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THE GENDARME ALGORITHM FOR ADAPTIVE MESH REFINEMENT

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Abstract. We present an adaptive two-level finite element method. The main idea of the algorithm is to guide the local mesh refinement of the coarser mesh by a finer mesh called gendarme mesh. Two different estimators are involved in the algorithm: a standard a posteriori error estimator of residual type is used to control the error in the gendarme mesh; a simpler estimator based on the difference of the solutions corresponding to the two meshes is employed to guide the local mesh refinement.

Keywords: Adaptive finite elements, convergence of adaptive algorithms

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§1. Introduction

The analysis of adaptive finite element methods has made important progress in recent years. Based on classical residual-based a posteriori error estimators [1, 5, 8] it has been shown by Dörfler [4] and Morin, Nochetto, and Siebert [6] that an adaptive mesh refinement algorithm converges towards the solution of the model problem. The model problem is the homogenuous Poisson equation in a two-dimensional bounded domain Ω with piecewise linear boundary $\partial\Omega$:

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1}$$

In [6] the importance of controlling oscillations in data not captured on a given finite element mesh is pointed out. In addition, the role of inner nodes - the existence of which is ensured by the newest vertex bisection algorithm - is accounted for.

An important further result is the estimation of the dimension of the adaptively constructed discrete spaces, first achieved for wavelet discretizations by Cohen, Dahmen, and DeVore [3]. This result is extended to a modified version of the algorithm of [6], including an additional coarsening step, by Binev, Dahmen, and DeVore in [2]. A further significant improvement has been achieved by Stevenson [7] who shows that the additional coarsening step is not necessary in order to prove optimal complexity.

The importance of the last-mentioned results lays in the fact that they show optimal complexity of certain adaptive algorithms: if the solution of (1) can be approximated by a given adaptive method at a certain rate (quotient of accuracy to number of unknowns), the iteratively constructed sequence of meshes will realize this rate up to a constant factor.

The purpose of this contribution is to define an algorithm based on successive solution of the problem on two nested spaces; such type of algorithms are widely used for numerical integration and solution of differential equations.

A simple criterion based on the difference of these solutions is used for local mesh refinement of the coarse mesh. We call this algorithm *Gendarme algorithm*. The techniques used here are based on the work reported on in the above mentioned articles, where lowest-order conforming finite elements on triangular meshes based on newest-vertex-bisection are considered.

1.1. Notation

Let $V \subset H_0^1(\Omega)$ denote a finite element space. We define the Ritz-projection $u_V = R_V u$ by the variational equation: find $u_V \in V$ such that

$$\int_\Omega \nabla u_V \cdot \nabla v \, dx = \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \forall v \in V. \tag{2}$$

Then the finite element approximation to the solution of (1) in space V is simply given by $R_V u$. In order to simplify notations we will generally suppress the traditional subscript h in the notation of finite element spaces ($V = V_h$), and write u_V instead of u_h for the Ritz projection.

We use the following notation for the $H^1(\Omega)$-semi-norm:

$$\|\nabla u\| := \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}, \tag{3}$$

and consider it as the norm on $H_0^1(\Omega)$. The dual space of $H_0^1(\Omega)$ is $H^{-1}(\Omega)$ and we recall that for $f \in H^{-1}(\Omega)$

$$\|f\|_{-1} = \sup_{v \in V \setminus \{0\}} \frac{(f,v)}{\|\nabla v\|}$$

denotes the dual norm.

§2. Motivation

In this section we give a motivation for our investigation, postponing precise definitions to the following sections. Suppose we are given two nested finite element spaces obtained by global refinement (which means that each cell in the underlying mesh has been refined)

$$\widehat{V}_k \subset V_k.$$

To start with we consider the following optimization problem: Find a space $\widehat{V}_{k+1}$ in between the given spaces,

$$\widehat{V}_k \subset \widehat{V}_{k+1} \subset V_k, \tag{4}$$

such that we are able to guarantee a certain error reduction with optimal complexity. To be more precise we define for given θ, $0 < \theta < 1$, the set of spaces

