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ASYMPTOTIC RATES FOR RECORDS AND δ-RECORDS

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Abstract. We consider random variables X_n, $n \geq 1$, and their associated counting process N_n of exceptional observations, such as records or δ-records. For most well know distributions, martingale tools show their worth in dealing with questions such as the law of large numbers and asymptotic normality.

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§1. Outstanding observations

Let $\{X_n,\ n \geq 1\}$ be a sequence of real valued random variables. We say that X_n is an outstanding observation if its value is exceptional, compared to previously observed ones. We will adopt the generic name of record for these observations although it is normally used in extreme value theory to designate values that are smallest or greatest.

Records can be identified by means of a sequence of indicators $\{I_n, n \geq 1\}$, with $I_n = 1$ if X_n is a record, $I_n = 0$ otherwise, and $I_1 = 1$ conventionally. Since X_n is declared exceptional with respect to preceding ones, we assume the I_n are adapted to the natural filtration of the X_n, that is, $I_n \in \mathscr{F}_n = \sigma(X_1,\dots,X_n)$, for $n \geq 1$. Times T_n at which successive records appear are called record times and are defined by $T_1 = 1$ and $T_{n+1} = \min\{k > T_n \mid I_k = 1\}$, $n \geq 1$. On the other hand, record values are defined as X_{T_n}, whenever T_n is finite.

The theory of records studies the behavior of the three sequences introduced above, namely indicators, record times and record values. Usual records (exceptionally large or small observations) have been a very active subject of research in extreme value theory, with interesting theoretical and applied results. See, for example, [1] and [17].

The notion of record can be further extended to random elements with values in partially ordered sets, using the following natural definition. Observation X_n is a record if $X_k \ngeq X_n$, for all $k = 1,\dots,n-1$.

1.1. Some examples

1.1.1. Strict records

This is by far the most studied case. Results have usually been obtained under the iid (independent and identically distributed) hypothesis and continuity of their common distribution F. The condition for X_n to be a strict upper record is $X_n > M_{n-1} := \max\{X_1,\dots,X_{n-1}\}$. For lower records, it is $X_n < \min\{X_1,\dots,X_{n-1}\}$.

1.1.2. Weak records

Weak record were introduced by Vervaat [19] in the context of integer valued random variables. Upper and lower weak records are obtained by relaxing the corresponding inequality of strict records. For example, weak upper records satisfy $X_n \geq M_{n-1}$. Of course, the definition makes sense only if ties occur with positive probability.

1.1.3. k-records

An observation X_n is defined to be a k-record if it has rank k among $X_1, X_2, \ldots, X_n$. In terms of indicators, the condition can we written as $\sum_{i=1}^{n} 1_{\{X_i \geq X_n\}} = k$. Note that the 1-record is simply the strict upper record, while the n-record is the strict lower record.

1.1.4. δ-records

The study of observations near the maximum or near-records, has attracted considerable attention in recent years and several definitions have been proposed, see [12] and references therein. A natural and tractable concept is that of δ-record, defined as follows: for a fixed $\delta \in \mathbb{R}$, X_n is said to be a (upper, additive) δ-record if $X_n > \delta + M_{n-1}$. Is is also possible to consider multiplicative versions, where the above condition is replaced by $X_n > \delta M_{n-1}$. These are called *geometric records* in a recent publication of Eliazar [8].

1.1.5. Multivariate records

Records have also been considered in a multivariate setting. However, there is no obvious way to extend the usual notion of record. Goldie and Resnick [9] introduce and discuss the merits of several plausible definitions. For example, suppose X_n has k real components $X_{n,1}, X_{n,2}, \ldots, X_{n,k}$, then X_n is defined to be a (upper) record if $X_{n,j} > M_{n-1,j} := \max\{X_{1,j}, \ldots, X_{n-1,j}\}$, for all $j = 1, \ldots k$, or, if $X_{n,j} > M_{n-1,j}$, for some $j = 1, \ldots k$.

