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Hydrodynamical modelling and multidimensional approximation of estuarian river flows

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Abstract

We propose here a new approach for deriving 2D and 1D hydrodynamical models, within the framework of mixed variational formulations. We thus obtain a 2D-horizontal model, as well as a 2D-vertical and a 1D model taking into account the geometry of the river. We analyze here only the 3D model and the 2D-horizontal one, for which we propose (after time discretization) low-order conforming finite element approximations. All the variational problems are well-posed. We intend to justify next a posteriori estimators between these models.

Keywords: Hydrodynamical modelling, error estimates, finite elements
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1 Introduction

We are interested in the hydrodynamical modelling and the multidimensional numerical approximation of an estuarian river flow. Near the estuary, one needs a quite fine modelling, which takes into account the tide effects, the salinity and the temperature of the water etc. The ideal model to be employed is a two-phase three-dimensional model, but due to the huge computational cost, it is obvious that one cannot use it on the whole length of the river.

Therefore, it would be interesting to employ different models on different zones of the river, and to couple them by the means of an *a posteriori* error estimators' technique. This is the purpose of our work : on the one hand, the derivation of multidimensional hydrodynamical models (3D, 2D and 1D) and their approximation (with respect to time and space) and, on the other hand, their coupling and the automatic determination of the 3D, 2D and 1D zones.

Concerning the modelization, the 3D model (to be employed in the estuarian zone) is a classical one, based on the instationary Navier-Stokes equations with physical boundary conditions. Usually, the 2D and 1D models used in fluvial hydrodynamics are of shallow-water type, but their mathematical justification is not very rigorous. We propose a new approach within the framework of variational formulations (so, one can now get error estimates between the continuous models). More precisely, we put the 3D problem in a mixed form and then obtain the lower-dimensional problems by a projection method.

In this manner, we get two bidimensional models which we call 2D-horizontal and 2D-vertical, whether they are written on the free surface or on the median longitudinal surface of the river. We also get a one-dimensional model, written on the median curve of the river. All these models provide a three-dimensional velocity (which is not the case for the shallow-water equations), and the pressure is not supposed to be hydrostatic but is now an unknown of the problem. Another advantage of this approach is that the 2D-vertical and the 1D models take into account the geometry of the river, since they are written in curvilinear coordinates; we can thus treat rivers with varying width and curvature.

In this paper, we detail only the 3D and the 2D-horizontal models. We show that the mixed velocity-pressure variational formulations corresponding to the time-discretized models are well-posed, thanks to the Babuska-Brezzi's theory. Their approximation is achieved by conforming low-order finite elements and numerical tests are carried on on the the river Adour (south-west of France), for which we dispose of its real bathymetry. These last topics are not presented here, but they are treated in the framework of a project with IFREMER. The other two models are briefly introduced. Their analysis, as well as the *a posteriori* estimators between the different models (at the continuous and the discrete level) will be presented in a future work.

2 The three-dimensional physical model

We begin by presenting the 3D model, where we have neglected the temperature and the salinity of the water and we have assumed that the density ρ is constant. We take into account the Coriolis force, the wind force at the free surface and the friction at the bottom. Then the conservation laws write, in the 3D domain $\Omega_F(t)$ of the fluid :

$$\begin{cases} div\mathbf{u} = 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \underline{\nabla \mathbf{u}}\,\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\Delta \mathbf{u} + \mathbf{g} + 2\omega \times \mathbf{u}, \end{cases}$$

to which we add initial conditions : $\mathbf{u}(0) = \mathbf{u}_0, \quad p(0) = p_0$ as well as the following physical boundary conditions :

$$\begin{aligned}
&\Gamma_B(t): \quad \mathbf{u}\cdot\mathbf{n}=0 \text{ (no}-\text{slip condition)}, \quad \mu\,\mathbf{curl}\,\mathbf{u}\wedge\mathbf{n}=-c_B\mathbf{u} \text{ (friction)}\\
&\Gamma_H(t): \quad p=p_a, \quad \mu\,\mathbf{curl}\,\mathbf{u}\wedge\mathbf{n}=\mathbf{w} \text{ (wind's effect)}\\
&\Gamma_M(t): \quad \mathbf{u}\cdot\mathbf{n}=k_M, \quad \mu\,\mathbf{curl}\,\mathbf{u}\wedge\mathbf{n}=0\\
&\Gamma_V(t): \quad \mathbf{u}\cdot\mathbf{n}=k_V, \quad \mu\,\mathbf{curl}\,\mathbf{u}\wedge\mathbf{n}=\beta_V.
\end{aligned}$$

