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A SIMPLIFIED MODEL FOR THE AEROELASTIC DESIGN OF QUASI-AXISYMMETRICAL BODIES

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Abstract. An axisymmetrical body embedded in an incompressible flow is considered in this paper. But the movement of the body can be non-axisymmetrical and therefore the flow perturbation is no more axisymmetrical. Using a perturbation method we construct an aeroelastic model which enables one to detect instabilities.

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§1. Introduction

An axisymmetrical airship set in an air flow, is considered in this paper. The reference configuration with respect to the wind direction is compatible with this property. It is assumed that the movements of the structure can be considered as a slight perturbation of the axisymmetrical configuration. Then using domain perturbation tools, we construct an asymptotic expansion of the solution to the unsymmetrical model using Fourier decomposition for the first order corrector terms. Therefore, the three dimensional model can be evaluated from a series of axisymmetrical models. Numerical results are also given in order to investigate the aeroelastic stability of the system. The first case that we consider in this paper, corresponds to the rigid eigenmodes of the structure. In fact, two of them are meaningful: the galloping and the pitching movements. The second case is more complicated and is obtained when flexible eigenmodes of the structure are taken into account. In the latter case, a progressive Euler-Lagrange formulation is suggested in order to give a consistent formulation of the coupling conditions between the structure and the fluid.
In each case, we develop a method which enables one to perform a stability analysis of the coupled system (the structure is represented by a finite number of eigen-modes), and the fluid which is modeled using an incompressible but viscous, flow model.

§2. Galloping and pitching movements

Let us first define few notations which will be used in the following. The three dimensional open set on which the aerodynamical model is set is denoted by $\Omega(0)$. It corresponds to an axisymmetrical configuration of the airship with respect to the flow direction. The boundary between $\Omega(0)$ and the structure is denoted by $S(0)$. The remaining part of this boundary is split into two parts: one denoted by Γ_0 is defined by

$$\Gamma_0 = \{x \in \mathbb{R}^3,\ \mathbf{e}(\alpha).\boldsymbol{\nu}(x) \leq 0\}, \tag{1}$$

where $\boldsymbol{\nu}(x)$ is the unit outwards normal to the boundary of $\Omega(0)$ at x and $\mathbf{e}(\alpha)$ is the wind direction. Let us prescribe the Dirichlet boundary condition on Γ_0 by

$$\mathbf{u} = U\mathbf{e}(\alpha)\; U > 0 \text{ on } \Gamma_0 \quad \text{and} \quad \mathbf{u} = 0 \text{ on } S(0) \text{ which is the structure.} \tag{2}$$

The velocity field $\mathbf{u}$ and the pressure field p are solution of

$$\begin{cases} (\mathbf{u}, p) \in V \times L^2(\Omega(0)) \text{ s.t. :} \\ \forall \mathbf{v} \in V,\; \displaystyle\int_{\Omega(0)} \rho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \otimes \nabla \mathbf{u}\right].\mathbf{v} - p\,\mathrm{div}(\mathbf{v}) + 2\mu\gamma(\mathbf{u}) : \gamma(\mathbf{v}) = 0, \\ \forall q \in L^2(\Omega(0)),\; -\displaystyle\int_{\Omega(0)} q\,\mathrm{div}(\mathbf{u}) = 0. \end{cases} \tag{3}$$

The following notations have been used (μ is the viscosity of the air, ρ the mass density):

$$\mathbf{u} = u_i \mathbf{e}_i,\; [\mathbf{u} \otimes \nabla \mathbf{u}]_j = \sum_{k=1,3} u_k \partial_k u_j,\; [\gamma(\mathbf{u})]_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \tag{4}$$

$$V = \{\mathbf{v} = v_i \mathbf{e}_i,\; v_i \in H^1(\Omega(0));\, v_i = 0 \text{ on } \Gamma_0 \cup S(0)\}. \tag{5}$$

The existence and uniqueness of a solution to (3) are not proved. The only results known are for the two dimensional case [7]. Concerning the solution method, the mixed finite elements are very reliable [5]. This is the method used in this paper. But the model that we consider is three dimensional. When $\mathbf{e}(\alpha)$ is parallel to the main axis of the ellipsoid, the solution is also axisymmetrical. One can use a cylindro-polar system of coordinates (r, β, x), for the Navier-Stokes model (figure 1). The velocity field $\mathbf{u}$ is expressed in the basis $(\mathbf{e}_r, \mathbf{e}_\beta, \mathbf{e}_x)$ by

$$\mathbf{u} = u_r \mathbf{e}_r + u_\beta \mathbf{e}_\beta + u_x \mathbf{e}_x.$$

When the flow is axisymmetrical all the derivatives with respect to β are zero and one has $u_\beta = 0$. In this case the solution is denoted $(\mathbf{u}^0, p^0)$ and is solution of the following system:

$$\begin{cases} \forall \mathbf{v} \in V_0,\; \rho \displaystyle\int_{\omega(0)} \frac{\partial \mathbf{u}^0}{\partial t}.\mathbf{v} + [\mathbf{u}^0 \otimes \nabla \mathbf{u}^0].\mathbf{v} \\ \qquad\qquad - \displaystyle\int_{\omega(0)} p^0 \;\mathrm{div}(\mathbf{v}) + 2\mu \int_{\omega(0)} \gamma(\mathbf{u}^0) : \gamma(\mathbf{v}) = 0, \\ \forall q \in L^2(\omega(0)),\; -\displaystyle\int_{\omega(0)} q\,\mathrm{div}(\mathbf{u}^0) = 0, \end{cases} \tag{6}$$

with the boundary conditions:

$$\mathbf{u}^0 = U e_x \text{ on } \gamma(0), \text{ and } \mathbf{u}^0 = 0 \text{ on } s(0). \tag{7}$$