Another interesting multidimensional extension, related to maximal layers and Pareto optima, was introduced by Devroye in [7]. Let f be a nonnegative and nondecreasing function on $[0,1]$ and $Z_i = (X_i, Y_i)$, $1 \leq i \leq n$, bivariate random variables distributed in the set $\{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq f(x)\}$. Observation Z_i is a defined to be a record if $Y_i = \max\{Y_j \mid X_j \leq X_i\}$. Note that here the event $\{Z_i \text{ is record}\}$ is not $\mathscr{F}_i$ measurable. In fact, it depends on the whole set of observations.

More generally along this line, Baryshnikov and Yukich [5] consider maximal points of a finite set of random elements $\mathscr{X} = \{X_1, \ldots, X_n\} \subseteq \mathbb{R}^d$, related to a cone $K \subseteq \mathbb{R}^d$. Observation X_i is said to be a record (maximal or Pareto optimal) if the cone $K \oplus X_i$ contains no other points in $\mathscr{X}$. That is, $K \oplus X_i \cap \mathscr{X} = \emptyset$. The collection of maximal points $M(\mathscr{X})$ is called the maximal layer. These outstanding points appear in pattern classification, multi-criteria decision theory, networks, etc.

1.1.6. Records on rooted trees

In the context of more exotic structures, S. Janson [13] introduces a definition of record on a rooted tree. It is related to the cutting down of trees, which consists in randomly pruning the tree until only the root is left. Real random variables X_v are attached to each vertex v of the tree. The value X_v is said to be a record if it is the largest (or smallest) value in the path from the root to v.

§2. Record rates

Among the many remarkable asymptotic properties of records, we focus in this paper on record rates, which have to do with the asymptotic behavior of the counting process of records. We will go into some details to show how martingale tools are useful in handling the 1-dimensional discrete case.

There are many applied instances where the counting process of record is relevant, especially in computer science. We can mention cost analysis of a simple sorting algorithm, best choice problems, insurance claims, cost analysis of algorithms for computing the maximal layer and cost analysis of skip lists.

The counting process of records associated to the sequence $\{X_n,\ n \geq 1\}$ is defined in terms of the indicators I_n, as $N_n = \sum_{i=1}^n I_i$. This definition applies to all forms of records mentioned above but, it is important to note that in some instances, such as the maximal layer model, the adaptability condition $I_n \in \mathscr{F}_n = \sigma(X_1,\ldots,X_n)$, $n \geq 1$, does not hold. The case of rooted trees is different but it seems natural to define N_n as the number of records in a tree T_n of height n, for example.

The asymptotic analysis of N_n is usually limited to the classical LLN (Law of Large Numbers) and the CLT (Central Limit Theorem). However, researchers in computer science pay significant attention to moments of N_n, obtaining very precise expansions with their tools from singularity analysis of generating functions.

We shall take a brief look at some remarkable results before getting into the more detailed analysis of the discrete case, which has nice features.

We begin with the simplest case of real iid random variables X_n, with common continuous distribution F. Renyi and Dwass established long ago that indicators $I_n = 1_{\{X_n > M_{n-1}\}}$ of usual (strict, upper) records are independent with $E(I_n) = 1/n$. The LLN and the CLT are therefore readily obtained for N_n: as $n \to \infty$,

$$\frac{N_n}{\log n} \xrightarrow{a.s.} 1 \quad \text{and} \quad \frac{N_n - \log n}{\sqrt{\log n}} \xrightarrow{D} N(0,1),$$

where $\xrightarrow{D}$ and $\xrightarrow{a.s.}$ stand for convergence in distribution and almost sure, respectively.

An interesting and tractable departure from the iid continuous case is the Nevzorov-Yang F^{α} model. Observations are independent but each X_n has distribution F^{α_n}, where $\alpha_1, \alpha_2, \ldots$ are positive constants and F is a fixed continuous distribution. As in the iid case, indicators I_n of usual records are independent with $E(I_n) = \alpha_n / \sum_{i=1}^n \alpha_i$, and again, the LLN and the CLT for N_n are easily obtained. See [17] for details.