The boundary $\Gamma_B(t)$ denotes the bottom, while $\Gamma_H(t)$ is the free surface of the river. $\Gamma_M(t)$ and $\Gamma_V(t)$ are the inflow, respectively the outflow boundary.

The free surface $\Gamma_H(t)$ will represent the 2D domain, therefore we want it to be fixed and meshed once for all. So we write the problem in a maximal domain $\Omega \supset \Omega_F(t)$, independent upon time and sufficiently smooth, whose horizontal surface is now denoted by Σ. Then we note by $\Sigma_F(t)$ the projection of $\Gamma_H(t)$ onto the fixed surface Σ . We also denote by $h(x,y,t)$ the height of the water (unknown), by $Z_B(x,y)$ the elevation of the bottom (given), and by $H(x,y)$ the elevation of the fixed horizontal surface Σ (given). So, the domain $\Omega_F(t)$ is characterized by : $(x,y)\in\Sigma_F(t)$, $Z_B\le z\le Z_B+h$ and the free surface $\Gamma_H(t)$ by : $(x,y)\in\Sigma_F(t)$, $z=Z_B(x,y)+h(x,y,t)$. We put $\partial\Omega=\Gamma_M\cup\Gamma_V\cup\Gamma_B\cup\Sigma$.

Let us also introduce a phase coefficient :

$$\alpha(x,y,z,t)=\begin{cases} 1 \text{ in } \Omega_F(t)\\ 0 \text{ in } \Omega\setminus\Omega_F(t)\end{cases},$$

with a given initial condition. This coefficient satisfies the following transport equation :

$$\frac{d\alpha}{dt}=0 \;\Leftrightarrow\; \frac{\partial\alpha}{\partial t}+\nabla\alpha\cdot\mathbf{u}=0 \quad \text{in } \Omega.$$

Finally, the boundary value problem writes, in the unknowns $(\alpha,\mathbf{u},p)$, as follows :

$$\begin{cases}
\frac{d\alpha}{dt}=0 & \text{in } \Omega\\
div\mathbf{u}=0 & \text{in } \Omega_F(t)\\
\frac{d\mathbf{u}}{dt}+\frac{\mu}{\rho}curl(\mathbf{curl}\,\mathbf{u})-2\omega\times\mathbf{u}=-\frac{1}{\rho}\nabla p+\mathbf{g} & \text{in } \Omega_F(t)
\end{cases}.$$

3 Time discretization and variational formulation

We discretize the transport terms by the characteristics' method (see for instance [3]) :

$$\begin{cases}
\alpha^{n+1}=\alpha^n\circ\chi^n & \text{in } \Omega\\
div\mathbf{u}^{n+1}=0 & \text{in } \Omega_{n+1}\\
\frac{\mathbf{u}^{n+1}-\mathbf{u}^n\circ\chi^n}{\Delta t}+\frac{\mu}{\rho}curl(\mathbf{curl}\,\mathbf{u}^{n+1})-2\omega\times\mathbf{u}^{n+1}=-\frac{1}{\rho}\nabla p^{n+1}+\mathbf{g} & \text{in } \Omega_{n+1}
\end{cases}$$

where the domain Ω_{n+1} occupied by the fluid at t_{n+1} is defined by $\alpha^{n+1}=1$. The function $\chi^n(x)$ represents an approximation of $X(x,t_{n+1};t_n)$, where $X(x,t;\tau)$ is the solution of the following Cauchy problem : $\frac{dX}{d\tau}=\mathbf{u}(X(\tau),\tau)$, $\quad X(t)=x$. For instance, one can use the Euler scheme : $\chi^n(x) = x - u^n(x)\Delta t$, which gives an approximation of order $O(\Delta t^2)$, or a Runge-Kutta scheme of $O(\Delta t^3)$.

