotes the unknown pressure. Among the works devoted to the system (1), which is called the Oseen equations, we can cite Finn [5], and more recently Farwig [4], Galdi [6]. The purpose of this paper is to study a simplified case of (1), the scalar Oseen equation:

$$-\nu\Delta u + k\frac{\partial u}{\partial x_1} = f \text{ in } \mathbb{R}^N, \;\; k > 0. \tag{2}$$

To prescribe the growth or the decay properties of functions at infinity, the problem is set in weighted Sobolev spaces. Since the fundamental solution $E(x)$ of (2),

$$E(x) = \frac{1}{4\pi\nu r}e^{-ks/2\nu}, \;\; r = |x|, \;\; s = r - x_1, \tag{3}$$

has anisotropic decay properties, we will deal with the anisotropic weights introduced by Farwig [3, 4]. The case $k = 0$ yields the Laplace's equation studied by Amrouche-Girault-Giroire [1] in weighted Sobolev spaces. In a first step, we establish anisotropically weighted Poincaré-type inequalities and,in a second part, we present some existence results.

2 Notations

In this paper, we will use the following notations:

$$r = r(x) = |x| = (x_1^2 + x_2^2 + ... + x_N^2)^{1/2}, \;\; x \in \mathbb{R}^N$$

$$s = s(x) = r - x_1, \;\; \rho = \rho(x) = (1 + r^2)^{1/2}.$$

For the anisotropic weights, we set

$$\eta_\beta^\alpha = (1 + r)^{\alpha/2}(1 + s)^{\beta/2}.$$

We will use the following spaces, $\alpha \in \mathbb{R}$, $1 < p < +\infty$,

$$W_\alpha^{1,p}(\Omega) = \left\{v \in \mathcal{D}'(\Omega), \rho^{\alpha-1}v \in L^p(\Omega), \rho^\alpha\nabla v \in \mathbf{L}^p(\Omega)\right\} \text{ if } n/p + \alpha \neq 1,$$

with its natural norm

$$\|v\|_{W_\alpha^{1,p}(\Omega)} = \left(\|\rho^{\alpha-1}v\|_{L^p(\Omega)}^p + \|\rho^\alpha\nabla v\|_{\mathbf{L}^p(\Omega)}^p\right)^{1/p},$$

and semi-norm

$$|v|_{W_\alpha^{1,p}(\Omega)} = \|\rho^\alpha\nabla v\|_{\mathbf{L}^p(\Omega)}.$$

For the anisotropically weighted Sobolev spaces, we set

$$H_{\alpha,\beta}^{1,p}(\Omega) = \left\{v \in \mathcal{D}'(\Omega), \eta_{\beta-1}^{\alpha-1}v \in L^p(\Omega), \eta_\beta^\alpha\nabla v \in \mathbf{L}^p(\Omega)\right\},$$

$$X_{\alpha,\beta}^{1,p}(\Omega) = \left\{v \in \mathcal{D}'(\Omega), \eta_\beta^{\alpha-2}v \in L^p(\Omega), \eta_\beta^\alpha\nabla v \in \mathbf{L}^p(\Omega)\right\},$$

$$W_{\alpha,\beta}^{1,p}(\Omega) = \left\{v \in \mathcal{D}'(\Omega), \eta_\beta^{\alpha-1}v \in L^p(\Omega), \eta_\beta^\alpha\nabla v \in \mathbf{L}^p(\Omega)\right\},$$

$$\overset{o}{W}{}_{\alpha,\beta}^{1,p}(\Omega) = \{v \in W_{\alpha,\beta}^{1,p}(\Omega), v = 0 \text{ on } \partial\Omega\},$$

equipped with their natural norms.
The dual of $\overset{o}{W}{}^{1,p}_{\alpha,\beta}(\Omega)$ is noted $W^{-1,p'}_{-\alpha,-\beta}(\Omega)$, with $1/p + 1/p' = 1$. If $\Omega = \mathbb{R}^N$, we have $\overset{o}{W}{}^{1,p}_{\alpha,\beta}(\Omega) = W^{1,p}_{\alpha,\beta}(\mathbb{R}^N)$.
Let $j = \min\{[-1/2 - N/p - \alpha/2], [-1 - N/p - (\alpha+\beta)/2]\}$, we have $\mathcal{P}_j \subset H^{1,p}_{\alpha,\beta}(\Omega)$. $\mathcal{P}_j$ stands for the space of polynomials of degree lower than j and $[a]$ for the integer part of a. We set $B_R = B(0,R)$ and $B'_R = \mathbb{R}^N \setminus \overline{B_R}$. Finally, in what follows, by $f \sim g$ in U, we mean the following: there exists $C_1, C_2 > 0$, such that