The following notations have been used (figure 1):

$$\Omega(0) = \{(r, \beta, x) | (r, x) \in \omega(0),\; \beta \in [0, 2\pi[\},\; \Gamma_0 = \gamma_0 \times [0, 2\pi[,\; S(0) = s(0) \times [0, 2\pi[$$

and

$$V_0 = \{\mathbf{v} = (v_r, v_x) \in [H^1(\omega(0))]^2,\; \mathbf{v} = 0 \text{ on } \gamma_0 \cup s(0)\}. \tag{8}$$

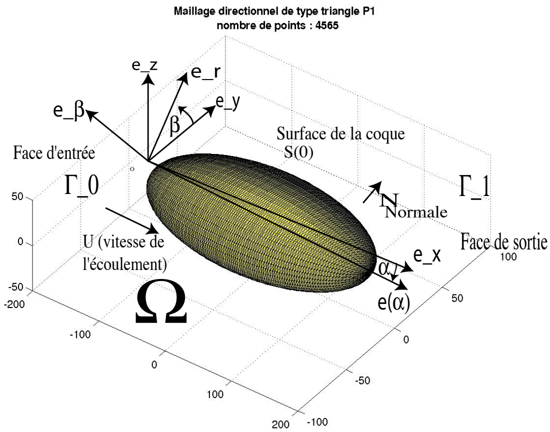

Figure 1: The geometry of the airship

Let us come back to the general case in which the pitching angle α is not zero. Nevertheless $\mathbf{e}(\alpha)$ is assumed to be close to $\mathbf{e}_x$. In fact the only non axisymmetrical condition is the Dirichlet condition satisfied by the velocity on the boundary Γ_0. Let us set

$$\begin{cases} \mathbf{e}(\alpha) = \cos(\alpha)\mathbf{e}_x + \sin(\alpha)\sin(\beta)\mathbf{e}_r + \sin(\alpha)\cos(\beta)\mathbf{e}_\beta, \\ \mathbf{u} = \mathbf{u}^0 + \mathbf{u}^\alpha, \;\; p = p^0 + p^\alpha. \end{cases} \tag{9}$$

Let us now formulate the linearized model with respect to α, $(\mathbf{u}^\alpha, p^\alpha)$ is the solution of which. Using a Fourier decomposition with respect to the angle β, one observes that only the harmonic 1 is different from zero. Hence, using a complex representation for sake of brevity, one obtains

$$(\mathbf{u}^\alpha, p^\alpha) \simeq (\mathbf{u}^1, p^1) = \sum_{\pm} (u_r^{\pm 1}\mathbf{e}_r + u_\beta^{\pm 1}\mathbf{e}_\beta + u_x^{\pm 1}\mathbf{e}_x, p^{\pm 1}) e^{\pm i\beta}, \tag{10}$$

where $(\mathbf{u}^1, p^1) \in V \times L^2(\Omega(0))$ is solution of an axisymmetrical model (hence 2D!) excepted concerning the boundary condition:

$$\begin{cases} \forall \mathbf{v} \in V, \; \rho \displaystyle\int_{\Omega(0)} \frac{\partial \mathbf{u}^1}{\partial t}.\mathbf{v} + [\mathbf{u}^0 \otimes \nabla \mathbf{u}^1 + \mathbf{u}^1 \otimes \nabla \mathbf{u}^0].\mathbf{v} \\ \qquad\qquad - \displaystyle\int_{\Omega(0)} p^1 \; \mathrm{div}(\mathbf{v}) + 2\mu \int_{\Omega(0)} \gamma(\mathbf{u}^1) : \gamma(\mathbf{v}) = 0, \\ \forall q \in L^2(\Omega(0)), \; - \displaystyle\int_{\Omega(0)} q \, \mathrm{div}(\mathbf{u}^1) = 0. \end{cases} \tag{11}$$

Furthermore these boundary conditions satisfied by $\mathbf{u}^1$ on Γ_0 are

$$\mathbf{u}^1 = U \sin(\alpha)[\sin(\beta)\mathbf{e}_r + \cos(\beta)\mathbf{e}_\beta]. \tag{12}$$