For the sake of simplicity, we note in the sequel α^{n+1} by α. In order to give a variational formulation of the problem in $(\mathbf{u}^{n+1}, p^{n+1})$, let us introduce the weighted space :

$$\mathbf{V} = \left\{\mathbf{v} \in H(\mathbf{curl}, div; \Omega, \alpha);\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_M \cup \Gamma_V \cup \Gamma_B,\ \mathbf{v}\wedge\mathbf{n} \in (L^2(\Gamma_B))^3\right\}$$

endowed with the norm :

$$\|\mathbf{v}\|_V^2 = \|\mathbf{v}\|_{0,\Omega,\alpha}^2 + \|div\mathbf{v}\|_{0,\Omega,\alpha}^2 + \|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega,\alpha}^2 + \|\mathbf{v}\wedge\mathbf{n}\|_{0,\Gamma_B,\alpha}^2 .$$

Remark : If one imposes homogeneous boundary conditions (of $\mathbf{v}\cdot\mathbf{n}$ or $\mathbf{v}\wedge\mathbf{n}$ type) on the whole boundary $\partial\Omega$, then one can establish (*cf.* Costabel) that $\mathbf{v} \in (L^2(\partial\Omega))^3$. In our case, due to the boundary conditions considered on the free surface $\Gamma_H(t)$, we can only obtain that $\mathbf{v} \in (L^2_{loc}(\Gamma_M \cup \Gamma_V \cup \Gamma_B))^3$. Then the time-discretized boundary value problem writes as :

$$\begin{cases} \text{find } \mathbf{u}^{n+1} \in \mathbf{V}^*,\ p^{n+1} \in L^2(\Omega,\alpha) \text{ such that} \\ \forall \mathbf{v} \in \mathbf{V}, \quad A(\mathbf{u}^{n+1}, \mathbf{v}) + B(p^{n+1}, \mathbf{v}) = F_n(\mathbf{v}) \\ \forall q \in L^2(\Omega,\alpha), \quad B(q, \mathbf{u}^{n+1}) = 0 \end{cases} \tag{1}$$

where :

$$\mathbf{V}^* = \{\mathbf{v} \in H(\mathbf{curl}, div; \Omega, \alpha); \\ \mathbf{v}\cdot\mathbf{n} = k \text{ on } \Gamma_M \cup \Gamma_V,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \Gamma_B, \mathbf{v}\wedge\mathbf{n} \in (L^2(\Gamma_B))^3\right\}$$

$$A(\mathbf{u},\mathbf{v}) = \int_\Omega \frac{\alpha\rho}{\Delta t}\mathbf{u}\cdot\mathbf{v} + \int_\Omega \alpha\mu\,\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} + \int_{\Gamma_B} \alpha c_B \mathbf{u}\wedge\mathbf{n}\cdot\mathbf{v}\wedge\mathbf{n} - 2\int_\Omega \alpha\rho(\omega,\mathbf{u},\mathbf{v})$$

$$B(p,\mathbf{v}) = -\int_\Omega \alpha p\, div\mathbf{v}$$

$$F_n(\mathbf{v}) = \int_\Omega (\frac{\alpha\rho}{\Delta t}\mathbf{u}^n \circ \chi^n + \alpha\rho\mathbf{g})\cdot\mathbf{v} + <\mathbf{v}\wedge\mathbf{n}, \beta\wedge\mathbf{n}>_{\Gamma_H(t)\cup\Gamma_V(t)} - <\mathbf{v}\cdot\mathbf{n}, \widetilde{p_a}>_{\partial\Omega_F(t)} .$$

We denoted by $\widetilde{p_a}$ a function of $H^{1/2}(\partial\Omega_F(t))$ whose restriction to $\Gamma_H(t)$ is equal to p_a. We have also denoted by β the function defined by $\beta = \beta_V$ on $\Gamma_V(t)$ and $\beta = \mathbf{w}$ on $\Gamma_H(t)$, under the hypothesis $\beta\wedge\mathbf{n} \in H_{00}^{1/2}(\Gamma_H(t)\cup\Gamma_V(t))$.