Another nice extension of the iid case is the sequence with linear trend $\{X_n, n \geq 1\}$, defined from an iid sequence $\{Y_n, n \geq 1\}$ by $X_n = Y_n + cn$, with $c > 0$. When the common distribution F of the Y_n's is continuous and $E(Y_1^+) < \infty$, Ballerini and Resnick [4] have established the following: as $n \to \infty$,

$$\frac{N_n}{n} \xrightarrow{a.s.} \int_{-\infty}^{\infty} \prod_{j=1}^{\infty} F(y+cj)F(dy).$$

2.1. The iid discrete case, strict upper records

We consider a sequence $\{X_n, n \geq 1\}$ of iid random variables, taking nonnegative integer values, and its associated sequence of (strict, upper) records. That is, $I_n = 1_{\{X_n > M_{n-1}\}}$, $n \geq 1$. The situation here is not as simple as in the continuous model since the independence property of indicators is lost. The case of geometric random variables received considerable attention since the early 1990's because of its connection with data structures known as skip lists. Using singularity analysis, Prodinger [18] obtained results such as

$$E(N_n) = p\left(-\frac{\log n}{\log(1-p)} + \frac{\gamma}{\log(1-p)} + \frac{1}{2} - \delta\left(-\frac{\log n}{\log(1-p)}\right)\right) + O\left(\frac{1}{n}\right), \tag{1}$$

where p is the parameter of the geometric distribution, γ is Euler's constant and δ is a periodic function. This type of expansion is still not available for other discrete models although the first order was reported in Gouet et al. [11].

The following CLT is essentially contained in Vervaat's paper [19] and was much later studied by Bai et al. [2] and by Gouet et al. [11], as a particular case of their martingale approach (in this case the process $N_n - pM_n$ is shown to be a martingale).

$$(\log n)^{-1/2}\left(N_n + \frac{p\log n}{\log(1-p)}\right) \xrightarrow{D} N\left(0, -\frac{p(1-p)}{\log(1-p)}\right).$$

The LLN was reported in Gouet et al. [10] and rediscovered by Key [15].

$$\frac{N_n}{\log n} \xrightarrow{a.s.} -\frac{p}{\log(1-p)}.$$

Extensions of the LLN to other discrete distributions, are based on the following well-known conditional Borel-Cantelli lemma.

Proposition 1. *Let $\{I_n,\ n \geq 1\}$ be a sequence of $\{0,1\}$ valued random variables, adapted to the sequence of increasing σ-fields $\{\mathscr{F}_n,\ n \geq 1\}$. Then the events $\{\sum_{n\geq 1} I_n < \infty\}$ and $\{\sum_{n\geq 1} E(I_n|\mathscr{F}_{n-1}) < \infty\}$ are a.s. equal and*

$$\frac{\sum_{k=1}^{n} I_k}{\sum_{k=1}^{n} E(I_k|\mathscr{F}_{k-1})} \xrightarrow{a.s.} 1, \tag{2}$$

on the set $\{\sum_{n\geq 1} I_n = \infty\}$.

When the result above is applied to the indicators of upper records, from iid random variables with common distribution F, we obtain $E(I_n|\mathscr{F}_{n-1}) = 1 - F(M_{n-1})$, where M_n is maximum of $X_1,\dots,X_n$ and $\mathscr{F}_n = \sigma(X_1,\dots,X_n)$. If $\sup\{x \geq 0 \mid F(x) < 1\} = \infty$, then it is easily seen that the total number of records is infinite, so that (2) applies and we obtain

$$\frac{N_n}{\sum_{k=1}^{n}(1-F)(M_{k-1})} \xrightarrow{a.s.} 1. \tag{3}$$

Since $1-F$ is decreasing, $(1-F)(M_{k-1})$ is in fact the minimum of iid random variables $Y_1,\dots,Y_{k-1}$, where $Y_j = (1-F)(X_j)$, $j \geq 1$ (M_0 can be arbitrarily defined).

Fortunately, the asymptotic behavior of sums of partial minima of iid random variables was studied long ago in contexts apparently not related to records. Deheuvels [6] gives a very complete picture in terms of the LLN, the CLT and almost sure bounds for $S_n = \sum_{k\leq n}\min\{Y_1,\dots,Y_k\}$, where the Y_k, $k \geq 1$, are nonnegative iid, with common distribution G. It is interesting but perhaps not surprising to see that the growth rate of S_n is $\sum_{k\leq n} G^-(1/k)$, where $G^-(y) = \inf\{x \mid G(x) \geq y\}$, for $0 \leq y \leq 1$, is the inverse of G.