$$\forall x \in U, \ \ C_1 f(x) \le g(x) \le C_2 f(x).$$

3 Weighted Hardy-type inequalities.

A fundamental property of the weighted Sobolev spaces $W^{1,p}_{\alpha}(\Omega)$ is that their elements satisfy Hardy-type inequalities. Amrouche-Girault-Giroire [2] proved that, for $\alpha \in \mathbb{R}$,
(i) the semi-norm $|.|_{W^{1,p}_{\alpha}(\Omega)}$ defines on $W^{1,2}_{\alpha}(\Omega)/\mathcal{P}_{j'}$ a norm which is equivalent to the quotient norm, where $j' = \inf(j, 0)$.
(ii) The semi-norm $|.|_{W^{1,p}_{\alpha}(\Omega)}$ defines on $\overset{o}{W}{}^{1,p}_{\alpha}(\Omega)$ a norm which is equivalent to the full norm $\|.\|_{W^{1,p}_{\alpha}(\Omega)}$.
We shall establish similar results in the case of anisotropically weighted Sobolev spaces.
We choose to consider the particular case $N = 3$, $p = 2$, but the results can be generalised to $N \ge 2$ and $p \ge 2$.
We consider the sector

$$S = S_{R,\lambda} = \{x \in \mathbb{R}^3; r \ge R, 0 \le s \le \lambda r\}, \quad R > 0, 0 < \lambda < 1. \tag{4}$$

In $\mathbb{R}^3 \setminus S$, we have $r \sim s$. Therefore, the spaces $H^{1,2}_{\alpha,\beta}(\mathbb{R}^3 \setminus S)$ and $W^{1,2}_{(\alpha+\beta)/2}(\mathbb{R}^3 \setminus S)$ coincide algebraically and topologically. It follows that, in $\mathbb{R}^3 \setminus S$, the previous results hold. Thus, it is enough to prove anisotropically weighted Hardy-type inequalities in S.
We first deal with the case $\beta > 0$.

Lemma 1 *Let $\alpha, \beta \in \mathbb{R}$ satisfy $\beta > 0$. Then there exists a constant $C > 0$, such that*

$$\forall u \in \overset{o}{H}{}^{1,2}_{\alpha,\beta}(S), \quad \|u\|_{H^{1,2}_{\alpha,\beta}(S)} \le C|u|_{H^{1,2}_{\alpha,\beta}(S)} \tag{5}$$

Idea of the proof. We first prove the inequality for $u \in \mathcal{D}(S)$, then by density, we prove it for all u in $\overset{o}{H}{}^{1,2}_{\alpha,\beta}(S)$. Since $\beta > 0$, it is enough to prove

$$I = \int_S (1+r)^{\alpha-1} s^{\beta-1} |u|^2 dx \le C \int_S (1+r)^{\alpha} s^{\beta} |\nabla u|^2 dx. \tag{6}$$

Using polar coordinates with $u(x) = v(r, \theta, \varphi)$, (6) is equivalent to the following inequality

$$\begin{aligned} I &= \int_0^{2\pi}\int_R^{+\infty}\int_0^{\theta_0} (1+r)^{\alpha-1}(r - r\cos\theta)^{\beta-1} r^2 \sin\theta |v|^2 d\theta dr d\varphi \\ &\leq C \int_0^{2\pi}\int_R^{+\infty}\int_0^{\theta_0} (1+r)^{\alpha}(r - r\cos\theta)^{\beta} \sin\theta |\frac{\partial v}{\partial \theta}|^2 d\theta dr d\varphi, \qquad (7) \end{aligned}$$

with

$$\theta_0 \text{ such that } \cos\theta_0 = 1-\lambda, \;\; 0 < \lambda < 1.$$

We set

$$J = \int_0^{\theta_0} (1-\cos\theta)^{\beta-1} \sin\theta |v|^2 d\theta.$$

An integration by parts yields

$$J = \frac{1}{\beta}[(1-\cos\theta)^{\beta}|v|^2]_0^{\theta_0} - \frac{2}{\beta}\int_0^{\theta_0} (1-\cos\theta)^{\beta}\frac{\partial v}{\partial \theta} v d\theta.$$