Let us point out that the component (see figure 1) $u^1_\beta \mathbf{e}_\beta$ is not zero along the axis of symmetry, but one has $(\mathbf{u}^\theta, \mathbf{e}_x) = 0$. In order to ensure that $\gamma(\mathbf{u}) \in L^2(\omega(0))$ it is necessary that $u_r + iu_\beta =$ 0 on this axis corresponding to $r = 0$. The solution to the previous system is proportional to $\sin(\alpha)$. It can be solved with a two dimensional problem but with a coupling with u_β. It is worth to recall the explicit expression of the strain tensor expressed in the basis $(\mathbf{e}_r, \mathbf{e}_\beta, \mathbf{e}_x)$. Let us set $\mathbf{u} = u_r\mathbf{e}_r + u_\beta\mathbf{e}_\beta + u_x\mathbf{e}_x$. Then,

$$\mathrm{div}(\mathbf{u}) = \frac{1}{r}\Big[\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\beta}{\partial \beta}\Big] + \frac{\partial u_x}{\partial x},$$

$$\nabla\mathbf{u} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{\partial u_\beta}{\partial r} & \dfrac{\partial u_x}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial u_r}{\partial \beta} - \dfrac{u_\beta}{r} & \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\beta}{\partial \beta} & \dfrac{1}{r}\dfrac{\partial u_x}{\partial \beta} \\ \dfrac{\partial u_r}{\partial x} & \dfrac{\partial u_\beta}{\partial x} & \dfrac{\partial u_x}{\partial x} \end{pmatrix}, \quad \text{with} \quad \nabla(.) = \begin{pmatrix} \dfrac{\partial}{\partial r}(.) \\ \dfrac{1}{r}\dfrac{\partial}{\partial \beta}(.) \\ \dfrac{\partial}{\partial x} \end{pmatrix},$$

$$\gamma(\mathbf{u}) = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\Big(\dfrac{\partial u_\beta}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \beta} - \dfrac{u_\beta}{r}\Big) & \dfrac{1}{2}\Big(\dfrac{\partial u_x}{\partial r} + \dfrac{\partial u_r}{\partial x}\Big) \\ \dfrac{1}{2}\Big(\dfrac{\partial u_\beta}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_r}{\partial \beta} - \dfrac{u_\beta}{r}\Big) & \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\beta}{\partial \beta} & \dfrac{1}{2}\Big(\dfrac{1}{r}\dfrac{\partial u_x}{\partial \beta} + \dfrac{\partial u_\beta}{\partial x}\Big) \\ \dfrac{1}{2}\Big(\dfrac{\partial u_x}{\partial r} + \dfrac{\partial u_r}{\partial x}\Big) & \dfrac{1}{2}\Big(\dfrac{1}{r}\dfrac{\partial u_x}{\partial \beta} + \dfrac{\partial u_\beta}{\partial x}\Big) & \dfrac{\partial u_x}{\partial x} \end{pmatrix}.$$

2.1. Quasi steady state

Let us consider two time scalings. The first one is connected to the frequencies f_s of the structure, and the second one to the flow velocity U and the wave length L of an eigenmode. Let us set

$$f_r = \frac{Lf_s}{U}. \tag{13}$$

The steady state approximation can usually be applied if $f_r \ll 1$. Furthermore it is assumed that the magnitude of the structural displacements is small enough. Nevertheless it is necessary to take into account the changes in the flow velocity due to these movements. This leads to the concept of the aerodynamical damping [4, 3]. Three terms are meaningful in the dynamical contribution. One is the classical relative acceleration. The second one is the acceleration of the frame connected to the structure, and the third one is the gyroscopic effect. Let us introduce the relative velocity $\mathbf{v}_a$ on Γ_0:

$$\mathbf{v}_a = U\mathbf{e}(\alpha) - \dot{\mathbf{Z}}^0 - \dot{\mathbf{R}} \wedge \mathbf{ox}. \tag{14}$$

The second term is the rigid body motion of the structure: $\dot{\mathbf{Z}}^0$ is the velocity at point o and $\dot{\mathbf{R}}$ is the rotation vector. The flow model is then similar to (3), excepted the new terms due to the acceleration and those on Γ_0 which become

$$\mathbf{u} = \mathbf{v}_a = U(\cos(\alpha)\mathbf{e}_x + \sin(\alpha)\mathbf{e}_z) - \dot{\mathbf{Z}}^0 - \dot{\mathbf{R}} \wedge \mathbf{ox}. \tag{15}$$

Thus, $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \otimes \nabla \mathbf{u}$ is replaced by $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \otimes \nabla \mathbf{u} + \gamma^e + 2\dot{\mathbf{R}} \wedge \mathbf{u}$, where γ^e is given by

$$\gamma^e = \ddot{\mathbf{Z}}^{\mathbf{0}} + \ddot{\mathbf{R}} \wedge \mathbf{ox} + \dot{\mathbf{R}} \wedge (\dot{\mathbf{R}} \wedge \mathbf{ox}).$$

Let us consider a movement of the structure which implies a translation $d_z \mathbf{e}_z$ and a rotation $\alpha \mathbf{e}_y$. The boundary condition becomes

$$\begin{aligned} \mathbf{u} = (U\cos(\alpha) - \dot{\alpha} r \sin(\beta))\mathbf{e}_x + (U\sin(\alpha) + \dot{\alpha} x - \dot{d}_z)\sin(\beta)\mathbf{e}_r \\ + (U\sin(\alpha) + \dot{\alpha} x - \dot{d}_z)\mathbf{e}_\beta. \end{aligned} \tag{16}$$