When F is continuous, the $Y_j = (1-F)(X_j)$, $j \geq 1$, are iid uniformly distributed in $[0,1]$, therefore the corresponding sum of minima has growth rate $\sum_{k\leq n} 1/k$. This yields with (3), $N_n/\log n \xrightarrow{a.s.} 1$, which is Renyi's LLN. This proof is admittedly more complicated that the original one based on the independence of the indicators. Once the continuity of F is abandoned the power of Proposition 1 is fully appreciated.

2.1.1. Strong laws for discrete models

Let us consider nonnegative, integer-valued iid random variables X_n, with common distribution function F. For $k \in \mathbb{Z}_+$, let $p_k = P[X_n = k] > 0$, $y_k = 1 - F(k)$ and $r_k = p_k/y_{k-1} = P[X_n = k]/P[X_n \geq k]$. Let also $m_n = \min\{j \in \mathbb{Z}_+ \mid y_j < 1/n\}$.

It is shown in [10] that the sum of minima in the denominator of (3) has growth rate essentially given by the cumulative discrete hazard function

$$\theta(m_n) = \sum_{k\leq m_n} r_k. \tag{4}$$

This result implies the strong LLN for the number of strict records N_n in a wide class of discrete models, including the geometric and Poisson distributions. For instance, when $\lim_{k\to\infty} r_k = 0$ it is shown that $\theta(m_n)$ is asymptotically logarithmic and we have

$$\frac{N_n}{\log n} \xrightarrow{a.s.} 1,$$

as in the continuous case. The Zeta distribution, with $p_k = (k+1)^{-a}/\zeta(a)$, for $k \in \mathbb{Z}_+$ and $a > 1$, is a classical example in this category.

Another important particular case, which includes the geometric and negative binomial models, is when $\lim_{k\to\infty} r_k = r > 0$. The rate is again logarithmic and we have

$$\frac{N_n}{\log n} \xrightarrow{a.s.} \frac{r}{-\log(1-r)}.$$

Distributions with light tails ($\lim_{k\to\infty} r_k = 1$) are harder to deal with. It is proved that if either r_k converges increasingly to 1 or $y_n e^{-nK} > 1$, for a given constant K and sufficiently large n, it holds

$$\frac{N_n}{m_n} \xrightarrow{a.s.} 1,$$

as $n \to \infty$, with m_n defined above. The Poisson model has increasing hazard rates so the result applies. Lighter tails should yield fewer records and this is the situation here, where m_n has growth rate $\log n / \log\log n$.

Expansions for moments of $E[N_n]$ are appreciated in computer sciences. With a moderate effort is shown in [11] that

$$E[N_n] = \theta(m_n) + o(b_n^{-1}),$$

where b_n is the normalizing sequence for the CLT. A general result but modest compared to Prodinger's expansion for the geometric distribution (1).

2.1.2. Asymptotic normality for discrete models

Building a martingale from N_n which is amenable to the classical CLT analysis is not difficult. It suffices to subtract the conditional expectation from I_n to obtain the martingale $N_n - \sum_{k\le n}(1-F)(M_{k-1})$, $k \ge 1$. However there is no obvious way to replace the centering process $\sum_{k\le n}(1-F)(M_{k-1})$ by a deterministic sequence so the resulting CLT is not really satisfactory.

Another less obvious square-integrable martingale was introduced in [11] as

$$N_n - \theta(M_n). \tag{5}$$

Such a simple martingale, which remained apparently unnoticed in the literature, is the basis of our analysis. It should be pointed out that Deheuvels' CLT for sums of minima in [6], although interesting, is of no use here. However, his weak LLN is still essential in our strategy.