Since $\beta > 0$ and $v \in \mathcal{D}(S)$, we have

$$J \leq \frac{2}{\beta}\int_0^{\theta_0} (1-\cos\theta)^{\beta}|\frac{\partial v}{\partial \theta}||v| d\theta.$$

Using the Cauchy-Schwarz inequality, we get

$$J \leq \frac{4}{\beta^2}\int_0^{\theta_0} (1-\cos\theta)^{\beta+1}|\frac{1}{\sin\theta}\frac{\partial v}{\partial \theta}|^2 d\theta.$$

This last inequality allows to have (7).■

Remark 2 *Inequality (5) is not valid for* $\beta \leq 0$. *For* $\beta = 0$, *Farwig [3] gave a counter-example with the case* $\alpha = 0$. *For* $\beta < 0$, *taking as counter-example* $v(r, \theta, \varphi) = v(r)$, *we can show that the inequality (7) does not hold.*

Nevertheless, for $\beta \leq 0$, we have the analogue of Lemma 1 in the anisotropically weighted Sobolev space $X^{1,2}_{\alpha,\beta}(S)$.

Lemma 3 *Let* $\alpha, \beta \in \mathbb{R}$ *satisfy* $\beta \leq 0$ *and* $\alpha + \beta + 2 > 0$. *Then there exists* $C > 0$, *such that*

$$\forall u \in \overset{o}{X}{}^{1,2}_{\alpha,\beta}(S), \quad \|u\|_{X^{1,2}_{\alpha,\beta}(S)} \leq C|u|_{X^{1,2}_{\alpha,\beta}(S)}.$$

Idea of the proof. Let $u \in \mathcal{D}(S)$ and $u(x) = v(r, \theta, \varphi)$. For $R > 0$ sufficiently large, it is enough to prove

$$\begin{aligned} I &= \int_0^{2\pi}\int_R^{+\infty}\int_0^{\theta_0} r^{\alpha+1}(1 + r - r\cos\theta)^{\beta} \sin\theta |v|^2 d\theta dr d\varphi \qquad (8) \\ &\leq C \int_0^{2\pi}\int_R^{+\infty}\int_0^{\theta_0} r^{\alpha+3}(1 + r - r\cos\theta)^{\beta} \sin\theta |\nabla u|^2 d\theta dr d\varphi. \end{aligned}$$

We set

$$J=\int_R^{+\infty} r^{\alpha+1}(1+r-r\cos\theta)^\beta|v|^2dr.$$

Since $\beta\leq 0$ and $\alpha+\beta+2>0$, we have

$$J\leq\frac{1}{\alpha+\beta+2}\int_R^{+\infty}\frac{\partial}{\partial r}[r^{\alpha+2}(1+r-r\cos\theta)^\beta]|v|^2dr.$$

An integration by parts and the Cauchy-Schwarz inequality yields

$$J\leq\frac{4}{(\alpha+\beta+2)^2}\int_R^{+\infty} r^{\alpha+3}(1+r-r\cos\theta)^\beta|\frac{\partial v}{\partial r}|^2dr,$$

which allows to obtain (8).■

By Lemma 1, we have the two following results.

Lemma 4 *Let $\alpha,\beta,R\in\mathbb{R}$ satisfy $\beta>0$, $\alpha+\beta+1\neq 0$ and $R>0$. Then, there exists a constant $C_R>0$ such that*

$$\forall u\in \overset{o}{H}{}^{1,2}_{\alpha,\beta}(B'_R),\quad \|u\|_{H^{1,2}_{\alpha,\beta}(B'_R)}\leq C_R|u|_{H^{1,2}_{\alpha,\beta}(B'_R)}. \tag{9}$$

In other words, the semi-norm $|.|_{H^{1,2}_{\alpha,\beta}(B'_R)}$ is a norm on $\overset{o}{H}{}^{1,2}_{\alpha,\beta}(B'_R)$ equivalent to the norm of $H^{1,2}_{\alpha,\beta}(B'_R)$.

