

THE DEFECT-CORRECTION TECHNIQUE APPLIED TO SINGULARLY PERTURBED ELLIPTIC PROBLEMS OF CONVECTION-DIFFUSION TYPE

C. Clavero, J. L. Gracia and E. O’Riordan

Abstract. In this work the defect–correction technique is used to solve some singularly perturbed elliptic boundary value problems of convection-diffusion type. In [3] it was proved that the use of standard defect–correction technique allows one improve the order of convergence of stable low order finite difference schemes. Nevertheless, the analysis of the uniform convergence is difficult for 1D problems and at the moment we do not know of any theoretical result proving uniform convergence for general 2D elliptic problems. In [4] a variant of this approach was proposed in order to simplify the analysis for 1D problems of convection-diffusion type, proving almost second order uniform convergence of the method. In this paper we show numerically that it is possible to extend the parameter-uniform method given in [4] to the case of a 2D elliptic boundary value problem.

Keywords: Singular perturbation, defect-correction, Shishkin mesh, second order.

AMS classification: 65M06, 65M12, 65L70, 65L12.

§1. Introduction

Singularly perturbed problems are characterised by the presence of a small parameter multiplying the highest order derivatives of the differential equation. The presence of the singular perturbation parameter results in the solution of these problems having a multiscale character. Narrow regions (boundary layers) appear close to the boundary where the solution has large gradients. So, it is desirable to design special numerical methods which generate numerical approximations converging to the exact solution independently of the value of the singular perturbation parameter. These methods are called uniformly convergent methods or parameter uniform methods.

In [2] some parameter uniform fitted mesh methods were presented for problems of this kind, using (almost) exclusively a special class of meshes introduced by Shishkin [9]. These meshes are simple to construct, are piecewise uniform and condense half the mesh points into the boundary layer regions. In the literature there are many first order finite difference schemes used to approximate efficiently a wide class of 1D and 2D singularly perturbed problems. Moreover, it is possible to apply some classical techniques to improve the uniform

order of convergence in the 1D case. Nevertheless, at this moment, for convection–diffusion elliptic problems given by

$$L_\varepsilon u \equiv \varepsilon \Delta u + a \cdot \nabla u - bu = f, \quad (x, y) \in \Omega = (0, 1)^2, \quad u = g \text{ on } \partial\Omega, \quad (1)$$

we only know [6], where the author proves almost second order uniform convergence in the maximum norm. In that paper, the Richardson extrapolation technique was applied to the upwind operator defined on a special Shishkin mesh. It is well known that the Richardson extrapolation technique requires one solve two discrete problems. The basic discrete problem associated with the original mesh and a second discrete problem associated with a new mesh composed of the mesh points of the original mesh and their midpoints. Note also that the analysis of the uniform convergence of this technique employs an asymptotic expansion of the error in powers of the discretization parameter.

This paper is motivated by our interest in high order parameter uniform finite difference schemes. Here, we consider the defect–correction technique to achieve almost second order uniform convergence. This technique was applied in [3] for 1D problems, where the upwind operator (stable) and the central difference operator (unstable) were combined; unfortunately, the analysis of the uniform convergence, proposed in that work, is complicated and it appears difficult to extend to 2D elliptic problems. In [4] a new basic stable scheme was proposed, simplifying the analysis of the uniform convergence in the case of one dimensional problems.

The paper is structured as follows. In §2 we describe this new method and in §3 we show that it can be extended to the 2D case. The numerical results suggest almost second order uniform convergence of the numerical method. Henceforth, C denotes a positive constant independent of the singular perturbation parameter ε and the discretization parameter N .

§2. Defect correction technique for 1D problems

In this section we consider 1D boundary value problems of convection–diffusion type given by

$$L_\varepsilon u \equiv \varepsilon u'' + au' - bu = f, \quad x \in \Omega = (0, 1), \quad u(0) = u_0, \quad u(1) = u_1, \quad (2)$$

where we assume that the singular perturbation parameter can take arbitrary small positive values, $0 < \varepsilon \leq 1$, the coefficients $a, b, f \in C^5(0, 1)$ and also that $a \geq \alpha > 0, b \geq 0, \forall x \in [0, 1]$. The solution of (2) has a boundary layer at $x = 0$ with a width $\mathcal{O}(\varepsilon \ln(1/\varepsilon))$; moreover (see [2]) its derivatives satisfy the bounds

$$|u^{(k)}(x)| \leq C(1 + \varepsilon^{-k} \exp(-\alpha x/\varepsilon)), \quad 0 \leq k \leq 4. \quad (3)$$

