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POLYNOMIAL SYSTEMS, H–BASES, AND AN APPLICATION FROM KINEMATIC TRANSFORMS

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Abstract. We review some algebraic methods to solve systems of polynomial equations and illustrate these methods with a real–world problem that comes from computing kinematic transforms in robotics.

Keywords: Gröbner basis, H–basis, polynomial system, kinematic transform.

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§1. Introduction

Polynomial systems of equations and the structure of their solutions play a crucial role in many fields of theoretical and applied mathematics. The importance of polynomial equations in applications is often due to the need to determine locations of points from given euclidian distances which obviously leads to quadratic equations.

The mathematical formulation is as follows: Suppose we are given a finite set $F \subset \mathbb{K}[x] = \mathbb{K}[x_1,\dots,x_n]$ of polynomials in the n variables $x_1,\dots,x_n$ with coefficients in the field $\mathbb{K}$, where usually $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, i.e., the rational, real or complex numbers. Given the equations F, the goal is to find the solutions $X \subset \overline{\mathbb{K}}^n$ of the system $F(X) = 0$ in the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, that is,

$$X = \left\{ x \in \overline{\mathbb{K}}^n \;:\; f(x) = 0,\, f \in F \right\}. \tag{1}$$

Note that there are two major differences to the "standard approach" for solving nonlinear equations by means of Newton's method: The number of equations, $\#F$, need not coincide with the number of variables, n, and we are not interested in a single solution, but in the set of *all* solutions of $F(X) = 0$.

The equations $f(X) = 0$, $f \in F$, trivially remain valid if each of them is multiplied by an arbitrary polynomial $q_f \in \mathbb{K}[x]$ and if any such modified equations are added. Hence,

$$F(X) = 0 \quad\Longleftrightarrow\quad \langle F\rangle(X) = 0, \qquad \langle F\rangle = \left\{ \sum_{f\in F} q_f\, f \;:\; q_f \in \mathbb{K}[x] \right\}, \tag{2}$$

where $\langle F\rangle$ is the *ideal generated by* F; recall that an ideal $\mathscr{I}$ is a subset of $\mathbb{K}[x]$ which is closed under addition and multiplication by arbitrary polynomials, cf. [4]. A subset G of an ideal $\mathscr{I}$ is called a *basis* for the ideal $\mathscr{I}$ if G generates the ideal, i.e., $\mathscr{I} = \langle G\rangle$. With this terminology at hand, we can rephrase (2) as that the solution X depends only on the ideal $\mathscr{I}$, but not on the individual basis F. This simple observation is the fundamental idea behind all the algebraic methods to solve polynomial systems by interpreting the original equations as a basis of an ideal and then computing another basis for the same ideal from which the solution of the polynomial system is more easily accessible. In other words: Algebraic methods transform a given system of equations into a simpler or more useful form.

§2. Gröbner bases, H–bases and eigenvalues

Gröbner bases as well as H–bases are special ideal bases which provide representations of minimal degree, where these two types of bases differ by being related to different notions of degree. For Gröbner bases, we need the concept of a *term order* "$<$" on $\mathbb{N}_0^n$, that is, a well–ordering on $\mathbb{N}_0^n$ which is compatible with addition, cf. [4]. With respect to this order, any polynomial

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha x^\alpha, \qquad f_\alpha \in \mathbb{K}, \qquad \#\{\alpha \,:\, f_\alpha \neq 0\} < \infty,$$

has a maximal nonzero coefficient f_α and α is called the *(multi)degree* of the polynomial while $f_\alpha x^\alpha$ is usually named the *leading term* of f. For H–bases, on the other hand, the degree is not a multiindex, but a number, namely the maximal length $|\alpha| = \alpha_1 + \cdots + \alpha_n$ of the indices of nonzero coefficients – the usual *total degree*. Nevertheless, we will write the degree of a polynomial f as $\delta(f)$, regardless of whether $\delta(f) \in \mathbb{N}_0^n$ or $\delta(f) \in \mathbb{N}_0$; indeed, there is a joint framework in terms of graded rings, see [5], and [10] for the application in ideal bases and interpolation. A finite set $H \subset \mathbb{K}[x]$ is called *Gröbner basis* or *H–basis*, depending on whether δ is based on on a term order or on the total degree, if any $f \in \langle H \rangle$ can be written as

$$f = \sum_{h \in H} f_h h, \qquad f_h \in \mathbb{K}[x], \qquad \delta(f) \geq \delta(f_h h), \qquad h \in H. \tag{3}$$