$$\mathscr{V}_k := \left\{ \widehat{V}_k \subset V \subset V_k : \ \left\| \nabla u_V - u_{V_k} \right\|^2 \le \theta \left\| \nabla u_{\widehat{V}_k} - u_{V_k} \right\|^2 \right\}, \tag{5}$$

and choose $\widehat{V}_{k+1} \in \mathscr{V}_k$ as a solution to the optimization problem

$$\inf_{V \in \mathscr{V}_k} \dim(V). \tag{6}$$

This is a finite-dimensional optimization problem which is for the following reasons unpractical to solve. The first difficulty associated with (6) is the evaluation of the error, which requires repeated solution of the Poisson equation. It turns out that a similar problem can be solved efficiently, provided we replace in the definition of $\mathscr{V}_k$ (5) the reduction of the error by the reduction of an hierarchical error estimator μ. The second difficulty is that solving (6) is not an advisable strategy, if the dominant error is not captured in V_k. In order to check if this is the case we use a standard residual error estimator η which is supposed to yield an upper bound of the error,

$$\left\|\nabla u - u_{V_k}\right\|^2 \leq \eta(u_{V_k}), \tag{7}$$

and a lower bound in the following sense: if the space V contains the space obtained from V_k by global refinement, we have

$$\eta(u_{V_k}) \leq C_\eta \left\|\nabla u_V + u_{V_k}\right\|^2 - \operatorname{osc}(V_k), \tag{8}$$

where $\operatorname{osc}(V_k)$ is a oscillation term similar to the one in [6]. We propose to check if for a fixed $\alpha > 0$ the following inequality is satisfied:

$$\left\|\nabla u_{\widehat{V}_k} - u_{V_k}\right\|^2 \geq \alpha\, \eta(u_{\widehat{V}_k}). \tag{9}$$

If (9) holds we consider the afore mentioned optimization problem, otherwise it turns out to be appropriate to refine the data oscillations.

This leads as to the following algorithm which we term *gendarme algorithm*:

(0) Choose parameters α, θ, σ. Let $\widehat{V}_0$ be a finite element space and set $k = 0$.

(1) Perform a global refinement to obtain V_k and compute the corresponding solutions $\widehat{u}_k := u_{\widehat{V}_k}$ and $u_k := u_{V_k}$.

(2) If $\|\nabla u_k - \widehat{u}_k\|^2 \geq \alpha\, \eta(\widehat{u}_k)$ define

$$\mathscr{V}_k = \left\{ \widehat{V}_k \subset V \subset V_k : \; \mu(V, V_k) \leq \theta\, \mu(\widehat{V}_k, V_k) \right\}, \tag{10}$$

otherwise define

$$\mathscr{V}_k = \left\{ \widehat{V}_k \subset V : \; \operatorname{osc}(V) \leq \sigma \operatorname{osc}(\widehat{V}_k) \right\}. \tag{11}$$

(3) Define $\widehat{V}_{k+1}$ as a solution to the optimization problem $\inf_{V \in \mathscr{V}_k} \dim(V)$ and go to (1).

The objective of this article is to analyze the above algorithm in case of conforming finite element spaces on quadrilateral meshes with local refinement based on isotropic division of cells allowing hanging nodes. A simple rule to avoid multiple layers of hanging nodes is given below.

In order to elaborate the structure of the algorithm, we first formulate the gendarme algorithm in an abstract setting. The results are then applied to the concrete situation described above.

§3. The gendarme algorithm

We consider the following ingredients:

i) A starting mesh T_0 which defines a tree $\tau(T_0)$ containing all possible locally refined meshes.

ii) The set of all *admissible* meshes $\mathscr{T}$ obtained from the tree. For $T \in \mathscr{T}$ we write $N(T) = \#T$.

iii) A global refinement algorithm $\mathscr{R}_{glob}(T) \in \mathscr{T}$.

iv) A local refinement algorithm $\mathscr{R}_{loc}(T, \mathscr{K}) \in \mathscr{T}$ taking as additional argument a subset $\mathscr{K} \subset T$ of cells to be refined ('marked cells').

v) An error function $\phi(T', T)$, error estimators $\eta(T)$, $\mathrm{osc}(T)$ and $\mu(T', T)$, the last taking as additional argument $T' \subset T$. All error functionals are supposed to have positive values.