To prove uniform convergence of the numerical scheme, it is convenient to use an appropriate decomposition of the exact solution u . Then, following [2], we write $u = v + w$, where v and w are the regular and singular components of the exact solution respectively. They are the solutions of the following boundary value problems:

$$L_\varepsilon v = f, \quad v(0) = v^*(0), \quad v(1) = u(1), \quad L_\varepsilon w = 0, \quad w(0) = u(0) - v^*(0), \quad w(1) = 0,$$

where $v^*(0)$ is taken so that

$$|v^{(j)}(x; \varepsilon)| \leq C, \quad 0 \leq j \leq 3, \quad \varepsilon |v^{(4)}(x; \varepsilon)| \leq C, \quad (4)$$

$$|w(x; \varepsilon)| \leq C \exp(-\alpha x / \varepsilon), \quad |w^{(j)}(x; \varepsilon)| \leq C \varepsilon^{-j}, \quad 1 \leq j \leq 4. \quad (5)$$

To approximate the solution of (2), we consider a finite difference scheme defined on a Shishkin mesh. Let N be the discretization parameter; then, the Shishkin mesh is defined by using the transition parameter

$$\sigma = \min \{1/2, (3/\alpha)\varepsilon \ln N\}, \quad (6)$$

and dividing uniformly each one of the subdomains $[0, \sigma]$, $[\sigma, 1]$ into $N/2$ intervals. Then, the set of mesh points $\bar{\Omega}_\varepsilon^N$ is given by

$$x_j = \begin{cases} jh, & \text{if } 0 \leq j \leq N/2, \\ \sigma + (j - N/2)H, & \text{if } N/2 \leq j \leq N, \end{cases} \quad (7)$$

where $h = 2\sigma/N$ is the fine mesh step and $H = 2(1 - \sigma)/N$ is the coarse mesh step. Below, we denote by $h_{j+1} = x_{j+1} - x_j$ for $j = 0, \dots, N - 1$, and $\tilde{h}_j = (h_j + h_{j+1})/2$ for $j = 1, \dots, N - 1$. On this piecewise uniform mesh, we consider the following hybrid three point finite difference scheme:

$$L^N U_j \equiv r_j^- U_{j-1} + r_j^c U_j + r_j^+ U_{j+1} = f_j, \quad 0 < j < N, \quad U_0 = u_0, \quad U_N = u_1, \quad (8)$$

where the discrete operator is defined by

$$L^N \equiv \begin{cases} L_{cd*}^N, & \text{if } 1 \leq j < N/2, \\ L_{cd*}^N, & \text{if } N/2 \leq j < N \text{ and } \|a\|N^{-1} < \varepsilon, \\ L_{up}^N, & \text{if } N/2 \leq j < N \text{ and } \|a\|N^{-1} \geq \varepsilon. \end{cases}$$

The central difference operator L_{cd*}^N (slightly modified at the transition point) and the upwind operator L_{up}^N are given by

$$\begin{aligned} L_{cd*}^N U_j &\equiv \varepsilon \delta^2 U_j + a_j D^\pm U_j - b_j U_j = f_j, \\ L_{up}^N U_j &\equiv \varepsilon \delta^2 U_j + a_j D^+ U_j - b_j U_j = f_j, \end{aligned}$$

where

$$\begin{aligned} \delta^2 U_j &= \frac{1}{\tilde{h}_j} \left(\frac{U_{j+1} - U_j}{h_{j+1}} - \frac{U_j - U_{j-1}}{h_j} \right), \quad D^+ U_j = \frac{U_{j+1} - U_j}{h_{j+1}}, \\ D^- U_j &= \frac{U_j - U_{j-1}}{h_j}, \quad D^\pm U_j = \frac{h_j}{h_j + h_{j+1}} D^+ U_j + \frac{h_{j+1}}{h_j + h_{j+1}} D^- U_j. \end{aligned}$$

Theorem 1 (See [4]). *The error associated with the hybrid scheme (8) satisfies*

$$|u(x_j) - U_j| \leq \begin{cases} CN^{-1}, & \text{if } \|a\|N^{-1} \geq \varepsilon, \\ CN^{-2}(\ln N)^3, & \text{if } \|a\|N^{-1} < \varepsilon, \end{cases} \quad (9)$$

and therefore it is a first order uniformly convergent method.