The crucial point of Gröbner bases and H–bases is the degree constraint in (3) which helps to avoid a certain redundancy: Assume that one term in the sum on the right hand side were of higher degree than f, then there must be at least a second term of the same or higher degree compensating its leading term, and the representation would be redundant, all the terms of degree higher than that of f unneeded. But the main practical advantage of Gröbner bases and the main reason for their development in [2] is the fact that they permit the *algorithmic computation* of a unique remainder r,

$$f = \sum_{h \in H} f_h h + r. \tag{4}$$

This can be extended to the grading by total degree [6, 9] and even to arbitrary gradings in such a way that the remainder r depends only on $\langle H \rangle$ and the parameters of the grading, see [11] for details. Thus, we have a method to compute a normal form $\nu_{\langle H \rangle}$ modulo $\langle H \rangle$ and to efficiently perform arithmetic in the quotient ring $\mathscr{P} := \mathbb{K}[x] / \langle H \rangle$. Moreover, $\mathscr{P}$ is a finite dimensional space if and only if the ideal $\mathscr{I} = \langle H \rangle$ has dimension zero which is in turn equivalent to a finite number of solutions X for $H(X) = 0$.

So here is the first part of the algebraic simplification: Starting with a finite set F of polynomial equations, one computes a Gröbner basis or H–basis H for the ideal $\langle F \rangle$ from which it can be decided whether $F(X) = 0$ has no solution (this happens if and only if $1 \in H$), a finite number of solutions or infinitely many solutions. It is even possible, see [4], to determine the dimension of the algebraic variety formed by the solutions. But in this paper let us assume that X were nonempty and finite.

The classical method [13], see also [1, 4], to find X is by means of *elimination ideals*: A purely lexicographical Gröbner basis for a zero dimensional ideal contains some univariate polynomials whose greatest common divisor vanishes at the projections of the common zeros to this coordinate. Solving and substituting the solutions eliminates the variable and continuing this process, one can systematically find all the common zeros. Unfortunately, this process has a terrible complexity and can be very sensitive to perturbations of the coefficients, cf. [7], which limits its use in practical applications.

There is, however, a different approach proposed by Möller and Stetter [8, 12] which is based on *multiplication tables* on the quotient space $\mathscr{P}$. To that end, observe that multiplication of $f, g \in \mathscr{P}$ is defined as $\nu_{\mathscr{I}}(fg)$ and that for fixed $g \in \mathbb{K}[x]$ the operation

$$f \mapsto M_g(f) := \nu_{\mathscr{I}}(fg)$$

is a *linear operator* on $\mathscr{P}$ that can be represented with respect to a basis of $\mathscr{P}$ by a matrix M_g – the so called *multiplication table*. For $j = 1, \dots, n$ let now M_j denote the multiplication table for the coordinate polynomials $g(x) = x_j$, then the M_j generalize the classical Frobenius companion matrix, form a commuting family of matrices, have joint eigenvectors and the respective eigenvalues are the coordinates of the common zeros. Thus, the solutions of $F(X) = 0$ can be found by relying on well–developed methods from Numerical Linear Algebra and the flexibility of H–bases now offers an approach that changes continuously with the parameters and thus is much less sensitive to perturbations, see again [7] for an example.

§3. Practical Examples

In this section we want to apply and illustrate the mathematical concepts of the preceding chapters. To that end, we take a look at three slightly different kinematics. First, we will consider a simple example in three dimensions to show how we obtain the equations needed as starting ideal basis for the computation of a Gröber basis or H–basis. Then we present a kinematic that still appears to be quite simple but leads monstrous Gröbner bases and H–bases and also point out how crucial it is to incorporate "implicit" physical restrictions into the system of equations.

All our kinematics follow the same basic layout: The *manipulator* (in most cases used for melding or milling) is connected to three (or more) rods of variable length. In the *inverse kinematic transform* we know the position of the manipulator and want to compute the "machine parameters", i.e., the lengths of the rods, while in the *forward kinematic transform* the location of the manipulator is to be determined from the lengths of the rods. In both cases the ideal basis which we first must construct is the same, namely the implicit system of equations. The only difference consists of the choice which of the parameters are considered variables to be solved.