Furthermore, the acceleration of the frame is

$$\gamma^e = \ddot{d}_z \mathbf{e}_z + \dot{d}_z \dot{\alpha} \mathbf{e}_x + \ddot{\alpha} \mathbf{e}_y \wedge \mathbf{ox} + \dot{\alpha}^2 \mathbf{e}_y \wedge (\mathbf{e}_y \wedge \mathbf{ox}).$$

The linearisation around the axisymmetrical solution gives

$$\frac{\partial \mathbf{u}^{\mathbf{1}}}{\partial t} + \mathbf{u}^{\mathbf{0}} \otimes \nabla \mathbf{u}^{\mathbf{1}} + \mathbf{u}^{\mathbf{1}} \otimes \nabla \mathbf{u}^{\mathbf{0}} + \gamma^{eL} + 2\dot{\alpha} \mathbf{e}_y \wedge \mathbf{u}^{\mathbf{0}}, \tag{17}$$

where the linearized acceleration is denoted by γ^{eL}, and is such that

$$\gamma^{eL} = \ddot{d}_z \mathbf{e}_z + \ddot{\alpha} \mathbf{e}_y \wedge \mathbf{ox} = \ddot{d}_z[\sin(\beta)\mathbf{e}_r + \cos(\beta)\mathbf{e}_\beta] + \ddot{\alpha}[\cos(\beta)\mathbf{e}_r - \sin(\beta)\mathbf{e}_\beta] \wedge \mathbf{ox}. \tag{18}$$

Concerning the gyroscopic term, one gets:

$$\gamma^c = 2\dot{\alpha}[\cos(\beta)\mathbf{e}_r - \sin(\beta)\mathbf{e}_\beta] \wedge \mathbf{u}^0. \tag{19}$$

All the addditional terms (six) are confined on the first Fourier harmonic with respect to the angle β. The solution method requires to solve seven independent linear $2D$ models:

- The first one corresponds to the axisymmetrical flow with a flow velocity at the infinity equal to $U\cos(\alpha)\mathbf{e}_x \simeq U\mathbf{e}$. The solution is denoted $(\mathbf{u}^0, p^0)$.
- The six following ones corresponds to the first Fourier harmonic in β.
  - i) One is dependent on α and the solution is proportional to $U\sin(\alpha) \simeq U\alpha$.
  - ii) Another one is proportional to $\dot{\alpha}$.
  - iii) In a similar way there is one proportional to $\dot{d}_z$.
  - iv) Then, one contribution is proportional to $\ddot{d}_z$,
  - v) and a similar one is proportional to $\ddot{\alpha}$.
  - vi) Finally the gyroscopic term is proportional to $\dot{\alpha}$.

Finally, let us set, using a complex representation for sake of brevity,

$$\mathbf{u} = \mathbf{u}^0 + \sum_{\pm} e^{\pm i\beta}[U\alpha \mathbf{u}^{\pm i} + \dot{\alpha}\mathbf{u}^{\pm t} + \dot{d}_z \mathbf{u}^{\pm g} + \ddot{d}_z u^{\pm ma} + \ddot{\alpha}\mathbf{u}^{\pm ia} + \dot{\alpha}\mathbf{u}^{\pm c}]. \tag{20}$$

§3. Computation of the aerodynamical forces

3.1. Slow movements

Let us consider the simpliest case where the structure is stationary versus the flow direction. The unit normal to $S(0)$ is denoted by $\mathbf{N}$ and its projection onto the plane $(\mathbf{e}_r, \mathbf{e}_x)$ is ν. The mechanical stress is

$$\mathbf{T} = p\mathbf{N} - 2\mu\gamma(\mathbf{u}).\mathbf{N} = \mathbf{T}^0 + \mathbf{T}^1,$$

with

$$\mathbf{T}^0 = p^0\mathbf{N} - 2\mu\gamma(\mathbf{u}^0).\mathbf{N}, \text{ and } \mathbf{T}^1 = p^1\mathbf{N} - 2\mu\gamma(\mathbf{u}^1).\mathbf{N}.$$

Because $\mathbf{T}^1$ is proportional to $\sin(\alpha)$ and thus to α after linearisation, one obtains a linear expression with respect to α. Let us now consider a movement of the rigid structure represented by

$$\delta\dot{\mathbf{Z}}(x) = \delta\dot{\mathbf{Z}}(0) + \delta\dot{\mathbf{R}} \wedge \mathbf{ox}.$$

The virtual work of the aerodynamical forces is

$$P(\delta\dot{\mathbf{Z}}) = \int_{S(0)} \mathbf{T}^0.\delta\dot{\mathbf{Z}}^0 + \int_{S(0)} (\mathbf{T}^1, \delta\dot{\mathbf{R}}, \mathbf{ox}).$$

But $\mathbf{T}^1$ is proportional to $\sin(\alpha)$, and α is constant along the structure, therefore

$$P(\delta\dot{\mathbf{Z}}) = P^0(\delta\dot{\mathbf{Z}}) + \sin(\alpha)P^1(\delta\dot{\mathbf{Z}}).$$

This enables one to formulate a stability model for the coupled system as follows:

$$J_0\ddot{\alpha} = \xi\sin(\alpha) \simeq \xi\alpha, \quad \text{for small } \alpha, \tag{21}$$

therefore the stability depends on the sign of ξ. In fact the question is to set the center of rotation denoted by $\mathbf{o}$, with respect to the aerodynamical centre.