To improve this order of uniform convergence, we apply the defect–correction technique. Note that the use of the hybrid scheme instead of the upwind scheme allows one prove almost second order in the case $\|a\|N^{-1} < \varepsilon$, and therefore in this case it is not necessary to modify the numerical solution given by scheme (8). Otherwise, it is necessary to correct the numerical solution in an appropriate way to obtain second order convergence rates. In [4] the following approximation to the solution of problem (2) was proposed

$$\begin{aligned} \widehat{U} &= U + \vartheta, & \text{if } \|a\|N^{-1} \geq \varepsilon, \\ \widehat{U} &= U, & \text{otherwise.} \end{aligned} \tag{10}$$

Here ϑ is the solution of the discrete problem

$$L^N \vartheta = f - L_{cd}^N U \text{ in } \Omega, \quad \vartheta = 0 \text{ in } \partial\Omega, \tag{11}$$

where L_{cd}^N is the classical central difference operator, which is defined by

$$L_{cd}^N Z_j \equiv \begin{cases} L_{cd}^N Z_j, & \text{if } j \neq N/2, \\ \varepsilon \delta^2 Z_j + a_j D^0 Z_j - b_j Z_j, & \text{if } j = N/2, \end{cases}$$

and $D^0 Z_j = (Z_{j+1} - Z_{j-1}) / (h_{j+1} + h_j)$. In [4] it was showed that if D^\pm is used instead of D^0 at the transition point to find the corrected solution, then the resulting numerical method is not parameter-uniformly convergent. The numerical solution of scheme (10) satisfies the following convergence result.

Theorem 2 (See [4]). *The error associated with the defect–correction scheme satisfies*

$$\|u - \widehat{U}\| \leq CN^{-2} \ln^3 N,$$

where u is the solution of the problem (2) and \widehat{U} is the numerical approximation given in (10). Therefore, the new method is almost second order uniformly convergent.

To corroborate the theoretical result of convergence proved in Theorem 2, we consider the following test problem:

$$\varepsilon u'' + u' = -e^x(1 + \varepsilon), \quad x \in (0, 1), \quad u(0) = 1, \quad u(1) = 1 - e, \tag{12}$$

whose exact solution is $u(x) = (e^{-x/\varepsilon} - e^{-1/\varepsilon}) / (1 - e^{-1/\varepsilon}) + 1 - e^x$. Then, for any value of N , the maximum pointwise errors $E_{\varepsilon,N}$ and the ε -uniform errors are calculated by $E_{\varepsilon,N} = \max_i |u(x_i) - \widehat{U}_i|$, $E_N = \max_\varepsilon E_{\varepsilon,N}$ respectively, where u is the exact solution of (12) and \widehat{U} is the numerical solution of the finite difference scheme (10). From these values the orders of convergence $p_{\varepsilon,N}$ and the order of ε -uniform convergence p_N are calculated by $p_{\varepsilon,N} = \log(E_{\varepsilon,N} / E_{\varepsilon,2N}) / \log 2$, $p_N = \log(E_N / E_{2N}) / \log 2$.

Table 1 displays, for problem (12), the ε -uniform maximum errors and the ε -uniform orders of convergence for the range of values $\varepsilon = 1, 2^{-2}, 2^{-4}, \dots, 2^{-30}$. From these results we deduce the almost second order uniform convergence in the maximum norm, which is in agreement with the bound given in Theorem 2.