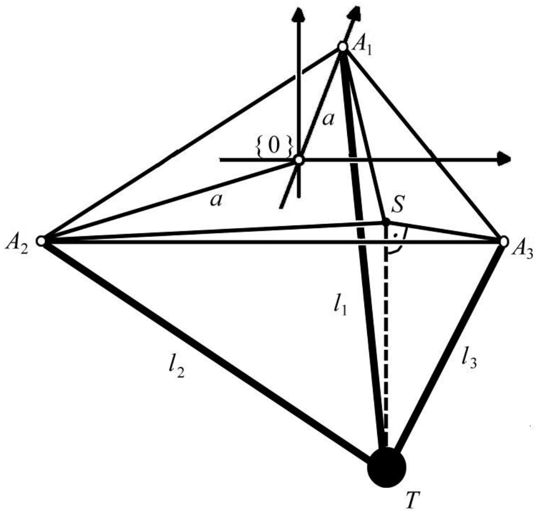

Figure 1: Simple 3D kinematic.

3.1. A Simple 3D-Kinematic

The first example is really easy to solve and we only use it to demonstrate how to obtain the equations from which we compute the Gröbner- or H-Basis. First we take a look at the construction. In figure 1 the construction is fixed in three points A_1, A_2 and A_3, coplanar with the origin $\{0\}$, and have the same distance a to $\{0\}$. Furthermore, the distance between every two points is constant. Now it is easy to see how to obtain the equations we need. Consider the projection S of $T = (x,y,z)$ in the plane generated by A_1, A_2 and A_3. With Pythagoras we have

$$l_i = y^2 + \|A_i - S\|_2^2, \quad i = 1,2,3,$$

which directly leads to the set of equations

$$\begin{aligned} y^2 + x^2 + (a-z)^2 - l_1^2 &= 0, \\ y^2 + \left(-\frac{\sqrt{3}}{2}a - x\right)^2 + \left(\frac{-1}{2}a - z\right)^2 - l_2^2 &= 0, \\ y^2 + \left(\frac{\sqrt{3}}{2}a - x\right)^2 + \left(\frac{-1}{2}a - z\right)^2 - l_3^2 &= 0. \end{aligned}$$

In `Maple` notation, the ideal is thus generated by

$$\begin{aligned} F := \Big[& x^2 + y^2 + (a-z)^2 - l_1^2,\ y^2 + \big(-\tfrac{\sqrt{3}}{2}a - x\big)^2 + \big(\tfrac{-1}{2}a - z\big)^2 - l_2^2, \\ & y^2 + (\tfrac{\sqrt{3}}{2}a - x)^2 + (\tfrac{-1}{2}a - z)^2 - l_3^2 \Big]. \end{aligned}$$

Because we used the (square of the) lengths l_1, l_2 and l_3 explicitly in our ideal basis we can give the solution of the inverse kinematic transform directly as

$$l_1 = \sqrt{y^2 + x^2 + (a-z)^2},$$
$$l_2 = \sqrt{y^2 + \left(-\frac{\sqrt{3}}{2}a - x\right)^2 + \left(\frac{-1}{2}a - z\right)^2},$$
$$l_3 = \sqrt{y^2 + \left(\frac{\sqrt{3}}{2}a - x\right)^2 + \left(\frac{-1}{2}a - z\right)^2}.$$

For the forward transform we switch the roles of variables and constants which are now declared as x, y, z and a, b, l_1, l_2, l_3, respectively. Without further problems we compute an H–basis of F as

$$H = \Big[9a^2y^2 - 3l_1^2a^2 + l_3^4 - l_3^2l_2^2 + l_2^4 + 9a^4 - l_2^2l_1^2 - 3a^2l_2^2 - 3a^2l_3^2 + l_1^4 - l_1^2l_3^2,$$
$$6az - l_2^2 + 2l_1^2 - l_3^2, 12ax + 2\sqrt{3}l_3^2 - 2\sqrt{3}l_2^2\Big]$$

and by means of multiplication tables of $\mathscr{P}$ and the corresponding eigenvectors we find that

$$x = \frac{\sqrt{3}(l_2^2 - l_3^2)}{6a},$$
$$y = \frac{\sqrt{-l_2^4 + 3l_1^2a^2 - l_3^4 + l_3^2l_2^2 + 3a^2l_3^2 - 9a^4 + l_2^2l_1^2 + 3a^2l_2^2 - l_1^4 + l_1^2l_3^2}}{-3a},$$
$$z = \frac{-2l_1^2 + l_2^2 + l_3^2}{6a}.$$

Note that the equations for x and z are significantly simpler than the one for y.

