Monografías del Seminario Matemático García de Galdeano **33**, 137–144 (2006)

NONLINEAR STABILITY NEAR A 1:3 RESONANCE

V. Lanchares and A. I. Pascual

Abstract. In this paper we investigate the evolution of the stability domain for a two degrees of freedom Hamiltonian system near a 1:3 resonance. In this way, it is proven that the lost of stability takes place when the size of the stability domain goes to zero. In fact, away of the 1:3 resonance, there always exists a stability domain which size diminishes as we approach the exact value of the resonance.

Keywords: Nonlinear stability, normal forms, resonances.

AMS classification: 34D20, 37J25, 70H14, 93D05.

§1. Introduction

The question of nonlinear stability of equilibrium positions in Hamiltonian systems is a classical one, and it is an important piece in the study of problems arising in different scientific branches, such as Classical Mechanics, Celestial Mechanics, Atomic Physics, etc. This problem is trivial for one degree of freedom Hamiltonian systems but it turns to be intricate for more degrees of freedom. In this paper we focus in two degrees of freedom Hamiltonian systems.

The starting point in the study of the stability consists of an expansion of the Hamiltonian function around the equilibrium position (we suppose, without loss of generality, the origin). We write

$$\mathscr{H} = \mathscr{H}_2 + \mathscr{H}_3 + \cdots,$$

where each $\mathscr{H}_i$ is a homogeneous polynomial of degree i in coordinates and momenta. In some cases, it is possible to deduce the stability or instability character from the analysis of the eigenvalues of the quadratic term. In this way, if at least one of the eigenvalues has nonzero real part, then the equilibrium position is nonlinear unstable. On the other hand, if all eigenvalues are pure imaginary, the corresponding linear system is semisimple and the quadratic form $\mathscr{H}_2$ sign defined, Dirichlet's theorem [12] ensures nonlinear stability. Otherwise, the study of the linear approximation is not sufficient to deduce the nonlinear stability and specialized theorems, based on KAM theory, are required.

In this paper we study the general elliptic case which is characterized by the existence of four different pure imaginary eigenvalues when the quadratic term is an undefined form. If we denote the eigenvalues $\pm i\omega_1$, $\pm i\omega_2$ with $\omega_1,\ \omega_2 > 0$ and $\omega_1 \neq \omega_2$, the quadratic part can be written as

$$\mathscr{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2,$$

where Ψ_1, Ψ_2 stand for the Poincaré variables.

All known results are based on the Birkhoff's normal form [2] associated to the Hamiltonian. However, the normal form depends on the rational dependence of the frequencies ω_1 and ω_2. In this sense we give the following definition.

Definition 1. We say that ω_1 and ω_2 satisfy a resonance condition of order s if there exist n and m relatively prime integers such that

$$n\omega_1 - m\omega_2 = 0, \qquad n+m = s.$$

This resonance is called n:m resonance.

In absence of resonances, Arnold's theorem [1, 10] assures the stability in the majority of the situations. For the resonant cases, Markeev [9] and Sokolsky [13, 14] established suitable theorems for particular resonances. These results were generalized by Cabral and Meyer [3], Elipe et al. [7, 8] and Pascual [11] for any order of the resonance for non degenerate cases. In particular, for the 1:3 resonance, the following stability criterion was established by Markeev [9].

Theorem 1. *Let us consider a Hamiltonian system under a 1:3 resonance whose normal form is written in terms of the Poincaré variables as*

$$\mathscr{H} = 3\omega_2\Psi_1 - \omega_2\Psi_2 + \delta\Psi_1^{1/2}\Psi_2^{3/2}\cos(\psi_1+3\psi_2) + \frac{1}{2}(A\Psi_1^2 + 2B\Psi_1\Psi_2 + C\Psi_2^2) + \overline{\mathscr{H}},$$

where $\overline{\mathscr{H}} = \overline{\mathscr{H}}(\Psi_1,\Psi_2,\psi_1,\psi_2) = O((\Psi_1+\Psi_2)^{5/2})$. *If we denote* $D = A+6B+9C$, *then, if* $6\sqrt{3}|\delta| > |D|$, *the equilibrium is unstable and, if* $6\sqrt{3}|\delta| < |D|$, *the equilibrium is stable.*

Taking into account Theorem 1, it is possible to deduce the stability of the equilibrium position, except for the case $|D| = 6\sqrt{3}|\delta|$ that is a degenerate one. Anyway, we observe that both stability or instability are likely to happen. However, if we move away from the resonance, Arnold's theorem ensures the stability for the equilibrium position. The natural question arising is how the transition between stability, outside the resonance, and instability, when $6\sqrt{3}|\delta| > |D|$ in the exact 1:3 resonance, takes place. To give an appropriate answer to this question it is better to follow a geometrical approach. In this way, the geometric stability criterion given by Elipe et al. [8] and Pascual [11] will help us to this purpose.