Thanks to the Babuska-Brezzi's theory for mixed formulation (see [1] for more details), we can easily show that this problem is well-posed.

4 The 2D-horizontal model

In order to obtain a hydrodynamical model written on the 2D plane domain $\Sigma_F(t)$, let us project the three-dimensional solution $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{V}^* \times L^2(\Omega,\alpha)$, calculated at t_{n+1}, on some conveniently chosen subspaces $\mathbf{V}_0 \subset \mathbf{V}$ and $M \subset L^2(\Omega, \alpha)$. The domain $\Sigma_F(t)$ is described by a new phase coefficient, defined as below :

$$\alpha_{2D}(x,y,t) = \begin{cases} 1 \text{ on } \Sigma_F(t) \\ 0 \text{ on } \Sigma \setminus \Sigma_F(t) \end{cases}, \forall (x,y) \in \Sigma.$$

This coefficient α_{2D} satisfies a transport equation, analogous to the one written in 3D :

$$\frac{\partial \alpha_{2D}}{\partial t} + \nabla \alpha_{2D} \cdot \mathbf{u}_{2D} = 0 \Leftrightarrow \frac{d\alpha_{2D}}{dt} = 0 \quad \text{in } \Sigma.$$

We next choose the weighted spaces :

$$\begin{aligned} M &= \left\{ q = q(x,y) \in L^2(\Sigma, h) \right\} \\ \mathbf{V}_0 &= \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} ;\ \mathbf{v}_{2D} \in H(curl, div; \Sigma, h),\ v_3 \in H^1(\Sigma, h),\ \mathbf{v}_{2D} \cdot \mathbf{n}_{2D} = 0 \text{ on } \Sigma_M \cup \Sigma_V \right\} \end{aligned}$$

where we have put : $\mathbf{v}_{2D} = \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}$ and $v_3(x,y) = \mathbf{v}_{2D} \cdot \nabla Z_B$. Then one has :

$$\begin{aligned} \forall \mathbf{v} \in \mathbf{V}_0, \quad \|\mathbf{v}\|_V^2 &= \|\mathbf{v}_{2D}\|^2_{H(curl,div;\Sigma,h)} + \|v_3\|^2_{H^1(\Sigma,h)} + \|\mathbf{v} \wedge \mathbf{n}\|^2_{L^2(\Sigma, \alpha_{2D}\sqrt{1+|\nabla Z_B|^2})} \\ \forall q \in M, \quad \|q\|_{0,\Omega,\alpha} &= \|q\|_{0,\Sigma,h} \,. \end{aligned}$$

Let us equally introduce, for a given k, the set :

$$\mathbf{V}_0^* = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} ;\ \mathbf{v}_{2D} \in H(curl, div; \Sigma, h),\ v_3 \in H^1(\Sigma, h),\ \mathbf{v}_{2D} \cdot \mathbf{n}_{2D} = k \text{ on } \Sigma_M \cup \Sigma_V \right\}.$$

Remark : Under the hypotheses $\nabla Z_B \in (L^\infty(\Sigma))^2$ and $\Gamma_M,\ \Gamma_V$ vertical, one can show that $\mathbf{V}_0 \subset \mathbf{V}$. Indeed, the condition $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_B (described by $z = Z_B(x,y)$) is obviously satisfied by construction of v_3, while $\mathbf{v} \cdot \mathbf{n} = 0$ on $\Gamma_M \cup \Gamma_V$ is equivalent to $\mathbf{v}_{2D} \cdot \mathbf{n}_{2D} = 0$ on $\Sigma_M \cup \Sigma_V$. On the other side, for any $\mathbf{v} \in \mathbf{V}_0$ one can easily see that $\mathbf{v} \wedge \mathbf{n}$ belongs to $(L^2(\Gamma_B, \alpha))^3$ since : $\int_{\Gamma_B} w^2 = \int_\Sigma w^2 \sqrt{1 + |\nabla Z_B|^2} < \infty$ for any $w \in L^2(\Sigma)$.