Since y appears quadratically in the H–basis, it follows that together with (x,y,z) also $(x,-y,z)$ is a solution of the system. However, this second solution is impossible in physical reality because the rods are flexible but fixed and cannot cross themselves. Unfortunately, it appears impossible to eliminate this unwanted “solution” a priori by adding more equations to the system; in fact, the only way to distinguish between the two solutions is by means of inequalities.

Remark 1. It is worthwhile to mention that not for all values of l_1, l_2 and l_3 the solution belongs to the real domain as in some cases the solution gains an additional imaginary part because the three rods have no common point. Though physically impossible this is absolutely correct mathematically. Finding additional constraints that eliminate complex solutions would consist of determining the associated *real ideal*.

3.2. The realistic problem

Now we want to take a close look at a slightly extended version of the latter three dimensional kinematic used in practical applications. In figure 2 the upper part of the construction equals

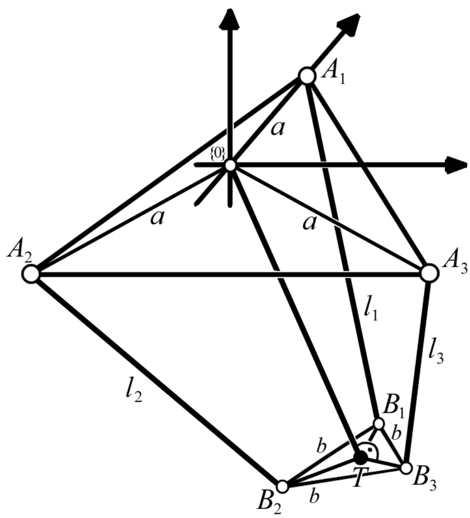

Figure 2: Complex 3D kinematic.

the one in figure 1 while the lower part differs with the manipulator being attached centrally under a platform which is held and moved by the rods. To make things simpler, we assume that the vertices B_1, B_2 and B_3 of the platform form an equilateral triangle with distance b between the points and barycenter $T = (x,y,z)$. To stabilize the construction, the platform is also linked to the origin $\{0\}$ by an additionally guiding rod which is attached perpendicular in T.

We will not discuss the ideal basis construction in full detail but should mention a few facts. First, it is not possible to compute the value of T directly, but it is easily found as midpoint of the triangle formed by B_1, B_2, B_3 once these locations are determined. The lengths l_1, l_2 and l_3 are just as easy to obtain as before from the equations

$$\|S - A_i\|_2^2 + \|S - B_i\|_2^2 = \|B_i - A_i\|_2^2, \quad i = 1,2,3,$$

in which S is the projection of T, leading to

$$\begin{aligned} x_1^2 + (z_1 - a)^2 + y_1^2 &= l_1^2, \\ \left(x_2 + \frac{\sqrt{3}a}{2}\right)^2 + \left(z_2 + \frac{a}{2}\right)^2 + y_2^2 &= l_2^2, \\ \left(x_3 - \frac{\sqrt{3}a}{2}\right)^2 + \left(z_3 + \frac{a}{2}\right)^2 + y_3^2 &= l_3^2. \end{aligned}$$

As mentioned previously the triangle is equilateral giving us the additional three equations

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = b^2, \quad 1 \le i < j \le 3.$$

The orthogonality of the system can finally be described by the inner products $(T-B_i,T)=0$, $i=1,\ldots,3$, which leads to

$$\begin{aligned}(x-x_1)\,x+(y-y_1)\,y+(z-z_1)\,z&=0,\\(x-x_2)\,x+(y-y_2)\,y+(z-z_2)\,z&=0,\\(x-x_3)\,x+(y-y_3)\,y+(z-z_3)\,z&=0.\end{aligned}$$