	N=32	N=64	N=128	N=256	N=512	N=1024	N=2048
E_N	1.641E-2	5.483E-3	1.781E-3	5.662E-4	1.758E-4	5.353E-5	1.604E-5
p_N	1.581	1.622	1.653	1.688	1.715	1.739	

Table 1: Defect–correction technique: Numerical results for problem (12)

§3. Defect correction technique for 2D problems

The aim of this section is to show that the ideas developed for the 1D case can be extended to elliptic boundary value problems given by

$$L_\epsilon u \equiv \epsilon \Delta u + a \cdot \nabla u - bu = f, \quad (x, y) \in \Omega = (0, 1)^2, \quad u = g \text{ on } \partial\Omega, \quad (13)$$

where the coefficients of the differential equation a_1, a_2, b and f are sufficiently smooth and they satisfy $a = (a_1(x), a_2(y)) \geq (\alpha_1, \alpha_2) > (0, 0)$, $b(x, y) \geq 0$, in $\bar{\Omega}$. Besides, we suppose that sufficient compatibility conditions hold in order that $u \in C^{4,\alpha}(\bar{\Omega})$ (space of Hölder continuous functions whose derivatives up to fourth order exist and they are Hölder continuous). To achieve this regularity, compatibility conditions up to second level are necessary (see [5]).

Using an appropriate change of variable and a classical analysis (see [7]), we can deduce the following crude bounds for the exact solution and its partial derivatives in the maximum norm

$$\|u^{(i,j)}\| \leq C\epsilon^{-(i+j)}, \quad 0 \leq i + j \leq 4.$$

Also, it is well-known (see [2]) that the exact solution of (13) has two regular boundary layers near the sides $x = 0$ and $y = 0$ of width $\mathcal{O}(\epsilon \ln 1/\epsilon)$.

To approximate the solution of (13), again we consider a piecewise uniform mesh $\bar{\Omega}^N = \{(x_i, y_j)\}_{i,j=0}^N$, which is the tensor product of the corresponding Shishkin mesh considered for the 1D problem. Then, the mesh points are

$$x_i = \begin{cases} ih, & 0 \leq i \leq N/2, \\ x_{N/2} + (i - N/2)H, & N/2 < i \leq N, \end{cases}$$

$$y_j = \begin{cases} jk, & 0 \leq j \leq N/2, \\ y_{N/2} + (j - N/2)K, & N/2 < j \leq N, \end{cases}$$

where $h = 2\sigma_x/N$, $k = 2\sigma_y/N$, $H = 2(1 - \sigma_x)/N$, $K = 2(1 - \sigma_y)/N$ and the transition parameters are defined by

$$\sigma_x = \min \{1/2, (3/\alpha_1)\epsilon \ln N\}, \quad \sigma_y = \min \{1/2, (3/\alpha_2)\epsilon \ln N\}.$$

We denote the local step sizes by

$$h_j = x_j - x_{j-1}, \quad k_j = y_j - y_{j-1}, \quad j = 1, \dots, N,$$

$$\bar{h}_j = (h_j + h_{j+1})/2, \quad \bar{k}_j = (k_j + k_{j+1})/2, \quad j = 1, \dots, N - 1.$$

On this mesh we define the following hybrid scheme

$$\begin{cases} L^N U_{i,j} \equiv \varepsilon(\delta_x^2 + \delta_y^2)U_{i,j} + a_1(x_i)D_x U_{i,j} + a_2(y_j)D_y U_{i,j} \\ \quad - b(x_i, y_j)U_{i,j} = f(x_i, y_j) \text{ in } \Omega^N, \\ U_{i,j} = g(x_i, y_j) \text{ on } \partial\Omega^N, \end{cases} \quad (14)$$

where

$$\begin{aligned} \delta_x^2 U_{i,j} &\equiv \frac{1}{h_i}(D_x^+ U_{i,j} - D_x^- U_{i,j}), & \delta_y^2 U_{i,j} &\equiv \frac{1}{k_j}(D_y^+ U_{i,j} - D_y^- U_{i,j}), \\ D_x U_{i,j} &\equiv \begin{cases} D_x^+ U_{i,j}, & \text{if } \varepsilon \leq \|a_1\|N^{-1}, \\ D_x^\pm U_{i,j}, & \text{if } \|a_1\|N^{-1} < \varepsilon \text{ or } x_i < \sigma_x, \end{cases} \\ D_y U_{i,j} &\equiv \begin{cases} D_y^+ U_{i,j}, & \text{if } \varepsilon \leq \|a_2\|N^{-1}, \\ D_y^\pm U_{i,j}, & \text{if } \|a_2\|N^{-1} < \varepsilon \text{ or } y_j < \sigma_y, \end{cases} \end{aligned}$$