We have a new unknown in the 2D model, which is the height of the water $h = h(x,y,t)$ defined by :

$$h(x,y,t) = \int_{Z_B}^{H} \alpha(x,y,z,t)dz,$$

so $h = 0$ in $\Sigma \setminus \Sigma_F(t)$. Integrating over depth the equation satisfied by α and taking the velocity in $\mathbf{V}_0$, one gets :

$$\frac{\partial h}{\partial t} + \nabla h \cdot \mathbf{u}_{2D} = 0 \Leftrightarrow \frac{dh}{dt} = 0 \quad \text{in } \Sigma_F(t).$$

Finally, after time discretisation, the 2D-horizontal model is described by the next equations : $\alpha_{2D}^{n+1} = \alpha_{2D}^n \circ \zeta^n$ in Σ and $h^{n+1} = h^n \circ \zeta^n$ in Σ_{n+1}, where Σ_{n+1} is now defined by $\alpha_{2D}^{n+1} = 1$ and where $\zeta^n(x,y)$ is obtained in a similar manner as $\chi^n(x,y,z)$, by simply replacing $\mathbf{u}(x,y,z,t)$ by $\mathbf{u}_{2D}(x,y,t)$. The approximated variational formulation writes as below :

$$\begin{cases} \text{find } \mathbf{u}_0^{n+1} \in \mathbf{V}_0^*,\ p_0^{n+1} \in M \text{ such that} \\ \forall \mathbf{v} \in \mathbf{V}_0, \quad A(\mathbf{u}_0^{n+1}, \mathbf{v}) + B(p_0^{n+1}, \mathbf{v}) = F_n^0(\mathbf{v}) \\ \forall q \in M, \quad B(q, \mathbf{u}_0^{n+1}) = 0 \end{cases} \tag{2}$$

with $\mathbf{u}_0^{n+1} = \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \mathbf{u}_{2D}^{n+1} \cdot \nabla Z_B \end{pmatrix}$. One can express the forms $A(.,.)$, $B(.,.)$ and $F_n^0(.)$ on the subspaces $\mathbf{V}_0$ and M. Thus, denoting for the sake of simplicity $h^{n+1}$by h, one has :

$$\begin{aligned} A(\mathbf{u}_0, \mathbf{v}) &= \int_\Sigma \frac{\rho h}{\Delta t}(\mathbf{u}_{2D} \cdot \mathbf{v}_{2D} + u_3 v_3) + \mu h(curl\mathbf{u}_{2D} \cdot curl\mathbf{v}_{2D} + \nabla u_3 \cdot \nabla v_3) \\ &\quad + \alpha_{2D} c_B (\mathbf{u}_{2D} \cdot \mathbf{v}_{2D} + u_3 v_3)\sqrt{1 + |\nabla Z_B|^2} - 2\rho h \omega_3 (u_1 v_2 - u_2 v_1) \\ B(q, \mathbf{v}) &= -\int_\Sigma hq div \mathbf{v}_{2D} \\ F_n^0(\mathbf{v}) &= \int_\Sigma \frac{\rho h}{\Delta t}[(\mathbf{u}_{2D}^n \circ \zeta^n) \cdot \mathbf{v}_{2D} + (u_3^n \circ \zeta^n) v_3] - \rho h g v_3 + \alpha_{2D} \mathbf{w} \cdot \mathbf{v} \sqrt{1 + |\nabla(Z_B + h)|^2} \\ &\quad - \int_\Sigma \alpha_{2D} p_a \mathbf{v}_{2D} \cdot \nabla h + < \mathbf{v} \wedge \mathbf{n}, \beta_V \wedge \mathbf{n} >_{\Gamma_V(t)}, \end{aligned}$$

where we have used that :

$$\int_{\Gamma_H(t)} p_a \mathbf{v} \cdot \mathbf{n} = \int_{\Sigma_F(t)} p_a \left(\mathbf{v}_{2D} \cdot \nabla Z_F - \mathbf{v}_{2D} \cdot \nabla(Z_F + h)\right) = -\int_\Sigma \alpha_{2D} p_a \mathbf{v}_{2D} \cdot \nabla h.$$