Finally we need the fact that the midpoint T of the triangle can be written as sum of the outer points $T=\frac{B_1+B_2+B_3}{3}$ yielding three more equations

$$(x_1+x_2+x_3)=3x,\qquad (y_1+y_2+y_3)=3y,\qquad (z_1+z_2+z_3)=3z.$$

Together, these twelve equations forms our initial ideal basis

$$\begin{aligned}F:=\Big[&x_1^2+(z_1-a)^2+y_1^2-l_1^2,\ \big(x_2+\tfrac{\sqrt{3}a}{2}\big)^2+\big(z_2+\tfrac{a}{2}\big)^2+y_2^2-l_2^2,\\&\big(x_3-\tfrac{\sqrt{3}a}{2}\big)^2+\big(z_3+\tfrac{a}{2}\big)^2+y_3^2-l_3^2,\ (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2-b^2,\\&(x_1-x_3)^2+(y_1-y_3)^2+(z_1-z_3)^2-b^2,\ (x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2-b^2,\\&(x-x_1)x+(y-y_1)y+(z-z_1)z,\ (x-x_2)x+(y-y_2)y+(z-z_2)z,\\&(x-x_3)x+(y-y_3)y+(z-z_3)z,\ (x_1+x_2+x_3)-3x,\ (y_1+y_2+y_3)-3y,\\&(z_1+z_2+z_3)-3z\Big].\end{aligned}$$

This time we begin with the more interesting forward kinematic transformation and are only interested in the dimension of the variety of the solutions $F(X)=0$. To do so, we substitute some numerical values for the constants l_1,l_2,l_3,a and b and compute a Gröbner basis which can be done without many problems but with a little bit of time (a tdeg ordered basis has no less than 56 elements). Computing the dimension, we surprisingly realize that the ideal is one–dimensional and not zero–dimensional as it should be if we wanted a finite number of solutions and to apply multiplication tables for their computation.

So the first question is why we found a one–dimensional variety. For convenience, we substitute (as before) $\{a=\sqrt{3},b=3,l_i=4 \mid i=1,2,3\}$ (see figure 3), and the desired final solution for the platform is

$$T=(0,4,0)^T,\ B_1=\big(0,4,\sqrt{3}\big)^T,\ B_2=\left(-\frac{3}{2},4,-\frac{\sqrt{3}}{2}\right)^T,\ B_3=\left(\frac{3}{2},4,-\frac{\sqrt{3}}{2}\right)^T.$$

If we rotate the lower triangle counterclockwise around the origin, so that B_2 is below A_3, B_1 below A_2 and B_3 below A_1 (see figure 4), we find that the point $T'=(0,\sqrt{7},0)^T$ resulting from

$$B_1'=\left(-\frac{3}{2},\sqrt{7},-\frac{\sqrt{3}}{2}\right)^T,B_2'=\left(\frac{3}{2},\sqrt{7},-\frac{\sqrt{3}}{2}\right)^T,B_3'=\big(0,\sqrt{7},\sqrt{3}\big)^T.$$

is another solution of our polynomial system.

Consequently, we obtain, by simple rotation, a one–parameter family of solutions and that is precisely the reason why our ideal is not zero–dimensional, so that we have add more

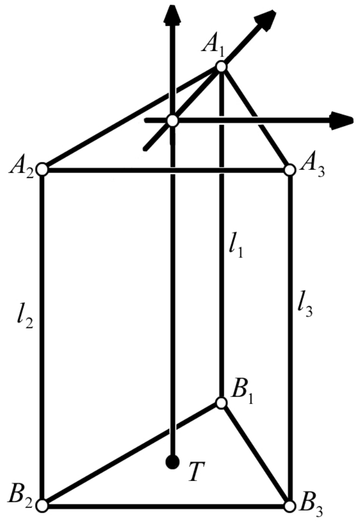

Figure 3: Simple Substitution.