Idea of the proof. It is enough to consider $u\in\mathcal{D}(B'_R)$. We use the following partition of unity

$$\varphi_1,\varphi_2\in\mathcal{C}^\infty(B'_R),\ \ 0\leq\varphi_1,\varphi_2\leq 1,\ \ \varphi_1+\varphi_2=1 \text{ in } B'_R,$$

with

$$\varphi_1=1 \text{ in } S_{R,\lambda/2},\ \ \text{supp}\varphi_1\subset S_{R,\lambda}.$$

We have

$$\|u\|_{H^{1,2}_{\alpha,\beta}(B'_R)}=\|\varphi_1u+\varphi_2u\|_{H^{1,2}_{\alpha,\beta}(B'_R)}\leq\|\varphi_1u\|_{H^{1,2}_{\alpha,\beta}(B'_R)}+\|\varphi_2u\|_{H^{1,2}_{\alpha,\beta}(B'_R)}.$$

Since $\beta>0$, Lemma 1 yields

$$\|\varphi_1u\|_{H^{1,2}_{\alpha,\beta}(B'_R)}=\|\varphi_1u\|_{H^{1,2}_{\alpha,\beta}(S_{R,\lambda})}\leq C|\varphi_1u|_{H^{1,2}_{\alpha,\beta}(S_{R,\lambda})}=C|\varphi_1u|_{H^{1,2}_{\alpha,\beta}(B'_R)}$$

Since $\alpha+\beta+1\neq 0$, using the following Hardy-type inequality

$$\forall v\in\mathcal{D}(]R,+\infty[),\ \int_R^{+\infty}(1+t)^\gamma t^\xi|v(t)|^pdt\leq(\frac{p|\gamma+\xi+1|}{c})^p\int_R^{+\infty}(1+t)^{\gamma+p}t^\xi|v'(t)|^pdt$$

with γ, ξ, $R\in\mathbb{R}$ such that $\xi>0$, $\gamma+\xi+1\neq 0$ and $(\gamma+\xi+1)^2R+\xi(\gamma+\xi+1)>0$, we get

$$|\varphi_1u|_{H^{1,2}_{\alpha,\beta}(B'_R)}\leq C|u|_{H^{1,2}_{\alpha,\beta}(B'_R)}.$$

Thus, we have

$$\|\varphi_1 u\|_{H^{1,2}_{\alpha,\beta}(B'_R)} \leq C|u|_{H^{1,2}_{\alpha,\beta}(B'_R)},$$

and by the same method, we get

$$\|\varphi_2 u\|_{H^{1,2}_{\alpha,\beta}(B'_R)} \leq C|u|_{H^{1,2}_{\alpha,\beta}(B'_R)},$$

which conclude the proof.■

Theorem 5 *Let $\alpha, \beta \in \mathbb{R}$ satisfy $\beta > 0$ and $\alpha + \beta + 1 \neq 0$. Let $j' = \inf(j, 0)$, where j is the highest degree of the polynomials contained in $H^{1,2}_{\alpha,\beta}(\Omega)$. Then the semi-norm $|.|_{H^{1,2}_{\alpha,\beta}(\Omega)}$ defines on $H^{1,2}_{\alpha,\beta}(\Omega)/\mathcal{P}_{j'}$ a norm which is equivalent to the quotient norm.*

4 Weak solutions of the scalar Oseen equation.

In this section, we propose to solve the scalar Oseen equation with $\nu = k = 1$, $N = 3$:

$$-\Delta u + \frac{\partial u}{\partial x_1} = f \text{ in } \mathbb{R}^3. \tag{10}$$

We introduce the concept of weak solution.

Definition 6 *A function $u : \mathbb{R}^3 \to \mathbb{R}$ is called a weak solution to (10) if*
(i) $u \in H^1_{loc}(\mathbb{R}^3)$,
(ii) u satisfies

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3), \ \int_{\mathbb{R}^3} \nabla u . \nabla \varphi dx - \int_{\mathbb{R}^3} u \frac{\partial \varphi}{\partial x_1} = [f, \varphi]. \tag{11}$$

We are, first, interested in existence of weak solutions when the data $f \in W_0^{-1,2}(\mathbb{R}^3)$, which is the dual of $W_0^{1,2}(\mathbb{R}^3)$.