§2. Geometric criterion

The geometric criterion is based on the structure of the reduced phase space after the normalization procedure. To this end, extended Lissajous variables [4, 5, 6] are preferable.

2.1. Extended Lissajous variables

Extended Lissajous variables constitute a set of variables suitable to handle resonant cases. Usually they are denoted by $(\phi_1,\phi_2,\Phi_1,\Phi_2)$, and for a n:m resonance, they are given in terms of Poincaré variables by

$$\begin{aligned} f\colon T^2\times\{\Phi_1>0\}\times\{|\Phi_2|\le\Phi_1\} &\mapsto \mathbb{R}^4\\ (\phi_1,\phi_2,\Phi_1,\Phi_2) &\mapsto (\psi_1,\psi_2,\Psi_1,\Psi_2)\end{aligned}$$

$$\Psi_1 = \frac{\Phi_1 + \Phi_2}{2m}, \qquad \psi_1 = m(\phi_1 + \phi_2),$$

$$\Psi_2 = \frac{\Phi_1 - \Phi_2}{2n}, \qquad \psi_2 = n(\phi_1 - \phi_2).$$

In this set of variables, the quadratic term of the normal form reduces to

$$\mathscr{H}_2 = \omega \Phi_2,$$

where

$$\omega = \frac{\omega_1}{m} = \frac{\omega_2}{n}.$$

Now, $\mathscr{H}_2$ is a formal integral of the normalized system and the normalization turns to be an average procedure because the Poisson's bracket is

$$(\mathscr{H}_2, \mathscr{H}_j) = \omega \frac{\partial \mathscr{H}_j}{\partial \phi_2}.$$

It can be seen that every term in the normal form can be written as a function of the so called invariants, we denote by (C, S, M_1, M_2). In terms of the extended Lissajous variables, they are given by

$$\begin{aligned} M_1 &= \frac{1}{2}\Phi_1, & C &= 2^{-(m+n)/2}(\Phi_1 - \Phi_2)^{m/2}(\Phi_1 + \Phi_2)^{n/2} \cos 2nm\phi_1, \\ M_2 &= \frac{1}{2}\Phi_2, & S &= 2^{-(m+n)/2}(\Phi_1 - \Phi_2)^{m/2}(\Phi_1 + \Phi_2)^{n/2} \sin 2nm\phi_1, \end{aligned}$$

where we highlight that M_2 is a formal integral. Using invariants, each term of the normal form can be expressed as

$$\mathscr{H}_j = \sum_{2(\gamma_1+\gamma_2)+(n+m)(\gamma_3+\gamma_4)=j} a_{\gamma_1\gamma_2\gamma_3\gamma_4} M_1^{\gamma_1} M_2^{\gamma_2} C^{\gamma_3} S^{\gamma_4}.$$

2.2. The reduced phase space

The invariants are not independent and they satisfy the equation

$$C^2 + S^2 = (M_1 + M_2)^n (M_1 - M_2)^m, \tag{1}$$

together with the restriction

$$M_1 \geq |M_2|. \tag{2}$$

Note that the reduced phase space is given by the equation (1) and the restriction (2). Since M_2 is a constant, (1) is a surface of revolution with a vertex in the point $M_1 = |M_2|, C = S = 0$. We can see in Figure 1 the different situations for the reduced phase space for the 1:3 resonance depending on the value of M_2.

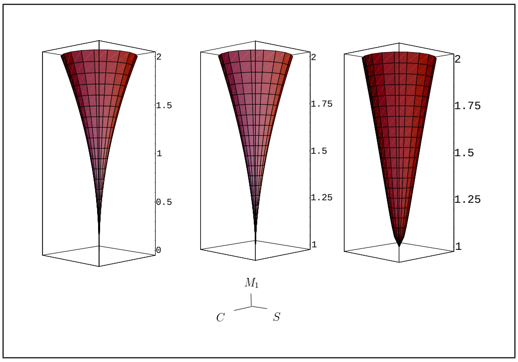

Figure 1: The reduced phase space of the 1:3 resonance for $M_2 = 0$, $M_2 > 0$ and $M_2 < 0$ respectively.