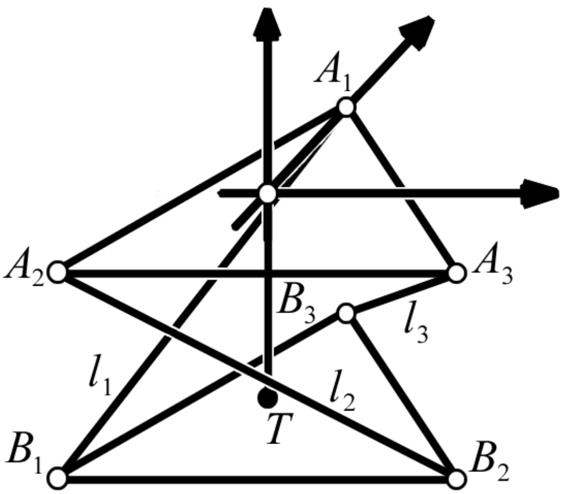

Figure 4: Simple Rotated Substitution.

equations to the ideal basis in order to prevent rotations. In such situations, it is a good idea to give a closer look to reality and indeed it turns out that such torsions of the robot are impossible since the guiding rod is connected to the upper part by a *universal joint* that can only move forwards/backwards and left/right but does not permit rotational movement.

Again, we will not discuss the modeling of the joint in detail, but here is the basic idea behind our approach: If we know the center $T = (x,y,z)$ of the triangle, the position of the outer points B_1, B_2, B_3 is fixed. So take a look at the point $S := (0, -\sqrt{x^2+y^2+z^2}, 0)$ which is just the position of T if the kinematic is not moved to any side (*"rest position"*). We can calculate the angle α between S and T, more precisely the term $c_\alpha = \cos\alpha$. Let the points B_1', B_2', B_3' be the vertices of the lower triangle in this rest position. With the help of rotation matrices and the angle α we can then compute the solution for the points B_1, B_2, B_3 explicitly. Doing so adds *eleven* further equations to our former ideal basis which makes us end up with

$$
\begin{aligned}
F := \Big[& x_1^2 + (z_1 - a)^2 + y_1^2 - l_1^2,\ \left(x_2 + \tfrac{\sqrt{3}a}{2}\right)^2 + \left(z_2 + \tfrac{a}{2}\right)^2 + y_2^2 - l_2^2, \\
& \left(x_3 - \tfrac{\sqrt{3}a}{2}\right)^2 + \left(z_3 + \frac{a}{2}\right)^2 + y_3^2 - l_3^2,\ (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - b^2, \\
& (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 - b^2,\ (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 - b^2, \\
& (x - x_1)x + (y - y_1)y + (z - z_1)z,\ (x - x_2)x + (y - y_2)y + (z - z_2)z, \\
& (x - x_3)x + (y - y_3)y + (z - z_3)z,\ (x_1 + x_2 + x_3) - 3x,\ (y_1 + y_2 + y_3) - 3y, \\
& (z_1 + z_2 + z_3) - 3z,\ \sqrt{3}dl(x - x1) - bxz,\ \sqrt{3}dl(y - y1) - byz,\ \sqrt{3}l(z - z1) + bd, \\
& \sqrt{3}lby + 2\sqrt{3}dl(x - x2) + bxz,\ -\sqrt{3}lbx + 2\sqrt{3}dl(y - y2) + byz,\ 2\sqrt{3}l(z - z2) - bd, \\
& -\sqrt{3}lby + 2\sqrt{3}dl(x - x3) + bxz,\ \sqrt{3}lbx + 2\sqrt{3}dl(y - y3) + byz,\ 2\sqrt{3}l(z - z3) - bd, \\
& x^2 + y^2 - d^2,\ x^2 + y^2 + z^2 - l^2 \Big],
\end{aligned}
$$

where $d = \sqrt{x^2+y^2}$ and $l = \sqrt{x^2+y^2+z^2}$.

To solve the inverse kinematic problem, we choose the variables as x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, l, d, l_1, l_2, l_3 and the constants as x, y, z, a, b. The H–basis can be easily computed as

$$
\begin{aligned}
H = \Big[& (y^2 + x^2)x_1 - 2xzz_3 - xy^2 + 2z^2x - x^3,\ z_1 + 2z_3 - 3z, \\
& (2x^2 + 2y^2)y_2 + 2zyz_3 + xbd - 2y^3 - 2yx^2 - 2yz^2,\ z_2 - z_3, \\
& (2x^2 + 2y^2)x_3 + 2xzz_3 + ybd - 2z^2x - 2xy^2 - 2x^3, \\
& (2x^2 + 2y^2)y_3 + 2zyz_3 - xbd - 2y^3 - 2yx^2 - 2yz^2, \\
& (2x^2 + 2y^2)x_2 + 2xzz_3 - ybd - 2z^2x - 2x^3 - 2xy^2, \\
& (y^2 + x^2)y_1 - 2zyz_3 - yx^2 - y^3 + 2yz^2, \\
& (z^2 + y^2 + x^2)d^2 - 2y^2x^2 - x^4 - x^2z^2 - y^4 - z^2y^2, \\
& (6z^2 + 6y^2 + 6x^2)z_3d + (b\sqrt{3}x^2 + b\sqrt{3}y^2)l + (-6z^3 - 6zy^2 - 6zx^2)d, \\
& (12z^2 + 12y^2 + 12x^2)z_3^2 + (-24zy^2 - 24zx^2 - 24z^3)z_3 + 12z^4 + 12x^2z^2 \\
& \qquad - b^2x^2 - y^2b^2 + 12z^2y^2,
\end{aligned}
$$