A few words about the martingale CLT are in order. The version of the theorem we use depends on two conditions on the martingale $\sum_{k\le n}\xi_k$. The first is convergence of the conditional variances process

$$\frac{1}{b_n^2}\sum_{k\le n} E[\xi_k^2|\mathscr{F}_{k-1}] \xrightarrow{P} 1, \tag{6}$$

and the second is the Lindeberg type hypothesis

$$\frac{1}{b_n^2}\sum_{k\le n} E[\xi_k^2 \mathbf{1}_{\{|\xi_k|>\varepsilon b_n\}}|\mathscr{F}_{k-1}] \xrightarrow{P} 0, \tag{7}$$

for all $\varepsilon > 0$, where $\xrightarrow{P}$ denotes convergence in probability. Then $\sum_{k\le n}\xi_k/b_n \xrightarrow{D} N(0,1)$.

Checking (6) and (7) turns out to be related again to sum of minima. This is so because $E[\xi_k^2|\mathscr{F}_{k-1}] = \sum_{i>M_{k-1}} r_i y_i$ is a decreasing function of M_{k-1} and this allows to write (6) as sum of minima of iid random variables. Also, the sum in (7) can be bounded by sums of minima so that Deheuvels' weak convergence results are applicable. We have the general result

§3. δ-records

The notion of δ-record is quite natural when one is interested in observations which are close to being records. Counting such observations may be relevant, for instance, in insurance applications. The study of observations near the maximum has drawn significant attention in recent years. See for example [3], [14] and [16].

The formal definition of (upper) δ-records is as follows. Let $\delta \in \mathbb{R}$, an observation X_n is called a δ-record if $X_n > M_{n-1} + \delta$, that is, if it is greater than the previous maximum plus a (negative or positive) fixed quantity. For $\delta < 0$, every record is a δ-record, while for $\delta > 0$ this is not the case. Usual records are obtained by taking $\delta = 0$ and, for integer-valued random variables, $\delta = -1$ yields weak records. We focus attention on the process N_n counting the number of δ-records among the first n observations.

In the most favourable setting of iid random variables with continuous distribution, the main difficulty is that independence of indicators and distribution-freeness are lost. However the Borel-Cantelli type result of Proposition 1 is still useful and probably the simplest way to deal with the LLN. For the CLT it is not readily seen what martingale could be used instead of (5).

Another particular feature of δ-records, for $\delta > 0$, is that N_n can have a finite limit even though the support of F is unbounded. As it will be seen below, well known light-tailed distributions exhibit this behavior.

3.1. The iid integer valued case

We consider nonnegative, integer-valued iid random variables $X_n, n \geq 1$ with common distribution F. Therefore, δ takes only integer values. The notation is that of Subsection 2.1.1.

3.1.1. Strong laws

Proposition 1 is applied to translate the problem from N_n to the sum of minima $\sum_{k\leq n}(1 - F)(M_{k-1} + \delta)$. Although we cannot expect to have results as general as for usual records, it is a conservative guess to expect a growth rate for δ-records similar to (4), with some kind of correction allowing for more or for less δ-records when δ is negative or positive respectively. It turns out to be

$$a_n = \sum_{k\leq m_n} \frac{y_{k+\delta}}{y_k} r_k. \tag{9}$$

As with usual records, light-tailed distributions ($r_k \to 1$) are more difficult to deal with. The general result is as follows.

Theorem 3. *If either*

(a) $\delta \in \mathbb{Z}$ *and* $\limsup r_k < 1$ *or*

(b) $\delta > 0$, $r_k \leq r_{k+1}$, *for all sufficiently large* k *and* $r_k \to 1$ *or*

(c) $\delta < 0$ *and* $1 - [(1 - r_k)/(1 - r_{k-1})]^{-\delta} \leq ck^{-\alpha}$, *for all sufficiently large* k *and constants* $c > 0$, $\alpha \in (1/2, 1)$.

Theorem 2. *If* $\sum_{k\geq 0}(1-r_k)=\infty$ *and if either* $\limsup r_k<1$ *or* $\liminf r_k>0$ *then*

$$\frac{N_n-\theta(m_n)}{b_n}\xrightarrow{D}N(0,1), \tag{8}$$

where $b_n^2=\sum_{k\leq m_n}z_kr_k/y_k$ *and* $z_k=\sum_{i>k}r_iy_i$.