2.3. The geometric criterion

Once the reduced phase space is determined, it is possible to know the flow of the normalized system, when it is truncated to a prescribed order. Indeed, the flow results as the intersection of the normalized Hamiltonian function with the surface defined by (1). Based on this consideration, the following stability result can be established (for more details, see [8, 11]).

Theorem 2. *Let us assume that the Hamiltonian is normalized up to a certain order $N \geq s$, being $\mathscr{H}_N$ the first term that does not vanish for $M_2 = 0$. Let us consider the two surfaces*

$$\mathscr{G}_1 = \{(C,S,M_1) \in \mathbb{R}^3; \mathscr{H}_N(C,S,M_1,0) = 0\},$$

and

$$\mathscr{G}_2 = \{(C,S,M_1) \in \mathbb{R}^3; C^2 + S^2 = M_1^s\}.$$

If the origin is an isolated intersection point of $\mathscr{G}_1$ and $\mathscr{G}_2$, then it is stable. If they intersect each other transversely, then the origin is unstable.

§3. The 1:3 resonance

For the case of the 1:3 resonance, the normal form must be computed up to order 4. Without loss of generality, it is expressed in terms of the invariants as

$$\mathscr{H}_4 = a_4 M_1^2 + a_2 M_1 M_2 + \gamma S.$$

Now, the two surfaces $\mathscr{G}_1$ and $\mathscr{G}_2$ are given by

$$\mathscr{G}_1 = \{(C,S,M_1) \in \mathbb{R}^3 \,; a_4 M_1^2 + \gamma S = 0\},$$

and

$$\mathscr{G}_2 = \{(C,S,M_1) \in \mathbb{R}^3 \,; C^2 + S^2 = M_1^4\}.$$

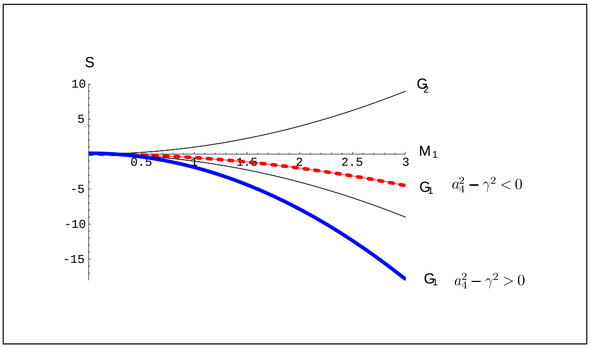

Figure 2: Surfaces $\mathscr{G}_1$ and $\mathscr{G}_2$ projected onto the plane $C=0$ in two cases: stable (not dashed line) and unstable (dashed line).

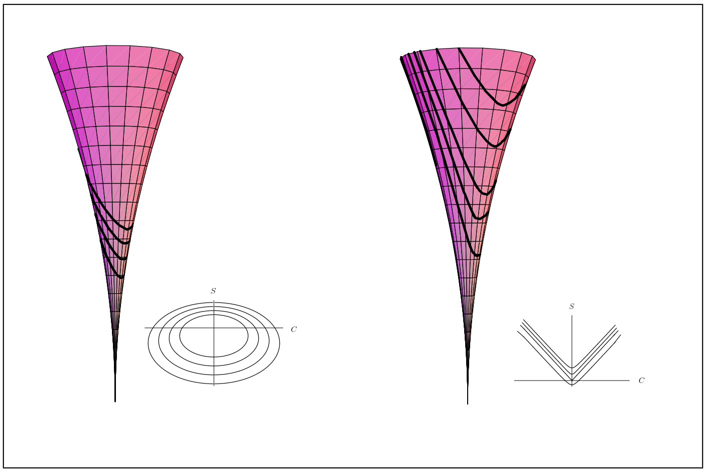

Figure 3: Orbits (and projections) in stable and unstable case respectively when $M_2=0$.

These two surfaces has the origin as the unique common point if $a_4^2-\gamma^2>0$ and they intersect transversely if $a_4^2-\gamma^2<0$ as we can see in Figure 2. Therefore, the first case corresponds to a stable situation whereas the second belongs to an unstable situation.