Let us also notice that the last term of $F_n(.)$ can be written, under the hypotheses β_V independent upon z and $\mathbf{v}_{2D} \in L^2(\Sigma_V)^2$, as : $< \mathbf{v} \wedge \mathbf{n}, \beta \wedge \mathbf{n} >_{\Gamma_V(t)} = \int_{\Sigma_V} h\beta_V \cdot \mathbf{v}$. The existence and uniqueness of the solution of (2) is insured by the Babuska-Brezzi' theory, under some non-restrictive hypotheses (e.g., Σ sufficiently smooth and $\partial_{ij} Z_B \in L^\infty(\Sigma)$). Let us notice here that one of the main difficulties of this approach consists in letting h vanish on the boundary. Then the *inf-sup* condition is established in a quite technical manner, using some recent regularity results in weighted spaces.

For the spatial discretization of the 3D model (1), we can employ the same spaces as for a Stokes problem (the main difference between the formulations is that in our case the velocity belongs to $H(div, rot)$ instead of H^1). For instance, we can consider as in [1] the MINI element, that is continuous $(P_1)^3$ plus bubbles for the velocity, respectively continuous P_1 for the pressure. This choice insures that the *inf-sup* condition for the discrete 3D formulation holds uniformly with respect to the discretization parameter. In order to get a uniform ellipticity of the bilinear form $A(\cdot, \cdot)$ on the discrete kernel of

$B(\cdot,\cdot)$, it is sufficient to replace $A(\cdot,\cdot)$ by $A(\cdot,\cdot)+\beta A_0(\cdot,\cdot)$ where β is a stabilization parameter and $A_0(\mathbf{u},\mathbf{v})=\int_\Omega div\mathbf{u} div\mathbf{v}$. Concerning the numerical approximation of the 2D-horizontal problem (2), one has to adapt the finite element space for the velocity and enrich the space $(P_1)^2$ with bubles multiplied by $\nabla(hp)$.

Comparison with the shallow-water equations

In terms of boundary value problem, we obtain the next equations (in the unknowns α_{2D} in Σ, respectively $h,\ \mathbf{u}_{2D},\ p$ in $\Sigma_F(t)$) :

$$\begin{gathered}
\tfrac{\partial \alpha_{2D}}{\partial t}+\nabla\alpha_{2D}\cdot\mathbf{u}_{2D}=0\\
\tfrac{\partial h}{\partial t}+\nabla h\cdot\mathbf{u}_{2D}=0\\
div\mathbf{u}_{2D}=0\\
h\left(\tfrac{d\mathbf{u}_{2D}}{dt}+\tfrac{du_3}{dt}\nabla Z_B\right)+\tfrac{\mu}{\rho}\mathbf{curl}(h\, curl\mathbf{u}_{2D})-\tfrac{\mu}{\rho}div\,(h\nabla u_3)\,\nabla Z_B\\
+\tfrac{c_B}{\rho}\sqrt{1+|\nabla Z_B|^2}\,(\mathbf{u}_{2D}+u_3\nabla Z_B)+\tfrac{h}{\rho}\nabla p+\tfrac{1}{\rho}(p-p_a-\rho g h)\\
=-hg\nabla(Z_B+h)+2h\omega_3\begin{pmatrix}-u_2\\ u_1\end{pmatrix}+\tfrac{1}{\rho}(\mathbf{w}_{2D}+w_3\nabla Z_B)\sqrt{1+|\nabla(Z_B+h)|^2},
\end{gathered}$$

with the boundary conditions :

$$\begin{aligned}
\Sigma_{lat}: &\quad h\mu(curl\mathbf{u}_{2D})\mathbf{t}+h\mu(\partial_n u_3)\nabla Z_B=hp\mathbf{n}\\
\Sigma_V: &\quad \mathbf{u}_{2D}\cdot\mathbf{n}=k_M \text{ and } h\mu curl\mathbf{u}_{2D}+h\mu\partial_n u_3\partial_t Z_B=h(\beta_{2D}+\beta_3\nabla Z_B)\cdot\mathbf{t}\\
\Sigma_M: &\quad \mathbf{u}_{2D}\cdot\mathbf{n}=k_V \text{ and } h\mu curl\mathbf{u}_{2D}+h\mu\partial_n u_3\partial_t Z_B=0.
\end{aligned}$$