$$3bld + (6x^2\sqrt{3} + 6\sqrt{3}z^2 + 6y^2\sqrt{3})z_3 - 6\sqrt{3}z^3 - 6\sqrt{3}zx^2 - 6y^2z\sqrt{3},$$
$$6z_3 l - 6zl + \sqrt{3}bd,\ l^2 - x^2 - y^2 - z^2,$$
$$3l_1^2 - 12az_3 - 3x^2 - b^2 - 3z^2 + 18za - 3y^2 - 3a^2,$$
$$6y^2 + 6x^2)l_2^2 + (6xz\sqrt{3}a - 6ax^2 - 6ay^2)z_3 - 3ayb\sqrt{3}d - 6x^2a^2 - 6xz^2a\sqrt{3} - 6y^4$$
$$- 6x^3a\sqrt{3} - 6xa\sqrt{3}y^2 - 6x^4 - 12y^2x^2 - 2b^2x^2 - 6a^2y^2 - 2y^2b^2 - 6z^2y^2 - 6x^2z^2,$$
$$(6y^2 + 6x^2)l_3^2 + (-6ay^2 - 6ax^2 - 6xz\sqrt{3}a)z_3 - 3ayb\sqrt{3}d - 2y^2b^2 - 6a^2y^2 - 6z^2y^2$$
$$- 6x^4 + 6x^3a\sqrt{3} - 6y^4 - 6x^2a^2 - 12y^2x^2 - 6x^2z^2 - 2b^2x^2 + 6xz^2a\sqrt{3} + 6xa\sqrt{3}y^2\Big].$$

The space $\mathscr{P}$ has dimension 32, so we will have 32 solutions, but most notably our ideal is zero-dimensional as desired. If we substitute some numerical values for the constants, we see why there are so many solutions (from physics only one would be expected): Because of squaring l and d we can have both positive and negative solutions. Since l and d are physical lengths they cannot be negative, however, but this cannot be fixed a priori. So if we select the correct results at the end, we get

