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# SEQUENCES OF CONTRACTIONS AND CONVERGENCE OF FIXED POINTS

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**Abstract.** Stability of fixed points of contraction mappings has been studied by Bonsall (cf. [2]) and Nadler (cf. [4]). These authors consider a sequence $(T_n)$ of maps defined on a metric space $(X,d)$ into itself and study the convergence of the sequence of fixed points for uniform or pointwise convergence of $(T_n)$, under contraction assumptions of the maps.

We will first consider $k$-contractions $T_n$ which are only defined on a subset $X_n$ of the metric space. We note that, in general, we cannot apply their results by using an extension theorem of contractions (cf. [1]). In this general setting, pointwise convergence cannot be defined (except when all $X_n$ are a same subset). We then introduce a new notion of convergence and we obtain a convergence result for the fixed points which generalizes Bonsall's theorem.

Secondly, after introducing another notion of convergence which generalizes uniform convergence, we obtain a stability result when only the limit map is a contraction. Some other results of stability of fixed points, which generalize Nadler's theorems, can be found in [3].

*Keywords:* Convergence of fixed points, contraction mappings in metric spaces.

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## §1. Introduction

We are interested in the convergence of a sequence of fixed points corresponding to a given sequence of contraction mappings $(T_n : X_n \to X)_{n\in\mathbb{N}}$ which converges (in some senses to be defined) to a contraction mapping $T_\infty : X_\infty \to X$, where all $X_n$ ($n \in \overline{\mathbb{N}} := \mathbb{N}\cup\{\infty\}$) are nonempty subsets of a metric space $(X,d)$. Recall that, given a constant $k \in (0,1)$, a map $T : Y \subset X \to X$ is a $k$-contraction mapping if $d(Tx,Ty) \le kd(x,y)$ for all $x,y \in Y$. The existence of the fixed points will be an assumption; for instance, the contraction mapping principle of Banach guarantees the existence of a unique fixed point of each contraction mapping of a complete metric space into itself. Thus we are only interested in stability properties.

Many results have been given when each map $T_n$ is defined on the whole metric space $X$. It was proved by Bonsall that pointwise convergence of a sequence of $k$-contraction mappings $(T_n : X \to X)_{n\in\mathbb{N}}$ to a $k$-contraction mapping $T_\infty : X \to X$ implies convergence of the sequence of fixed points associated to $(T_n)_{n\in\mathbb{N}}$ to the fixed point of $T_\infty$ where $X$ is supposed to be a complete metric space (cf. [2]). Nadler proved a similar result under uniform convergence on the domain $X$ of a sequence of mappings to a contraction mapping (cf. [4]).

Our main purpose is to generalize these two classical results when considering a sequence of contraction mappings $(T_n : X_n \to X)_{n\in\mathbb{N}}$ which converges to a contraction mapping $T_\infty$ :

$X_\infty \to X$. Since the domains $X_n \subset X$ are assumed to be different, pointwise and uniform convergence cannot be defined. It will be necessary to introduce and study two new notions of convergence for this type of problem.

We note that another important stability result can be obtained for a sequence of contraction mappings but without a uniform Lipschitz constant as considered by Bonsall. When the metric space $X$ is locally compact and each $T_n$ is a $k_n$-contraction mapping from $X$ into itself ($\forall n \in \overline{\mathbb{N}}$), convergence of the fixed points is a consequence of pointwise convergence of the contraction mappings (cf. [4]). A generalization of this theorem of Nadler can be found in [3].

## §2. Stability and generalization of pointwise convergence

Let us introduce a first notion of convergence as follows (where $Gr$ is the symbol of graph):

$$(G) \qquad Gr(T_\infty) \subset \liminf Gr(T_n):$$

$$\forall x \in X_\infty,\ \exists (x_n)_{n\in\mathbb{N}} \in \Pi_{n\in\mathbb{N}} X_n : x_n \to x \text{ and } T_n x_n \to T_\infty x.$$

We will say that $T_\infty$ is a $(G)$-limit of the sequence $(T_n)_{n\in\mathbb{N}}$ when property $(G)$ is satisfied by the family $(T_n)_{n\in\overline{\mathbb{N}}}$.