with

$$\begin{aligned} D_x^\pm U_{i,j} &\equiv \frac{h_i}{h_i + h_{i+1}}D_x^+ U_{i,j} + \frac{h_{i+1}}{h_i + h_{i+1}}D_x^- U_{i,j}, \\ D_y^\pm U_{i,j} &\equiv \frac{k_j}{k_j + k_{j+1}}D_y^+ U_{i,j} + \frac{k_{j+1}}{k_j + k_{j+1}}D_y^- U_{i,j}, \\ D_x^+ &= \frac{U_{i+1,j} - U_{i,j}}{h_{i+1}}, & D_x^- &= \frac{U_{i,j} - U_{i-1,j}}{h_i}, \\ D_y^+ &= \frac{U_{i,j+1} - U_{i,j}}{k_{j+1}}, & D_y^- &= \frac{U_{i,j} - U_{i,j-1}}{k_j}. \end{aligned}$$

Similarly to [8], it is not difficult to prove that the error associated to the hybrid scheme (14) satisfies $\|u - U\| \leq CN^{-1}$, proving first order of uniform convergence. To improve this order, similarly to the 1D problems, we propose the following defect–correction scheme

$$\begin{aligned} \widehat{U} &= U + \vartheta, & \text{if } \max\{\|a_1\|, \|a_2\|\}N^{-1} \geq \varepsilon, \\ \widehat{U} &= U, & \text{otherwise.} \end{aligned} \quad (15)$$

where now ϑ is the solution of the discrete problem

$$L^N \vartheta = f - L_{cd}^N U \text{ in } \Omega, \quad \vartheta = 0 \text{ on } \partial\Omega, \quad (16)$$

with

$$\begin{aligned} L_{cd}^N U_{i,j} &\equiv \varepsilon(\delta_x^2 U_{i,j} + \delta_y^2 U_{i,j}) + a_1(x_i)D_x^0 U_{i,j} + a_2(y_j)D_y^0 U_{i,j} - b(x_i, y_j)U_{i,j}, \\ D_x^0 U_{i,j} &= \frac{U_{i+1,j} - U_{i-1,j}}{h_{i+1} + h_i}, & D_y^0 U_{i,j} &= \frac{U_{i,j+1} - U_{i,j-1}}{k_{j+1} + k_j}. \end{aligned}$$

To illustrate numerically the uniform convergence of this defect–correction method, we show the numerical results obtained for two test problems, whose exact solution in both cases

	N=16	N=32	N=64	N=128	N=256
D_N	0.310E-1	0.127E-1	0.485E-2	0.329E-2	0.228E-2
q_N	1.288	1.388	0.561	0.528	

Table 2: Defect–correction technique: Numerical results for problem (19)

is unknown. To estimate the pointwise errors of the solution $\{\widehat{U}^N\}$, we use a variant of the double mesh principle [1]; then, we calculate a new approximation $\{\widehat{V}^{2N}\}$ on the mesh $\widehat{\Omega}^{2N} = \{(\tilde{x}_i, \tilde{y}_j)\}$ that uses the mesh points of the original Shishkin mesh and their midpoints, i.e.,

$$\tilde{x}_{2i} = x_i, \quad i = 0, \dots, N, \quad \tilde{x}_{2i+1} = (x_i + x_{i+1})/2, \quad i = 0, \dots, N-1, \quad (17)$$

$$\tilde{y}_{2j} = y_j, \quad j = 0, \dots, N, \quad \tilde{y}_{2j+1} = (y_j + y_{j+1})/2, \quad j = 0, \dots, N-1. \quad (18)$$

In this way, we compare both numerical solutions at the mesh points of the coarse mesh, i.e., we calculate $D_{\varepsilon,N}(x_i, y_j) = |\widehat{U}_{i,j}^N - \widehat{V}_{2i,2j}^{2N}|$. For each fixed value of ε , the maximum errors and the numerical orders of convergence are computed by $D_{\varepsilon,N} = \max_{i,j} D_{\varepsilon,N}(x_i, y_j)$ and $q_{\varepsilon,N} = \log(D_{\varepsilon,N}/D_{\varepsilon,2N})/\log 2$ respectively. From these values we calculate the ε -uniform errors and the ε -uniform orders of convergence $D_N = \max_{\varepsilon} D_{\varepsilon,N}$ and $q_N = \log(D_N/D_{2N})/\log 2$.