When tails are light enough to make $\sum_{k\geq 0}(1-r_k)<\infty$, the martingale $N_n-\theta(M_n)$ converges a.s. This fact can be shown to imply the following "tightness" property: $(N_n-m_n)/c_n\xrightarrow{P}0$, for any sequence $c_n\to\infty$, as $n\to\infty$. See [11] for proofs.

For illustration, consider the following examples.

1. *Converging hazard rates* $r_k\to 0$, *with* $\sum_{k=1}^{\infty}r_k^2<\infty$. (Zeta distribution).

$$(\log n)^{-1/2}\left(N_n-\log n\right)\xrightarrow{D}N(0,1).$$

2. *Converging hazard rates* $r_k\to r$, $0<r<1$, *with* $\sum_{i=1}^{n}|r_i-r|/\sqrt{n}\to 0$. (Geometric or negative binomial distributions).

$$(\log n)^{-1/2}\left(N_n+\frac{r\log n}{\log(1-r)}\right)\xrightarrow{D}N\left(0,-\frac{r(1-r)}{\log(1-r)}\right).$$

3. *Alternating hazard rates.* $r_{2k}=p$ and $r_{2k+1}=q$, $0<p<q<1$, $k\geq 0$.

$$(\log n)^{-1/2}\left(N_n+\frac{(p+q)\log n}{\log(1-p)(1-q)}\right)\xrightarrow{D}N\left(0,-\frac{p(1-p)+q(1-q)}{\log(1-p)(1-q)}\right).$$

4. *Poisson distribution with parameter* λ $(r_k\to 1)$.

$$(\log\log n)^{-1/2}\left(N_n-m_n+\lambda\log(m_n)\right)\xrightarrow{D}N(0,\lambda),$$

with $m_n\log\log n/\log n\to 1$.

Remark 1. The results provided by Theorem 2 are quite complete. The only case not covered is when hazard rates r_k have both 0 and 1 as accumulation points. Observe that unlike continuous distributions, the CLT for integer random variables depends on the parent distribution F via the hazard rates. Moreover, for distribution with very light tails ($\sum_{k\geq 0}(1-r_k)<\infty$) N_n is not asymptotically normal.

2.1.3. The discrete Nevzorov-Yang F^{α} model

When looking for tractable departures from the iid hypothesis, the F^{α} model emerges as natural candidate. Up to the authors' knowledge, the discrete F^{α} model has not received any attention from researchers in extreme value theory. Concerning the behavior of N_n, preliminary calculations show that a straightforward application of Proposition 1 leads to the analysis of sum of minima for non iid random variables. No such extension of Deheuvels' results is yet available. In general, it appears that our martingale approach is not easily adaptable to situations of nonstationary or dependent observations.

Then

$$\frac{N_n}{a_n} \xrightarrow{a.s.} 1.$$

A few examples follow.

1. For the Zeta distribution, with $p_k = (k+1)^{-a}/\zeta(a)$, for $k \in \mathbb{Z}_+$ and $a > 1$, it can be shown that $N_n/\log n \xrightarrow{a.s.} 1$, regardless of the value or sign of δ.

2. The geometric distribution, with $p_k = p(1-p)^k$, $y_k = (1-p)^{k+1}$ and $r_k = p$, for $k \geq 0$ and $p \in (0,1)$ yields $N_n/\log n \xrightarrow{a.s.} -p(1-p)^{\delta}/\log(1-p)$.

3. The Poisson distribution. We recall that in this case $r_k \to 1$ Furthermore, it can be shown that $y_{k+\delta}/y_k$ is asymptotically equivalent to $(\lambda/k)^{\delta}$. This means that N_n converges a.s to a finite limit for $\delta > 1$. Otherwise, for $\delta = 1$, $N_n/\log m_n \xrightarrow{a.s.} \lambda$ and, if $\delta < 0$, $N_n/m_n^{1-\delta} \xrightarrow{a.s.} \lambda^{\delta}/(1-\delta)$.