We can remark that a $(G)$-limit map of a sequence $(T_n)_{n\in\mathbb{N}}$ is not necessarily unique. Consider $X_n := \mathbb{R}$ ($n \in \overline{\mathbb{N}}$) and the family $(T_n : \mathbb{R} \to \mathbb{R})_{n\in\overline{\mathbb{N}}}$ of mappings defined by $T_n x := \frac{nx}{1+nx}$ for $x \in \mathbb{R}$ and $T_\infty x := 1$ for $x \in \mathbb{R}^*$, $T_\infty 0 := 0$. It is clear that $T_\infty$ is a $(G)$-limit of $(T_n)$. Let $T'_\infty : \mathbb{R} \to \mathbb{R}$ be defined by $T'_\infty x := T_\infty x$ if $x \neq 0$ and $T'_\infty 0 := \frac{1}{2}$. Then $T'_\infty$ is also a $(G)$-limit of $(T_n)$; indeed, the point $x = 0$ is the limit of the sequence $(x_n)_{n\in\mathbb{N}^*} := (\frac{1}{n})_{n\in\mathbb{N}^*}$ such that $(T_n x_n)$ converges to $T'_\infty 0$.

We now give a sufficient condition for uniqueness of the $(G)$-limit map.

**Proposition 1.** *Let $(X,d)$ be a metric space, $(X_n)_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of $X$ and $(T_n : X_n \to X)_{n\in\mathbb{N}}$ a sequence of $k$-Lipschitz mappings. If $T_\infty : X_\infty \to X$ is a $(G)$-limit of $(T_n)$ then $T_\infty$ is the unique one (defined on $X_\infty$) .*

*Proof.* Assume that $T_\infty : X_\infty \to X$ and $T'_\infty : X_\infty \to X$ are $(G)$-limit maps of the sequence $(T_n)$. For any point $x \in X_\infty$, there exist two sequences $(x_n)$ and $(y_n)$ in $\Pi_n X_n$ converging to $x$ such that $(T_n x_n)$ converges to $T_\infty x$ and $(T_n y_n)$ converges to $T'_\infty x$. From the Lipschitz condition, the sequence $(d(T_n x_n, T_n y_n))$ converges to 0 and since for all $n$ we have

$$d(T_\infty x, T'_\infty x) \leq d(T_\infty x, T_n x_n) + d(T_n x_n, T_n y_n) + d(T_n y_n, T'_\infty x)$$

we deduce that $T_\infty x = T'_\infty x$. $\square$

The following statement is our first stability result.

**Theorem 2.** *Let $(X,d)$ be a metric space, $(X_n)_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of $X$ and $(T_n : X_n \to X)_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying property $(G)$ and such that, for all $n \in \mathbb{N}$, $T_n$ is a $k$-contraction from $(X_n,d)$ into $(X,d)$. If, for all $n \in \overline{\mathbb{N}}$, $x_n$ is a fixed point of $T_n$ then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_\infty$.*

*Proof.* Let $x_n$ be a fixed point of $T_n$ for each $n \in \overline{\mathbb{N}}$. Since property $(G)$ holds and $x_\infty \in X_\infty$, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in X_n$ $(\forall n \in \mathbb{N})$, $y_n \to x_\infty$ and $T_n y_n \to T_\infty x_\infty$. Thus, by the contraction condition we get:

$$\begin{aligned} d(x_n, x_\infty) &\leq d(T_n x_n, T_n y_n) + d(T_n y_n, T_\infty x_\infty) \\ &\leq k d(x_n, y_n) + d(T_n y_n, T_\infty x_\infty) \\ &\leq k d(x_n, x_\infty) + k d(x_\infty, y_n) + d(T_n y_n, T_\infty x_\infty). \end{aligned}$$

We conclude that $(x_n)_{n\in\mathbb{N}}$ converges to $x_\infty$ from the following error estimate:

$$d(x_n, x_\infty) \leq (1-k)^{-1}(k d(x_\infty, y_n) + d(T_n y_n, T_\infty x_\infty)). \quad \square$$