3.2. Steady aeroelasticity

From Section 2, one can write the forces applied to the structure as follows:

$$\mathbf{T} = U\cos(\alpha)T^a + U\sin(\alpha)\mathbf{T}^i + \dot{\alpha}\mathbf{T}^t + \dot{d}_z\mathbf{T}^g + \ddot{\alpha}\mathbf{T}^{ia} + \ddot{d}_z\mathbf{T}^{ma} + \dot{\alpha}\mathbf{T}^c, \tag{22}$$

where $\mathbf{T}^a$ is the stress vector due to the axisymmetrical flow $(\cos(\alpha) \simeq 1)$, $\mathbf{T}^i$ is the stress vector due to the pitching angle $\alpha \simeq \sin(\alpha)$, $\mathbf{T}^t$ is the stress vector due to the pitching velocity, $\mathbf{T}^g$ is the stress vector due to the galloping, $\mathbf{T}^{ia}$ is the added inertia for the pitching, $\mathbf{T}^{ma}$ is the added mass due to the galloping, and $\mathbf{T}^c$ is the stress vector due to the gyroscopic effect.

After a linearization of the full system with respect to the pitching angle (α) and the galloping (d_z), one obtains

$$M^a\begin{pmatrix}\ddot{\alpha}\\ \ddot{d}_z\end{pmatrix} + C^a\begin{pmatrix}\dot{\alpha}\\ \dot{d}_z\end{pmatrix} + K^a\begin{pmatrix}\alpha\\ d_z\end{pmatrix} = 0, \tag{23}$$

where:

- M^a is the full mass matrix,
- C^a is the damping (symmetrical part) and the gyroscopic (skew part) matrix,
- $K^a = \begin{pmatrix} k_t & 0 \\ k_g & 0 \end{pmatrix}$ is the aerodynamical stiffness matrix.

Because K^a not symmetrical, one can observe (at a critical velocity of the steady flow), a flutter instability (mode crossing). If the damping becomes negative, one says that there is a wake-flutter instability. Hence the stability analysis consists in studying the real part of λ solution to

$$\det(-\lambda^2 M^a + i\lambda C^a + K^a) = 0. \tag{24}$$

§4. Progressive Euler-Lagrange formulation for a flexible structure

The coupling equation between the fluid and the structure should be written in the deformed configuration. Because the steady state is not neglectible, additional terms due to the rotation of the normal appear in the model. The best way to write correctly this compatibility condition in our opinion, is to use a progressive Euler-Lagrange frame. First of all let us recall the formulation of the shell model for the structure.

4.1. The shell model

The classical Koiter model has been used in order to compute the eigenmodes $\mathbf{w}_n$. The corresponding frequencies are denoted by f_n.

Let us define by $\mathbf{z}$ the displacement field of the structure. If $m(.,.)$ is the inertia bilinear form, $a(.,.)$ the stiffness one and $\mathscr{W}$ the admissible displacement space, the eigenvalue problem consists in finding $(\lambda_n = (2\pi f_n)^2, \mathbf{w}_n)$ such that

$$\begin{cases} \mathbf{w}_n \in \mathscr{W},\ \lambda_n \in \mathbb{R}^+, \ \text{ such that:} \\ \forall v \in \mathscr{W},\ \lambda_n m(\mathbf{w}_n, v) = a(\mathbf{w}_n, v). \end{cases} \tag{25}$$

Let us assume that the structural movement is well represented by a space of N eigenmodes denoted by $\mathscr{W}_N$:

$$\mathbf{z} = \sum_{n=1,N} \kappa_n(t)\mathbf{w}_n. \tag{26}$$

Let us denote by $\mathbf{N}$ the unit normal to the shell oriented towards the inside of the fluid. From shell theory the deformed normal becomes

$$\mathbf{N}' = \mathbf{N} + \zeta(t), \tag{27}$$

where $\zeta(t)$ is the inplane rotation which depends on $\mathbf{z}$. Let us extend $\mathbf{z}$ inside the fluid by:

$$\left|\begin{array}{rl} \text{i)} & \theta = (\theta_i),\ i = 1,2,\ \theta_i \in W^{1,\infty}(\Omega(0)), \\ \text{ii)} & \text{the support of } \theta \text{ being included in a neighbourhood of the shell } S(0), \\ \text{iii)} & \theta = \mathbf{z} \text{ on } S(0). \end{array}\right. \tag{28}$$

Let us now define the mapping F^θ from $\Omega(0)$ onto $\Omega(\mathbf{z})$ (deformed configuration):

$$x \in \Omega(0) \mapsto x^\theta = x + \theta(x) \in \Omega(\mathbf{z}). \tag{29}$$