The first test problem is given by

$$\varepsilon \Delta u + (1+x)u_x + (2+y)u_y - u = (1-x) + (1-y) \quad \text{in } \Omega = (0,1)^2, \quad u = 1 \quad \text{on } \partial\Omega. \quad (19)$$

The data of this problem do not satisfy sufficient compatibility conditions in order that the exact solution has the required regularity (see [5]). Concretely, we only have compatibility conditions of level zero, that is, the boundary conditions are continuous in the four corners of the unit square. Table 2 displays the numerical results for the same range of values of ε as before, giving in the first row the ε -uniform errors and in the second one the corresponding uniform orders of convergence. From this table we clearly see that we do not have a second order rate of uniform convergence.

The second test problem is given by

$$\varepsilon \Delta u + (1+x)u_x + (2+y)u_y = (1-x) + (1-y) \quad \text{in } \Omega = (0,1)^2, \quad u = 0 \quad \text{on } \partial\Omega. \quad (20)$$

Now, we again have compatibility conditions of level zero at three corners of the unit square, and we can check that in the inflow corner (1,1) the first level of compatibility is satisfied. Table 3 displays that the scheme (15) is almost second order uniformly convergent. Therefore, we can conjecture that the defect correction technique improves the order of convergence of the basic hybrid scheme when a sufficient level of compatibility holds.

Acknowledgements

This research has been partially supported by the project MEC/FEDER MTM2004-01905, the Diputación General de Aragón and a grant EUROPA XXI of Caja de Ahorros de la Inmaculada.

	N=16	N=32	N=64	N=128	N=256
D_N	0.225E-1	0.827E-2	0.283E-2	0.925E-3	0.298E-3
q_N	1.445	1.547	1.612	1.633	

Table 3: Defect–correction technique: Numerical results for problem (20)

References

- [1] FARRELL, P. A., AND HEGARTY, A. F. On the determination of the order of uniform convergence. In *Proceedings of the 13th IMACS World Congress for Computation and Applied Mathematics*, R. Vichnevetsky and J. J. H. Miller, Eds. IMACS, 1991, pp. 501–502.
- [2] FARRELL, P. A., HEGARTY, A. F., MILLER, J. J. H., O’RIORDAN, E., AND SHISHKIN, G. I. *Robust Computational Techniques for Boundary Layers*. Chapman and Hall/CRC, Boca Raton, U.S.A., 2000.
- [3] FROHNER, A., LINSS, T., AND ROOS, H.-G. Defect correction on Shishkin-type meshes. *Numerical Algorithms* 26 (2001), 281–299.
- [4] GRACIA, J. L., AND O’RIORDAN, E. A defect–correction parameter-uniform numerical method for a singularly perturbed convection-diffusion problem in one dimension. *Numerical Algorithms* 41 (2006), 359–385.
- [5] HAN, H., AND KELLOGG, R. B. Differentiability properties of solutions of the equation $-\varepsilon^2 \Delta u + ru = f(x, y)$ in a square. *SIAM J. Math. Anal.* 21 (1990), 394–408.
- [6] KOPTEVA, N. Error expansion for an upwind scheme applied to a two–dimensional convection–diffusion problem. *SIAM J. Numer. Anal.* 5 (2003), 1851–1869.
- [7] LADYZHENSKAYA, O. A., AND URAL’TSEVA, N. N. *Linear and Quasilinear Elliptic Equations*. Academic Press, New York and London, 1968.
- [8] LINSS, T., AND STYNES, M. A hybrid difference scheme on a Shishkin mesh for linear convection–diffusion problems. *Appl. Numer. Math.* 31 (1999), 255–270.
- [9] SHISHKIN, G. I. *Discrete Approximation of Singularly Perturbed Elliptic and Parabolic Equations*. Russian Academy of Sciences, Ural section, Ekaterinburg, 1992. (In Russian)

C. Clavero and J. L. Gracia
 Department of Matemática Aplicada
 University of Zaragoza
 clavero@unizar.es and jlgracia@unizar.es

E. O’Riordan
 School of Mathematical Sciences
 Dublin City University
 eugene.oriordan@dcu.ie