When all the subsets $X_n$ are equal to the space $X$, we obtain, as a consequence, the theorem of Bonsall (cf. [2]):

**Corollary 3.** *Let $X$ be a nonempty complete metric space and let $(T_n : X \to X)_{n\in\overline{\mathbb{N}}}$ be a family of contraction mappings with the same Lipschitz constant $k < 1$ and such that the sequence $(T_n)_{n\in\mathbb{N}}$ converges pointwise to $T_\infty$. Then, for all $n \in \overline{\mathbb{N}}$, $T_n$ has a unique fixed point $x_n$ and the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_\infty$.*

Let us point out some properties. The first one says that the $(G)$-limit map $T_\infty : X_\infty \to X$ is a $k$-contraction as soon as each map $T_n : X_n \to X$ is a $k$-contraction. More generally:

**Proposition 4.** *Let $(X,d)$ be a metric space, $(X_n)_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of $X$ and $(T_n : X_n \to X)_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying property $(G)$ and such that, for any $n \in \mathbb{N}$, $T_n$ is $k_n$-Lipschitz with $(k_n)_{n\in\mathbb{N}}$ a bounded (resp. convergent) sequence. Then $T_\infty$ is $k$-Lipschitz with $k := \sup_{n\in\mathbb{N}} k_n$ (resp. $k := \lim k_n$).*

*Proof.* Given two points $x$ and $y$ in $X_\infty$, by $(G)$ there exist two sequences $(x_n) \in \Pi_n X_n$ and $(y_n) \in \Pi_n X_n$ converging respectively to $x$ and $y$ and such that the sequences $(T_n x_n)$, $(T_n y_n)$ converge respectively to $T_\infty x$ and $T_\infty y$. For any $n \in \mathbb{N}$, we deduce from the Lipschitz condition that

$$\begin{aligned} d(T_\infty x, T_\infty y) &\leq d(T_\infty x, T_n x_n) + d(T_n x_n, T_n y_n) + d(T_n y_n, T_\infty y) \\ &\leq d(T_\infty x, T_n x_n) + k_n d(x_n, y_n) + d(T_n y_n, T_\infty y). \end{aligned}$$

Since $\limsup k_n d(x_n, y_n) \leq k d(x,y)$, we conclude that $d(T_\infty x, T_\infty y) \leq k d(x,y)$. $\square$

When all the subsets $X_n$ ($n \in \overline{\mathbb{N}}$) are equal to a nonempty subset $M$ of $X$, we can compare the notion of $(G)$-convergence with the pointwise convergence for a sequence of maps $(T_n : M \to X)$ to a map $T_\infty : M \to X$. We will prove that property $(G)$ is more general than the pointwise convergence but these two notions are equivalent when the sequence $(T_n)_{n\in\mathbb{N}}$ is equicontinuous on $M$.

It is clear that pointwise convergence implies property $(G)$. The converse is false. Consider the family $(T_n : \mathbb{R}_+ \to \mathbb{R})_{n\in\overline{\mathbb{N}}}$ defined by: $T_n x := \frac{nx}{1+nx}$ and $T_\infty x := 1$ for all $x \in \mathbb{R}_+$. The map $T_\infty$ is a $(G)$-limit of $(T_n)$ but pointwise convergence is not satisfied. The problem arises at the point $x := 0$; we can take the sequence $(x_n) := (1/\sqrt{n})_{n\in\mathbb{N}^*}$ to verify that property $(G)$ is satisfied at this point.

In the next result, a sufficient condition is given in order that the two notions of convergence become equivalent.