3.1.2. Asymptotic normality

Given the previous results on the LLN, it is reasonable to think that only minor changes in the strategy of 2.1.2 are needed to work out a CLT for δ-records. We will see that in fact, the main ideas still apply but some difficulties and lengthy calculations are the prize to pay for the greater generality. The first challenge is to properly generalize martingale (5). This is done by introducing the concept of δ-hazard rate as

$$s_k = \frac{p_{k+\delta}}{y_{k-1}} = \frac{P[X_1 = k+\delta]}{P[X_1 \geq k]}.$$

The process

$$N_n - \theta^{\delta}(M_n) \equiv N_n - \sum_{k=0}^{M_n} s_k, \quad n \geq 1, \tag{10}$$

is a square integrable martingale. Moreover, it is also cubic integrable depending on the sign of δ and asymptotic properties of the r_k. This reinforced integrability is needed because we have to replace the Lindeberg hypothesis (7) by the more tractable Lyapunov condition.

$$\frac{1}{b_n^3} \sum_{k \leq n} E[|\xi_k|^3 | \mathscr{F}_{k-1}] \xrightarrow{P} 0. \tag{11}$$

Evaluation of conditional expectations required to check either (6) or (11) is not a simple task. And, to make matters worse, the process of conditional variances in (6) cannot always be written as sums of minima, as required to use Deheuvels' LLN. However, the following general CLT is obtained. Versions for negative and positive δ are presented separately; for technical details and proofs, see [12].

Theorem 4. *Let $\delta < 0$ and $z_k = \sum_{i>k} s_i (y_{i+\delta} + y_{i+\delta-1} - y_{i-1})$.*

(a) If $\limsup r_k < 1$*, then*

$$\frac{N_n - \theta^\delta(m_n)}{\sqrt{\sum_{k=0}^{m_n} z_k r_k / y_k}} \xrightarrow{D} N(0,1). \tag{12}$$

(b) If $\lim r_k = 1$ *and* $\lim(1-r_k)/(1-r_{k-1}) = 1$*, then*

$$\frac{N_n - \theta^\delta(m_n)}{\sqrt{\sum_{k=0}^{m_n} (1-r_k)^{2\delta}}} \xrightarrow{D} N(0,1). \tag{13}$$

In the following central limit theorem for $\delta > 0$, we restrict our attention to converging r_k.

Theorem 5. *Let* $\delta > 0$ *and* $\lim_{k\to\infty} r_k = r \in [0,1]$.

(a) If $r < 1$ *then*

$$\frac{N_n - \theta^\delta(m_n)}{\sigma_r \sqrt{\log n}} \xrightarrow{D} N(0,1),$$

where $\sigma_r^2 = -r(1-r)^\delta((1-r)^{\delta+1} - (1+2\delta r)(1-r)^\delta + 1)/\log(1-r)$ *for* $r \neq 0$ *and* $\sigma_0 = 1$.

(b) If $r = 1$ *then, defining* $e_k = (1-r_k)\cdots(1-r_{k+\delta-1})$,

$$\frac{N_n - \theta^\delta(m_n)}{\sqrt{\sum_{k=0}^{m_n} e_k}} \xrightarrow{D} N(0,1),$$

whenever $\sum_{k=0}^\infty e_k = \infty$*. If* $\sum_{k=0}^\infty e_k < \infty$ *then* N_n *converges almost surely to a finite limit* .

Remark 2. Observe that Theorem 5(a) is more restrictive than Theorem 4(a), concerning the behavior of the failure rates r_k. This is because the process of conditional variances can always be written as partial sums of minima only when $\delta < 0$. For positive δ we were able to analyze the case of converging r_k.

On the other hand, comparing results of Theorem 4(b) and Theorem 5(b), on light-tailed models, we find more generality in the positive case since we do not impose any condition on the rate of convergence of r_k to 1. This is not surprising in view of the structure of the s_k, with $1-r_k$'s in the denominator when δ is negative.

Remark 3. When $\delta > 0$, unlike the negative case, it is not guaranteed that the number of δ-records is infinite. Nevertheless, when this happens, this number is always asymptotically normal in contrast to the situation of usual records, which can grow to infinity without having a limiting normal distribution; see [11].

Examples of application of the above results to well-known distributions are given next.

1. Zeta distribution.

$$(\log n)^{-1/2}(N_n - \log n) \xrightarrow{D} N(0,1).$$

   Note that the normalizing sequences in this example do not depend on the value of δ, positive or negative.