In Figure 3 we show the orbits in the reduced phase space and the respective projection onto the plane $M_1=0$ in both cases (stable and unstable) when $M_2=0$.

§4. Near the 1:3 resonance

In order to see how the instability takes place when passing from a stable configuration near the 1:3 resonance, we introduce a suitable detunning parameter ε. In this way, we write

$$\omega_1=3\omega_2+\varepsilon,$$

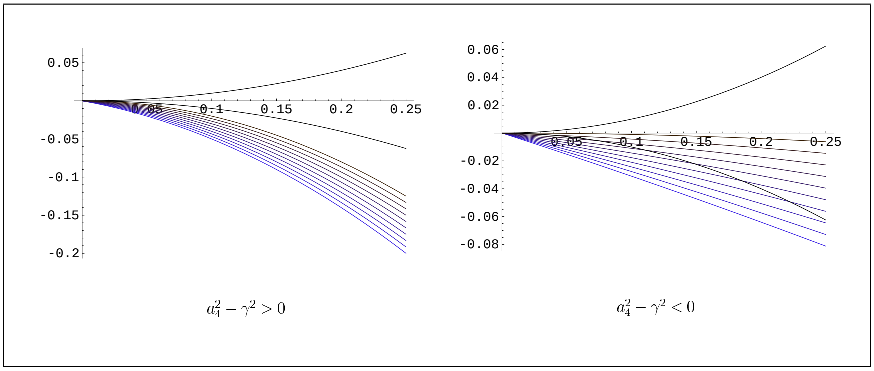

Figure 4: Surfaces $\mathscr{G}_1$ and $\mathscr{G}_2$ projected onto the plane $C = 0$ in two cases: stable (left) and unstable (right) for different values of ε ($\varepsilon \to 0$).

where $\varepsilon \ll 1$ is chosen to be of the appropriate order, in this case $\mathscr{O}(M_1)$.

Under this assumption, the quadratic term in the expansion of the Hamiltonian around the equilibrium position becomes

$$\mathscr{H}_2 = 2\omega_2 M_2 + \frac{2\varepsilon}{3}(M_1 + M_2).$$

Due to the fact that $\varepsilon = \mathscr{O}(M_1)$ we can regard $\frac{2\varepsilon}{3}(M_1 + M_2)$ as a perturbation and consider the problem as a perturbed Hamiltonian system under 1:3 resonance. Now, $\mathscr{H}_4$ in the normal form expresses as

$$\mathscr{H}_4 = a_4 M_1^2 + a_2 M_1 M_2 + \gamma S + \frac{2\varepsilon}{3}(M_1 + M_2),$$

where we have taken into account that $\varepsilon = \mathscr{O}(M_1)$.

We note that the reduced phase space is the same corresponding to the 1:3 resonance, that is, it is given by (1) and (2). However, the surface $\mathscr{G}_1$ in Theorem 2 is written now as

$$\mathscr{G}_1 = \{(C, S, M_1) \in \mathbb{R}^3\,;\, a_4 M_1^2 + \gamma S + \frac{2\varepsilon}{3} M_1 = 0\}.$$

It is worth to note that $\mathscr{G}_1$ and $\mathscr{G}_2$ have the origin as an isolated intersection point if it is verified that $a_4^2 - \gamma^2 > 0$, whereas they intersect transversely if $a_4^2 - \gamma^2 < 0$. These are, precisely, the conditions for stability or instability in the resonant case ($\varepsilon = 0$). In Figure 4 we can see several projections of the intersection onto the plane $C = 0$ for several values of ε close to 0.

When $a_4^2 - \gamma^2 > 0$, there is only one equilibrium point and all orbits in the surface of revolution $M_2 = 0$ are bounded surrounding the vertex. When $a_4^2 - \gamma^2 < 0$, it is worth to note that the orbits in the surface $M_2 = 0$ evolve from a stability (out of the 1:3 resonance) to instability condition (in the 1:3 resonance). This evolution can be seen in Figure 5. Near the 1:3 resonance, there always exists a stability domain which size is reduced as we approach the exact value of the resonance. The lost of stability takes place when the size of the stability domain goes to zero.