$$l_1 = \Big(-\big(-6z^2y^2 - y^2b^2 - b^2x^2 + 6z^3a + 6zay^2 + 6zax^2 + 2la\sqrt{3}bd$$
$$- 3a^2y^2 - 3x^2a^2 - b^2z^2 - 6x^2z^2 - 6y^2x^2 - 3x^4 - 3y^4 - 3a^2z^2 - 3z^4\big)\Big)^{1/2}$$
$$\times \left(3z^2 + 3y^2 + 3x^2\right)^{-1/2},$$
$$l_2 = \Big(-\big(-6z^4x^2 - 6y^2z^4 - 6xa\sqrt{3}y^2z^2 - 6z^3ax^2 - 6zax^4 - 6z^3ay^2 - 6zay^4 - 6y^6 - 6x^6$$
$$- 6xa\sqrt{3}y^4 - 6x^2a^2z^2 - 12x^2a^2y^2 - 12x^3a\sqrt{3}y^2 - 24y^2x^2z^2 - 2b^2x^2z^2 - 4b^2x^2y^2$$
$$- 6a^2y^2z^2 - 2y^2b^2z^2 - 6x^4a^2 - 12y^4z^2 - 18y^4x^2 - 12x^4z^2 - 18x^4y^2 - 2b^2x^4$$
$$- 6a^2y^4 - 2y^4b^2 - 12zax^2y^2 - 6x^5a\sqrt{3} - 3ay^3b\sqrt{3}d - 3ayb\sqrt{3}dz^2 - 3ayb\sqrt{3}dx^2$$
$$- 6x^3a\sqrt{3}z^2 - 3laxzbd + lax^2\sqrt{3}bd + lay^2\sqrt{3}bd\big)\Big)^{1/2}$$
$$\times \left(6z^2y^2 + 6y^4 + 12y^2x^2 + 6x^2z^2 + 6x^4\right)^{-1/2},$$
$$l_3 = \Big(-\big(-6z^4x^2 - 6y^2z^4 + 6xa\sqrt{3}y^2z^2 - 6z^3ax^2 - 6zax^4 - 6z^3ay^2 - 6zay^4 - 6y^6 - 6x^6$$
$$+ 6xa\sqrt{3}y^4 - 6x^2a^2z^2 - 12x^2a^2y^2 + 12x^3a\sqrt{3}y^2 - 24y^2x^2z^2 - 2b^2x^2z^2 - 4b^2x^2y^2$$
$$- 6a^2y^2z^2 - 2y^2b^2z^2 - 6x^4a^2 - 12y^4z^2 - 18y^4x^2 - 12x^4z^2 - 18x^4y^2 - 2b^2x^4$$
$$- 6a^2y^4 - 2y^4b^2 - 12zax^2y^2 + 6x^5a\sqrt{3} - 3ay^3b\sqrt{3}d - 3ayb\sqrt{3}dz^2 - 3ayb\sqrt{3}dx^2$$
$$+ 6x^3a\sqrt{3}z^2 + 3laxzbd + lax^2\sqrt{3}bd + lay^2\sqrt{3}bd\big)\Big)^{1/2}$$
$$\times \left(6z^2y^2 + 6y^4 + 12y^2x^2 + 6x^2z^2 + 6x^4\right)^{-1/2},$$

where $d = \sqrt{x^2 + y^2}$ and $l = \sqrt{x^2 + y^2 + z^2}$.

For the forward transform, the variables are x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, x, y, zl, d and the constants l_1, l_2, l_3. Because both Computer Algebra systems we used, `Singular` and `Maple`, cannot even compute a Gröbner basis for the ideal as it is given in this form, we had to relocate the points A_1, A_2 and A_3 to the next integer grid value. Furthermore, we will substitute $\{a = 2, b = 4, l_i = 3 \ : \ i = 1,2,3\}$ because the symbolic solution is still too complex, thus changing the ideal to

$$\begin{aligned} F = \Big[& x_1^2 + y_1^2 + (2 - z_1)^2 - 9,\ (-2 - x_2)^2 + y_2^2 + (-1 - z_2)^2 - 9, \\ & (2 - x_3)^2 + y_3^2 + (-1 - z_3)^2 - 9,\ (x - x_1)x + (y - y_1)y + (z - z_1)z, \\ & (x - x_2)x + (y - y_2)y + (z - z_2)z,\ (x - x_3)x + (y - y_3)y + (z - z_3)z, \\ & (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - 13,\ (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 - 16, \\ & (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 - 13,\ x_1 + x_2 + x_3 - 3x, y_1 + y_2 + y_3 - 3y, \\ & z_1 + z_2 + z_3 - 3z,\ 2dl(x - x_1) - 4xz,\ 2dl(y - y_1) - 4yz,\ 2l(z - z_1) + 4d, \\ & 8ly + 4dl(x - x_2) + 4xz,\ -8lx + 4dl(y - y_2) + 4yz,\ 4l(z - z_2) - 4d, \\ & -8ly + 4dl(x - x_3) + 4xz,\ 8lx + 4dl(y - y_3) + 4yz,\ 4l(z - z_3) - 4d, \\ & x^2 + y^2 - d^2, x^2 + y^2 + z^2 - l^2 \Big]. \end{aligned}$$

A (tdeg–ordered) Gröbner basis contains no less than 83 elements and therefore cannot be called very small. But at least we can figure out that there are 40 solutions to the equations and with the algorithm from [3, p. 134ff] we can compute the number of real solutions and discover that there are only four of them, thus, up to symmetry, the desired solution and probably one with crossed rods as before.

In summary one can say that presently the realistic problem is inaccessible, but its terrible complexity originates from "contamination" by the 36 complex solutions which correspond to physically impossible configurations. This is one more major drawback of algebraic methods which can find the solutions only in the algebraic closure of the original field.

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