Theorem 7 *Given a function $f \in W_0^{-1,2}(\mathbb{R}^3)$, the problem (10) has a weak solution $u \in W_0^{1,2}(\mathbb{R}^3)$ such that*

$$\|\nabla u\|_{\mathbf{L}^2(\mathbb{R}^3)} \leq \|f\|_{W_0^{-1,2}(\mathbb{R}^3)}. \tag{12}$$

More over

$$\frac{\partial u}{\partial x_1} \in W_0^{-1,2}(\mathbb{R}^3). \tag{13}$$

Idea of the proof. For $R > 0$, we consider the following equations

$$\begin{cases} -\Delta u + \dfrac{\partial u}{\partial x_1} = f \text{ in } B_R \\ u = 0 \text{ on } \partial B_R, \end{cases} \tag{14}$$

Since $f \in W_0^{-1,2}(\mathbb{R}^3)$, we have $f \in H^{-1}(B_R)$, thus, by Lax-Milgram theorem, we prove the existence of a unique weak solution $u_R \in H_0^1(B_R)$ to problem (14) such that

$$\|\nabla u_R\|_{\mathbf{L}^2(B_R)} \leq \|f\|_{W_0^{-1,2}(\mathbb{R}^3)}, \tag{15}$$

then, it suffices consider a sequence of problems analogous to (14) and to choose a weakly convergent subsequence.■

We now look for weak solutions when the data $f \in W_{\alpha,\beta}^{-1,2}(\mathbb{R}^3)$.

Theorem 8 *Let $\alpha, \beta \in \mathbb{R}$ satisfy $\beta > 0$ and $\beta > |\alpha|$. Then for a function $f \in W_{\alpha,\beta}^{-1,2}(\mathbb{R}^3)$, there exists a weak solution $u \in W_{\alpha,\beta}^{1,2}(\mathbb{R}^3)$ to (10) such that*

$$\|u\|_{W_{\alpha,\beta}^{1,2}(\mathbb{R}^3)} \leq C\|f\|_{W_{\alpha,\beta}^{-1,2}(\mathbb{R}^3)}. \tag{16}$$

Idea of the proof. Let $R > 0$ be given and let $u_R \in H_0^1(B_R)$ be the unique weak solution of (14). We need to prove the uniform estimate

$$\|u_R\|_{W_{\alpha,\beta}^{1,2}(B_R)} \leq C\|f\|_{W_{\alpha,\beta}^{-1,2}(\mathbb{R}^3)}, \tag{17}$$

which allows to end the proof as in the previous Theorem. In the variationnal equation

$$\forall \varphi \in H_0^1(B_R), \quad \int_{B_R} \nabla u_R . \nabla \varphi dx + \int_{B_R} \frac{\partial u_R}{\partial x_1} \varphi dx = [f, \varphi],$$

we use the test function $\varphi = \eta_{2\beta}^{2\alpha} u_R$, thus, by an integration by parts, we get

$$\int_{B_R} \eta_{2\beta}^{2\alpha} |\nabla u_R|^2 dx + \int_{B_R} u_R \nabla u_R . \nabla \eta_{2\beta}^{2\alpha} - \frac{1}{2}\int_{B_R} |u_R|^2 \frac{\partial \eta_{2\beta}^{2\alpha}}{\partial x_1} dx = [f, \eta_{2\beta}^{2\alpha} u_R].$$

The Young inequality implies that

$$\int_{B_R} \eta_{2\beta}^{2\alpha} |\nabla u_R|^2 dx + \frac{1}{2}\int_{B_R} \left(-\frac{\partial \eta_{2\beta}^{2\alpha}}{\partial x_1} - \frac{|\nabla \eta_{2\beta}^{2\alpha}|^2}{\eta_{2\beta}^{2\alpha}} \right) |u_R|^2 dx \leq [f, \eta_{2\beta}^{2\alpha} u_R].$$

Introducing the equivalent anisotropic weight functions

$$\eta_\beta^\alpha = (1 + \delta r)^{\alpha/2} (1 + \varepsilon s)^{\beta/2} \tag{18}$$

with sufficiently small positive constants δ and ε, Farwig [3] proved that if α, $\beta \in \mathbb{R}$ satisfy $\beta > 0$ and $|\alpha| < \beta$, then there are positive numbers $c_1(\delta, \varepsilon) = O(\delta) + O(\varepsilon)$, $c_2(\delta) = O(\delta)$, such that

$$-\frac{\partial \eta_{2\beta}^{2\alpha}}{\partial x_1} - \frac{|\nabla \eta_{2\beta}^{2\alpha}|^2}{\eta_{2\beta}^{2\alpha}} \geq (((\beta - |\alpha|) - c_1(\delta, \varepsilon))\delta\varepsilon s(x) - c_2(\delta))\eta_{2\beta-2}^{2\alpha-2}(x), \quad x \in \mathbb{R}^3. \tag{19}$$

This result with Theorem 5 yield (17).■

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