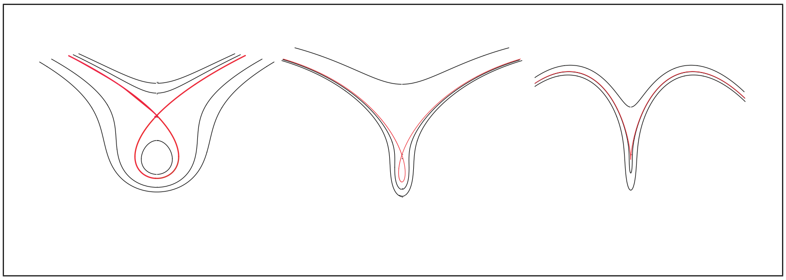

Figure 5: Evolution of the stability domain when $a_4^2 - \gamma^2 < 0$.

§5. Conclusions

Arnold's Theorem guarantees the stability of the origin in most situations when the frequencies of the system are not in resonance. In the particular case of the 1:3 resonance, Markeev's Theorem gives the conditions for stability or instability. These conditions also can be deduced from a geometric criterion. The advantage of the geometric criterion is that it allows to show how the stable situation, away the resonance, turns into instability for the resonant situation. This is because the stability domain shrinks and it disappears at the resonance when the parameters satisfy the instability condition.

Acknowledgements

We acknowledge financial support from Ministerio de Ciencia y Tecnología (project of reference BFM2002-03157) and Gobierno de Navarra (project of reference Resolution 18/2005 and title *Métodos cualitativos y cuantitativos en sistemas dinámicos con aplicaciones en física e ingeniería)*.

References

[1] ARNOLD, V. I. The stability of the equilibrium position of a Hamiltonian system of ordinary differential equations in the general elliptic case. *Soviet Math. Dokl. 2* (1961), 247–249.

[2] BIRKHOFF, G. D. *Dynamical Systems*. American Mathematical Society, Providence, Rhode Island, 1991.

[3] CABRAL, H. E., AND MEYER, K. R. Stability of equilibria and fixed points of conservative systems. *Nonlinearity 12* (1999), 1351–1362.

[4] ELIPE, A. Complete reduction of oscillators in resonance p:q. *Phys. Rev. E 61* (2000), 6477–6484.

[5] ELIPE, A. Extended Lissajous variables for oscilators in resonance. *Mathematics and Computers in Simulation 57* (2001), 217–226.

[6] ELIPE, A., AND DEPRIT, A. Oscillators in resonance. *Mech. Res. Commun. 26* (1999), 635–640.

[7] ELIPE, A., LANCHARES, V., LÓPEZ-MORATALLA, T., AND RIAGUAS, A. Non linear stability in resonant cases: A geometrical approach. *J. Nonlinear Sci. 11* (2001), 211–222.

[8] ELIPE, A., LANCHARES, V., AND PASCUAL, A. I. On the stability of equilibria in two degrees of freedom Hamiltonian systems under resonances. *J. Nonlinear Sci. 15* (2005), 305–319.

[9] MARKEEV, A. P. Stability of a canonical system with two degrees of freedom in the presence of resonance. *Prikh. Mat. Mech. 32*, 4 (1968), 738–744.

[10] MEYER, K. R., AND SCHMIDT, D. S. The stability of the Lagrange triangular point and a theorem of Arnold. *Journal of Differential Equations 62* (1986), 222–236.

[11] PASCUAL, A. I. *Sobre la estabilidad de sistemas hamiltonianos de dos grados de libertad bajo resonancias.* Doctoral Thesis, Logroño, La Rioja, Spain, 2005.

[12] SIEGEL, C. L., AND MOSER, L. K. *Lectures on Celestial Mechanics.* Springer-Verlag, New York, 1971.

[13] SOKOLSKY, A. G. On the stability of an autonomous Hamiltonian system with two degrees of freedom in the case of equal frequencies. *Prikh. Mat. Mech. 38*, 5 (1974), 791–799.

[14] SOKOLSKY, A. G. On the stability of an autonomous Hamiltonian system with two degrees of freedom under first–order resonance. *Prikh. Mat. Mech. 41*, 1 (1977), 24–33.

V. Lanchares and A. I. Pascual
Departamento de Matemáticas y Computación
Universidad de La Rioja
c/ Luis de Ulloa, s/n
Logroño (La Rioja)
vlancha@unirioja.es and aipasc@unirioja.es