**Proposition 5.** *Let $M$ be a nonempty subset of a metric space $(X,d)$, $(T_n : M \to X)_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying property $(G)$ and such that the sequence $(T_n)_{n\in\mathbb{N}}$ is equicontinuous on $M$. Then the sequence $(T_n)_{n\in\mathbb{N}}$ converges pointwise to $T_\infty$.*

*Proof.* Assume that the sequence $(T_n)_{n\in\mathbb{N}}$ is equicontinuous on $M$ and converges to $T_\infty$ in the sense of $(G)$. Given $x \in M$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset M$ such that the sequences $(x_n)$ et $(T_n x_n)$ converge respectively to $x$ and $T_\infty x$. Since $(T_n)$ is equicontinuous we have $d(T_n x_n, T_n x) \to 0$ and thus $d(T_n x, T_\infty x) \to 0$. We conclude that the sequence $(T_n)$ converges pointwise to $T_\infty$. □

The existence of a fixed point for a $(G)$-limit mapping is characterized by the following result when it is a contraction.

**Corollary 6.** *Let $(X,d)$ be a metric space, $(X_n)_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of $X$ and $(T_n : X_n \to X)_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying property $(G)$ and such that, for any $n \in \mathbb{N}$, $T_n$ is a $k$-contraction from $(X_n,d)$ into $(X,d)$. Assume that, for any $n \in \mathbb{N}$, $x_n$ is a fixed point of $T_n$. Then:*

$$\begin{aligned} T_\infty \textit{ admits a fixed point } &\iff (x_n) \textit{ converges and } \lim x_n \in X_\infty \\ &\iff (x_n) \textit{ admits a subsequence converging to a point of } X_\infty. \end{aligned}$$

*Proof.* From Theorem 2, we only have to prove the sufficient condition. Consider a subsequence $(x_{s(n)})$ of $(x_n)$ such that $\lim x_{s(n)} = x_\infty \in X_\infty$. By $(G)$, there exists a sequence $(y_n)$ in $X$ such that $y_n \in X_n$, $y_n \to x_\infty$ and $T_n y_n \to T_\infty x_\infty$. Since, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_\infty, T_\infty x_\infty) &\le d(x_\infty, x_{s(n)}) + d(T_{s(n)} x_{s(n)}, T_{s(n)} y_{s(n)}) + d(T_{s(n)} y_{s(n)}, T_\infty x_\infty) \\ &\le d(x_\infty, x_{s(n)}) + k d(x_{s(n)}, y_{s(n)}) + d(T_{s(n)} y_{s(n)}, T_\infty x_\infty) \end{aligned}$$

we deduce that $x_\infty = T_\infty x_\infty$. □

*Remark* 1. Under the assumptions of Corollary 6, and if:

(i) $\liminf X_n \subset X_\infty$ (i.e., the limit of any convergent $(z_n) \in \Pi_{n\in\mathbb{N}} X_n$ is in $X_\infty$) then:

$$T_\infty \text{ admits a fixed point } \iff (x_n) \text{ converges.}$$

(ii) $\limsup X_n \subset X_\infty$ (i.e., any cluster point of any $(z_n) \in \Pi_{n\in\mathbb{N}} X_n$ is in $X_\infty$) then:

$$T_\infty \text{ admits a fixed point } \iff (x_n) \text{ admits a convergent subsequence.}$$

Under a compactness assumption, the existence of a fixed point of the $(G)$-limit map can be obtained from the existence of fixed points of the contraction mappings $T_n$:

**Theorem 7.** *Let $(X_n)_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(X,d)$ and $(T_n : X_n \to X)_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying property $(G)$ and such that, for any $n \in \mathbb{N}$, $T_n$ is a $k$-contraction. Assume that $\limsup X_n \subset X_\infty$ and $\bigcup_{n\in\mathbb{N}} X_n$ is relatively compact. If, for any $n \in \mathbb{N}$, $T_n$ admits a fixed point $x_n$ then the $(G)$-limit map $T_\infty$ admits a fixed point $x_\infty$ and the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_\infty$.*

*Proof.* Let $x_n$ be the fixed point of $T_n$ for $n \in \mathbb{N}$. From compactness condition, there exists a convergent subsequence $(x_{s(n)})$. From the above remark, $T_\infty$ admits a fixed point $x_\infty$ and by Theorem 2 the sequence $(x_n)$ converges to $x_\infty$. □

Let us remark that a variant of Theorem 7 can be proved under the following assumptions: $\limsup T_n(X_n) \subset X_\infty$ and $\bigcup_{n\in\mathbb{N}} T_n(X_n)$ is relatively compact.