We define the set of meshes having at most $N \in \mathbb{N}$ elements:

$$\mathscr{T}_N := \{T \in \mathscr{T} : N(T) \leq N\},$$

and the error associated to T by

$$\phi(T) := \sup_{T' \supset T} \phi(T, T').$$

We make the following assumptions: there exist constants C_ϕ, c_ϕ, C_{osc}, c_μ, and C_μ such that

(H1) $\phi(T_1, T_3) = \phi(T_1, T_2) + \phi(T_2, T_3)$ for all $T_1 \subset T_2 \subset T_3$.

(H2) $\phi(T_1, T_1 \cup T_2) = \phi(T_1 \cap T_2, T_2)$ for all $T_1, T_2 \in \mathscr{T}$.

(H3) $\phi(T) \leq \eta(T)$ for all $T \in \mathscr{T}$.

(H4) $\eta(T) \leq C_\eta(\phi(T, T') + \mathrm{osc}(T))$ if $T' \supset \mathscr{R}_{glob}(T)$.

(H5) $\mathrm{osc}(T) \leq C_{osc}\phi(T)$ for all $T \in \mathscr{T}$.

(H6) $c_\mu\, \phi(T', \mathscr{R}_{glob}(T)) \leq \mu(T', \mathscr{R}_{glob}(T)) \leq C_\mu\, \phi(T', \mathscr{R}_{glob}(T))$ for all $T \subset T' \subset \mathscr{R}_{glob}(T)$.

In addition, we make the assumption that the error can be decreased at a certain rate $s > 0$:

$$\sup_{N \in \mathbb{N}} N^{-s} \inf_{T \in \mathscr{T}_N} \phi(T) < +\infty. \tag{H7}$$

Before proceeding any further, we make some simple conclusions of our assumptions. First we note that (H1) and the definition of $\phi(T)$ imply that

$$T_1 \subset T_2 \quad \Longrightarrow \quad \phi(T_1) \leq \phi(T_2). \tag{12}$$

Similarly, we find that

$$T_1 \subset T_2 \quad \Longrightarrow \quad \phi(T_1) = \phi(T_2) + \phi(T_1, T_2). \tag{13}$$

The gendarme algorithm reads:

(0) Choose parameters α, θ, σ. Set $\widehat{T}_1 := T_0 \in \mathscr{T}$ and set $k = 1$.

(1) Set $T_k := \mathscr{R}_{glob}(\widehat{T}_k)$.

(2) If $\phi(T_k, \widehat{T}_k) \geq \alpha\, \eta(\widehat{T}_k)$ define

$$\mathscr{T}_k = \left\{ T \in \mathscr{T} :\ \widehat{T}_k \subset T \subset T_k \text{ and } \mu(T, T_k) \leq \theta\, \mu(\widehat{T}_k, T_k) \right\}, \tag{14}$$

otherwise define

$$\mathscr{T}_k = \left\{ T \in \mathscr{T} :\ \widehat{T}_k \subset T :\ \operatorname{osc}(T) \leq \sigma \operatorname{osc}(\widehat{T}_k) \right\}. \tag{15}$$

(3) Define $\widehat{T}_{k+1}$ as a solution to the optimization problem $\inf_{T \in \mathscr{T}_k} N(T)$ and go to (1).

Our convergence result for this algorithm is expressed in the following theorem.

Theorem 1. *Suppose that (H1)-(H7) are fulfilled. Let the constants α, θ, σ be chosen such that*

$$0 < \alpha < 1/(2C_\eta), \quad \theta < c_\mu/(2C_\mu), \quad \sigma < 1/(4C_\eta C_{osc}). \tag{16}$$

Then, the sequence of meshes $(T_k)_k$ generated by the gendarme algorithm has the following properties. There exist constants C and $\rho < 1$ such that

$$\phi(T_{k+1}) \leq C\rho^k, \qquad k = 1, 2, \ldots. \tag{17}$$

In addition there exists another constant C such that for any $\varepsilon > 0$ we have the following implication:

$$\phi(T_k) \leq \varepsilon \quad \Longrightarrow \quad N(T_k) \leq C\varepsilon^{-1/s}. \tag{18}$$

A proof of this theorem will be presented elsewhere.

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