- **Change of functions**: Let φ be a function defined on $\Omega(\mathbf{z})$. We set $\varphi^\theta(x) = \varphi o F^\theta(x)$.
- **Changes in the integrals**: $\displaystyle\int_{\Omega(\mathbf{z})} \varphi = \int_{\Omega(0)} \varphi^\theta \det(I + D\theta)$, where $D\theta$ is the Jacobian matrix associated to θ. Its transpose in the polar coordinate system $(\mathbf{e}_r, \mathbf{e}_\beta, \mathbf{e}_x)$ is $\nabla\theta$.
- **Changes in the derivatives**: $\displaystyle\left(\frac{\partial\varphi}{\partial x^\theta}\right)^\theta = \frac{\partial\varphi}{\partial x} o (I + D\theta)^{-1}$.
- **Divergence for a vector p**:

$$(\mathrm{div}(\mathbf{p}))^\theta = \frac{1}{\det(I + D\theta)} \mathrm{div}((I + D\theta)^{-1} \mathbf{p}^\theta \det(I + D\theta)).$$

- **Change in the convection term**: $(\mathbf{u} \otimes \nabla\mathbf{u})^\theta = \mathbf{u}^\theta \otimes (I + {}^tD\theta)^{-1} \nabla\mathbf{u}^\theta$.
- **Changes in the strain rates**:

$$(\gamma(\mathbf{u}))^\theta = \gamma^\theta(\mathbf{u}^\theta) = \frac{1}{2}\big((I + {}^tD\theta)^{-1} \nabla\mathbf{u}^\theta + {}^t\nabla\mathbf{u}^\theta (I + D\theta^{-1})\big).$$

These formulae enables one to formulate an equivalent flow problem but set on $\Omega(0)$.

4.2. Progressive Euler-Lagrange formulation

Using the mapping F^θ and setting $(\mathbf{u}^\theta, p^\theta) = (\mathbf{u}, p) o F^\theta$, we derive the following model:

$$\begin{cases} \text{Find } (\mathbf{u}^\theta, p^\theta) \in V \times L^2(\Omega(0)) \text{ such that:} \\ \forall \mathbf{v} \in V,\, \rho \displaystyle\int_{\Omega(0)} \Big[\frac{\partial \mathbf{u}^\theta}{\partial t}.\mathbf{v} + \mathbf{u}^\theta \otimes (I + {}^t D\theta)^{-1} \nabla\mathbf{u}^\theta.\mathbf{v}\Big] \det(I + D\theta) \\ \qquad - \displaystyle\int_{\Omega(0)} p^\theta \ \mathrm{div}\big((I + D\theta)^{-1} \mathbf{u}^\theta \det(I + D\theta)\big) \\ \qquad + \mu \displaystyle\int_{\Omega(0)} \big((I + {}^t D\theta)^{-1} \nabla\mathbf{u}^\theta + {}^t\nabla\mathbf{u}^\theta (I + D\theta^{-1})\big) \\ \qquad\qquad : (I + {}^t D\theta)^{-1} \nabla\mathbf{v} \det(I + D\theta) = 0, \\ \forall q \in L^2(\Omega(0)),\ - \displaystyle\int_{\Omega(0)} q \ \mathrm{div}\big((I + D\theta)^{-1} \mathbf{u}^\theta \det(I + D\theta)\big) = 0. \end{cases} \tag{30}$$

If $\theta = 0$, the obtained model is exactly the axisymmetrical one. Let us introduce a linearization with respect to θ, which is a linear function of $\mathbf{z}$. Let us set

$$(\mathbf{u}^\theta, p^\theta) = (\mathbf{u}^0, p^0) + (\mathbf{u}^1, p^1) + \cdots$$

and by introducing this approximation into (30) one obtains that $(\mathbf{u}^1, p^1)$ is solution of

$$
\begin{cases}
\text{find } (\mathbf{u}^{\mathbf{1}}, p^1) \in V \times L^2(\Omega(0)) \text{ such that:} \\
\forall \mathbf{v} \in V,\ \rho \displaystyle\int_{\Omega(0)} \frac{\partial \mathbf{u}^1}{\partial t}.\mathbf{v} + [\mathbf{u}^0 \otimes \nabla \mathbf{u}^1 + \mathbf{u}^1 \otimes \nabla \mathbf{u}^0].\mathbf{v} - \int_{\Omega(0)} p^1 \ \mathrm{div}(\mathbf{u}^1) \\
\qquad\qquad\qquad\qquad\qquad\qquad + \mu \displaystyle\int_{\Omega(0)} \gamma(\mathbf{u}^1) : \gamma(\mathbf{v}) \\
\qquad = -\rho \displaystyle\int_{\Omega(0)} \Big[\Big(\frac{\partial \mathbf{u}^0}{\partial t} + \mathbf{u}^0 \otimes \nabla \mathbf{u}^0 \Big) \mathrm{div}(\theta) - \mathbf{u}^0 . {}^t D\theta . \nabla \mathbf{u}^0 \Big] . \mathbf{v} \\
\qquad\quad + \displaystyle\int_{\Omega(0)} p^0 \, \mathrm{div}(\mathbf{u}^0) \, \mathrm{div}(\theta) - p^0 \, \mathrm{div}(D\theta . \mathbf{v}) \\
\qquad\quad - \mu \displaystyle\int_{\Omega(0)} 2\gamma(\mathbf{u}^0) : \gamma(\mathbf{v}) \, \mathrm{div}(\theta) - ({}^t D\theta . \nabla \mathbf{u}^0 + {}^t \nabla \mathbf{u}^0 . D\theta) . \nabla \mathbf{v} \\
\qquad\quad + 2\mu \displaystyle\int_{\Omega(0)} \gamma(\mathbf{u}^0) : {}^t D\theta . \nabla \mathbf{v}, \\
\forall q \in L^2(\Omega(0)),\ - \displaystyle\int_{\Omega(0)} q \ \mathrm{div}(\mathbf{u}^1) = \int_{\Omega(0)} q \, \mathrm{div}(\mathbf{u}^0) \, \mathrm{div}(\theta) - p^0 \, \mathrm{div}(D\theta . \mathbf{v}).
\end{cases}
\tag{31}
$$