2. Geometric distribution. For $\delta < 0$,

$$(\log n)^{-1/2}(N_n + pq^{\delta}\log n/\log q) \xrightarrow{D} N\left(0, -pq^{\delta}(q^{\delta+1} + q^{\delta} - 1)/\log q\right).$$

For $\delta = -1$ (weak records)

$$(\log n)^{-1/2}(N_n + (p/q)\log n/\log q) \xrightarrow{D} N\left(0, -(p/q^2)/\log q\right).$$

For positive δ,

$$(\log n)^{-1/2}(N_n + pq^{\delta}\log n/\log q) \xrightarrow{D} N\left(0, -pq^{\delta}(q^{\delta+1} - (1+2\delta p)q^{\delta} + 1)/\log q\right).$$

3. Poisson distribution. For $\delta < 0$,

$$m_n^{\delta-1/2}\left(N_n - \lambda^{\delta} m_n^{1-\delta}/(1-\delta)\right) \xrightarrow{D} N\left(0, \lambda^{2\delta}/(1-2\delta)\right).$$

For $\delta = 1$,

$$(\log\log n)^{-1/2}(N_n - \lambda \log m_n) \xrightarrow{D} N(0, \lambda).$$

Finally, when $\delta > 1$, the number of δ-records converges a.s. to a finite limit.

3.2. The iid continuous case

We consider nonnegative, iid random variables $X_n, n \geq 1$ with common continuous distribution F, such that $\sup\{x \geq 0 \mid F(x) < 1\} = \infty$. Parameter δ can take any real value.

3.2.1. Strong laws

As mentioned at the beginning of Section 3, independence of indicators I_n and distribution-freeness are lost but Proposition 1 is applicable. Hazard rates r_k were a crucial element in our analysis of the discrete case. In the continuous setting we have to turn to the classical hazard measure. In order to simplify the presentation, we shall assume that F is strictly increasing in $\mathbb{R}_+$, with density f, so that the hazard measure has density given by $\lambda(x) = f(x)/(1-F(x))$.

As in the discrete case, we have to find the growth rate of $\sum_{k \leq n}(1-F)(M_{k-1}+\delta)$. It turns out to be the exact continuous analog of (9) and is given by

$$a_n = \int_0^{m_n} \frac{(1-F)(x+\delta)}{(1-F)(x)} \lambda(x)\, dx, \tag{14}$$

where $m_n = (1-F)^{-1}(1/n)$.

When $\lambda(x)$ is bounded, it can be shown that $a_n = O(\log n)$. For example, the exponential distribution with parameter μ has hazard rate $\lambda(x) = \mu$ and the following strong LLN holds:

$$N_n/\log n \xrightarrow{a.s.} e^{-\delta\mu}.$$

Another interesting case is the Pareto distribution $1 - F(x) = (a/x)^k$, $x \geq a$, $k > 0$, with $\lambda(x) = k/x$. The LLN for δ-records (δ positive or negative) is $N_n/\log n \xrightarrow{a.s.} 1$.

The continuous analog of the “difficult” discrete case $r_k \to 1$ is $\lambda(x) \to \infty$. As before, $\delta > 0$ may lead to a finite number of δ-records, and we don’t have to look far to find an example. Take the standard normal distribution Φ. Using the well-known approximation for the tail $1-\Phi(x) \sim \Phi'(x)/x$, as $x \to \infty$, we find that, for any $\delta > 0$, N_n converges a.s. to a finite limit. However, for $\delta = 0$ (usual records) the rate of N_n is $\log n$, from Renyi’s classical result. For negative δ, we can only obtain a weak LLN from Deheuvels’ results, showing that

$$\frac{N_n}{\sqrt{2\log n}\, e^{-\delta\sqrt{2\log n}}} \xrightarrow{P} -\frac{e^{-\delta^2/2}}{\delta}.$$

3.2.2. Asymptotic normality

The question of asymptotic normality for δ-records in the continuous setting has not been studied. However, a good starting point would be the process

$$N_n - \int_0^{M_n} \frac{f(x+\delta)}{1-F(x)} dx,$$

a martingale which is the continuous version of (10).

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