Let us remark that the classical shallow-water equations write as follows (see [2]) :

$$\frac{\partial h}{\partial t}+div\,(h\mathbf{u}_m)=0$$

$$h\frac{\partial \mathbf{u}_m}{\partial t}+h\nabla\mathbf{u}_m\,\mathbf{u}_m-\gamma\Delta(h\mathbf{u}_m)+gh\mathbf{J}=-gh\nabla(Z_B+h)+2h\omega_3\begin{pmatrix}-u_{2m}\\ u_{1m}\end{pmatrix}+kW^2\begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}$$

where $\mathbf{u}_m=\begin{pmatrix}u_{1m}\\ u_{2m}\end{pmatrix}$ represents the bidimensional average velocity and where the last term represents the wind's effect (of norm W and angle ψ with Ox).

First of all, let us notice that combining the second and the third equation of our final system gives the same continuity equation as in the Saint-Venant's system : $\frac{\partial h}{\partial t}+div\,(h\mathbf{u}_{2D})=0$. $\mathbf{J}$ is modelling the bottom friction and in practice several formulae are available (Manning-Strickler, ChÉzy etc.). However, the choice of J is not rigorous from a mathematical point of view. Another drawback of the shallow-water equations is that the pressure is considered to be hydrostatic, i.e. $p=p_a+\rho g(Z_B+h-z)$ and the vertical component of the velocity is supposed to be null. Moreover, it is very difficult to obtain an error estimate between the 3D model and the shallow-water equations.

5 The 2D-vertical and 1D curvilinear models

In a similar way, we obtain a 2D-vertical and a 1D model, which both take into account the geometry of the river. We present here only the general idea and the choice of the projection subspaces. Let $C(t)$ be the median curve of the free surface $\Gamma_H(t)$, and C its projection on the horizontal surface Σ. We suppose that C is described by a smooth function $\varphi : [0, A] \to C$. We work in the three-dimensional orthonormal basis $\{\tau(s), \nu(s), \mathbf{e}_3\}$ and in curvilinear coordinates (s, l, z), where $\{\tau(s), \nu(s)\}$ is the local Frenet basis of the curve C. We assume that : $h = h(s,t)$, $Z_B = Z_B(s)$. In fact, $Z_B(.)$ represents the average elevation of the bottom. The domain $\Omega_F(t)$ is then characterized by: $s \in [0, A]$, $\quad -L(s) \leq l \leq L(s)$, $\quad Z_B(s) \leq z \leq Z_B(s) + h(s,t)$, where $L(\cdot) : [0, A] \to R_+$ is the mid-width of the river, independent upon time and given by the bathymetry. The 2D-vertical model is written on the domain $\omega_F(t) = \{s \in [0, A]\,,\ z \in [Z_B, Z_B + h]\}$. We look for a projected velocity $\mathbf{u} = \begin{pmatrix} u_1(s,z)(1 - lr) \\ \frac{lL'}{L} u_1(s,z) \\ u_3(s,z) \end{pmatrix}$ with $r = r(s)$ the curvature of C and $u_3(s, Z_B) = u_1(s, Z_B) Z_B'(s)$. This choice insures a conforming approximation of the 3D time-discretized model. So, at each time step, the unknowns of the problem are the pressure $p = p(s,z)$ and $\mathbf{u}_V(s,z) = (u_1, u_3)$.

The 1D model is obtained directly from the 2D-vertical one, by taking the semi-discretized pressure $p = p(s)$ and the velocity $\mathbf{u}(s,z) = \begin{pmatrix} u_1(s)(1 - lr) \\ \frac{lL'}{L} v_1(s) \\ U_3(s,z) \end{pmatrix}$ where $U_3(s,z) = u_1(s) Z_B' + (z - Z_B) u_3(s)$. So, the unknowns are now $p(s,t)$, $\mathbf{u}_\mathbf{V}(s,t) = (u_1, u_3)$ and also the transverse section $\sigma(s,t) = 2L(s)h(s,t)$, which satisfies a transport equation. Under usual assumptions of regularity on the given functions L and r, one can show that the variational 1D and 2D-vertical problems are well-posed. Their discretization derive from the one of the Stokes problem by the MINI element.

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