4.3. Fourier decomposition

In order to simplify the three dimensional flow model we make use of a Fourier decomposition in β. The only harmonics which are different from zero are those which are contained in the structural displacement $\mathbf{z}$.

4.4. Kinematical continuity between the fluid and the structure

In the deformed configuration one has

$$\mathbf{u}(F^\theta(\mathbf{x},t),t) = \frac{\partial \mathbf{z}}{\partial t}(\mathbf{x},t),\ \forall \mathbf{x} \in S(0), \tag{32}$$

which is equivalent in $\Omega(0)$ to the following relation:

$$\mathbf{u}^\theta(\mathbf{x},t) = \frac{\partial \mathbf{z}}{\partial t}(\mathbf{x},t). \tag{33}$$

Let us consider for instance the normal component to the shell. The unit normal to the surface $S(\mathbf{z})$ is denoted by $\mathbf{N}'$ and we already point out that: $\mathbf{N}' = \mathbf{N} + \zeta(\mathbf{z})$, where ζ is the inplane rotation. Let us set $\mathbf{u} = \mathbf{u}^0 + \mathbf{u}^1$, and thus we derive the kinematical continuity condition:

$$(\mathbf{u}^1, \mathbf{N}) + (\zeta(\mathbf{z}), \mathbf{u}^0) = \Big(\frac{\partial \mathbf{z}}{\partial t}, \mathbf{N} \Big). \tag{34}$$

Let us point out that the time derivative of $\mathbf{u}^1$ appears in the model, and this implies the second order time derivative of $\mathbf{z}$ which acts an an added mass term.

The Fourier decomposition of $\zeta(\mathbf{z})$ only implies the harmonics contained in $\mathbf{z}$. The unit normal $\mathbf{N}$ can be written as follows: $\mathbf{N} = \cos(\kappa)\mathbf{e}_r + \sin(\kappa)\mathbf{e}_x$, where κ is the angle between $\mathbf{N}$ and $\mathbf{e}_r$. Hence, all the harmonics in β are decoupled.

§5. Forces due to fluid and applied to the structure

5.1. Forces due to the eigenmodes

On the surface $S(\mathbf{z})$ the stress vector is $\mathbf{T} = -p\mathbf{N}' + 2\mu\gamma(\mathbf{u}).\mathbf{N}'$. Using the mapping $\mathbf{T}^\theta$, this quantity becomes at order one on $S(0)$:

$$\mathbf{T}^\theta = -p^0\mathbf{N} + 2\mu\gamma(\mathbf{u}^0) - p^1\mathbf{N} - p^0\zeta(\mathbf{z}) + 2\mu\gamma(\mathbf{u}^1) - \mu({}^tD\theta.\nabla\mathbf{u}^0 +^t \nabla\mathbf{u}^0.D\theta). \qquad (35)$$

The two first terms correspond to the axisymmetrical flow. The four next ones are due to the dynamical behaviour. Let us assume that the reduced frequency is small enough in order to justify the use of the steady flow. In fact they are proportional to $\mathbf{z}$, $\partial\mathbf{z}/\partial t$ and $\partial^2\mathbf{z}/\partial t^2$. Let us set here again

$$\mathbf{z} = \sum_{n=1,N} \kappa_n(t)\mathbf{w}_n.$$

This enables one to write at order one:

$$\mathbf{T}^\theta = \mathbf{T}^0 + \sum_{n=1,N}\Big[\kappa_n(t)\mathbf{T}^{0z} + \frac{\partial\kappa_n}{\partial t}\mathbf{T}^{1z} + \frac{\partial^2\kappa_n}{\partial t^2}\mathbf{T}^{2z}\Big]. \qquad (36)$$

Hence, in order to compute the previous term, one has to solve $3N+1$ axisymmetrical and independent problems. For instance concerning the harmonic n one has the following expression to compute:

$$\mathscr{F}_n = \int_{S(0)}\Big[(\mathbf{T}^0,\mathbf{w}_n) + \kappa_n(t)(\mathbf{T}^{0z},\mathbf{w}_n) + \frac{\partial\kappa_n}{\partial t}(\mathbf{T}^{1z},\mathbf{w}_n) + \frac{\partial^2\kappa_n}{\partial t^2}(\mathbf{T}^{2z},\mathbf{w}_n)\Big]. \qquad (37)$$

Let us denote by $\mathscr{F}$ the vector in $\mathbb{R}^N$ the component of which are $\mathscr{F}_n$. Then,

$$\mathscr{F}_n = \mathbf{F}_n^0 + \kappa_n(t)\mathbf{F}_n^{0z} + \dot{\kappa}_n(t)\mathbf{F}_n^{1z} + \ddot{\kappa}_n(t)\mathbf{F}_n^{2z}. \qquad (38)$$

5.2. The aeroelastic model

Let us denote by $\mathbf{Z}$ the vector in $\mathbb{R}^N$ the component of which being ζ_n and which are the coefficients of the eigenvectors $\mathbf{w}_n$. Then,

$$M\frac{\partial^2\mathbf{Z}}{\partial t^2} + K\mathbf{Z} = \mathscr{F}\Big(\mathbf{Z},\frac{\partial\mathbf{Z}}{\partial t},\frac{\partial^2\mathbf{Z}}{\partial t^2}\Big). \qquad (39)$$

One should add initial conditions. Furthermore, the right hand side $\mathscr{F}$ depends on $\mathbf{Z}$ and its time derivatives. The matrices M and K are diagonal in the eigenvector basis. Let us set

$$\mathscr{F} = \mathscr{F}^0 - K^a\mathbf{Z} - C^a\frac{\partial\mathbf{Z}}{\partial t} - M^a\frac{\partial^2\mathbf{Z}}{\partial t^2}. \qquad (40)$$

The coupled system becomes, with already mentioned notations:

$$(M+M^a)\frac{\partial^2\mathbf{Z}}{\partial t^2} + C^a\frac{\partial\mathbf{Z}}{\partial t} + (K+K^a)\mathbf{Z} = \mathscr{F}^0. \qquad (41)$$

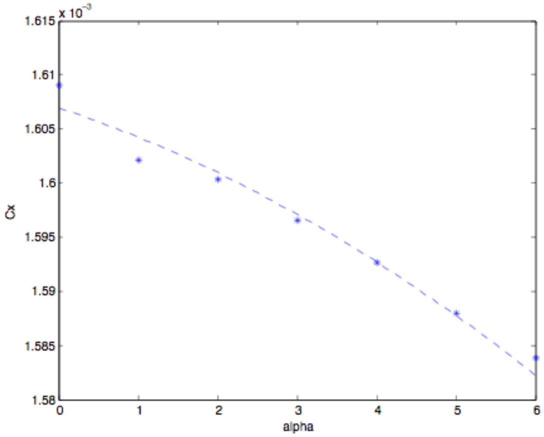

Figure 2: Drag coefficient with respect to the velocity

The aeroelastic study consists in computing the eigenvalues λ with respect to U:

$$\det(-\lambda^2(M+M^a)+i\lambda C^a+(K+K^a))=0. \tag{42}$$

An instability can occur if the imaginary part of λ is negative.

5.3. Discussion about the influence of the various terms appearing in the aeroelastic model and example

The effect of the added mass matrix is to reduce the eigenfrequencies. The even part of the matrix C is an aerodynamical damping. It can contribute to a so-called wake flutter. The odd part of C is the Corriolis effect and in most cases, stabilizes the system. The matrix $K+K^a$ is the augmented stiffness and is no more symmetrical because of the aerodynamical forces. A classical flutter instability can appear if two eigevalues are crossing each other. Let us give a simple example. It corresponds to a pitching or a galloping movement of the airship. The eigenvalues have been computed for several values of the angle of attack α and taking into account the aerodynamical forces due to $\dot{d}_z$ and $\dot{\alpha}$. Furthermore, the lift and the pitching moment coefficients have been computed (see figure 3). One can see on figure 3 left, that the airship is stable –from the static point of view– versus a pitching movement ($c_m \leq 0$). The aerodynamical centre is located in the front part of the airship. The drag coefficient has been plotted on figure 2. Even if it decrease, it is not meaning full beacause of the scale used. But concerning the aerodynamical damping it is quite zero for very small angle of attack. Then it is slightly negative for $\alpha \simeq 4$. But it becomes positive for larger value of α (see on the right figure 3).

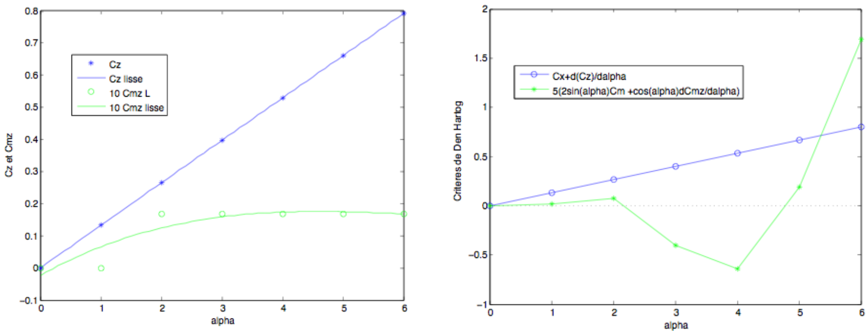

Figure 3: Left: lift and pitching moment at the center versus α. Right: imaginary part of λ for a pitching movement (green) and a galloping (blue)

§6. Conclusion

A simplified method for studying the aeroelasic stability of an axisymmetrical body is suggested. The method enables to take account the small perturbation with respect to the axis of symmetry in an aeroelastic analysis.

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