Let us introduce another notion of convergence which is weaker than $(G)$-convergence as follows:

$$(G^-) \qquad Gr(T_\infty) \subset \limsup Gr(T_n) :$$
$$\forall x \in X_\infty,\ \exists (x_n)_{n\in\mathbb{N}} \in \Pi_{n\in\mathbb{N}} X_n,\ \exists s \in \mathbb{S} \colon x_{s(n)} \to x \text{ and } T_{s(n)} x_{s(n)} \to T_\infty x,$$

where $\mathbb{S}$ denotes the set of all increasing maps $s : \mathbb{N} \to \mathbb{N}$.

Convergence of the sequence of fixed points to a fixed point of a $(G^-)$-limit map $T_\infty$ does not necessarily hold. Consider the family $(T_n : \mathbb{R} \to \mathbb{R})_{n\in\overline{\mathbb{N}}}$ given by $T_n x := (-1)^n$ and $T_\infty x := 1$. It is clear that the sequence $(T_n)$ converges to $T_\infty$ in the sense of $(G^-)$. The sequence of fixed points corresponding to $(T_n)$ is the divergent sequence $((-1)^n)$.

The map defined by $T'_\infty x := -1$ for $x < 0$ and $T'_\infty x := 1$ for $x \geq 0$ proves that a $(G^-)$-limit is not necessarily unique and the Lipschitz property is not preserved in general (this $(G^-)$-limit is discontinuous).

We shall establish in the next result that a fixed point of a $(G^-)$-limit map is then a cluster point of the sequence of fixed points associated with $(T_n)$.

**Theorem 8.** *Let $(X_n)_{n\in\overline{\mathbb{N}}}$ be a family of subsets in a metric space $(X,d)$ and $(T_n : X_n \to X)_{n\in\overline{\mathbb{N}}}$ a family of k-contraction mappings satisfying property $(G^-)$. If, for any $n \in \overline{\mathbb{N}}$, $x_n$ is a fixed point of $T_n$ then $x_\infty$ is a cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$.*

*Proof.* By property $(G^-)$, there exists a sequence $(y_n) \in \Pi_n X_n$ which has a subsequence $(y_{s(n)})$ such that $y_{s(n)} \to x_\infty$ and $T_{s(n)} y_{s(n)} \to T_\infty x_\infty$. Since each map $T_{s(n)}$ is a $k$-contraction, we have:

$$\begin{aligned} d(x_{s(n)}, x_\infty) &\leq d(T_{s(n)} x_{s(n)}, T_{s(n)} y_{s(n)}) + d(T_{s(n)} y_{s(n)}, T_\infty x_\infty) \\ &\leq k d(x_{s(n)}, y_{s(n)}) + d(T_{s(n)} y_{s(n)}, T_\infty x_\infty) \\ &\leq (1-k)^{-1}(k d(x_\infty, y_{s(n)}) + d(T_{s(n)} y_{s(n)}, T_\infty x_\infty)). \end{aligned}$$

Thus $(x_{s(n)})$ converges to $x_\infty$ the fixed point of $T_\infty$. □

## §3. Stability and generalization of uniform convergence

When the constants of contraction are not uniform, it was remarked by Nadler that pointwise convergence of $(T_n : X \to X)$ is not a sufficient condition to get a stability result as in Bonsall's theorem. First, Nadler proved that, under a uniform convergence assumption of $(T_n : X \to X)_{n\in\mathbb{N}}$ to a contraction $T_\infty : X \to X$, any sequence of fixed points corresponding to $(T_n)$ converges to the fixed point of the limit map. Let us note that later on, this author obtained a stability result assuming that the metric space $X$ is locally compact and that the sequence of contractions $(T_n : X \to X)_{n\in\mathbb{N}}$ converges pointwise to a contraction $T_\infty : X \to X$.

In this paper, we are interested in giving a generalization of the first result of Nadler (cf. [4, Theorem 1]); some generalizations of the second one are obtained in [3].

Let us introduce a second convergence property as follows:

$$(H) \qquad \forall (x_n)_{n\in\mathbb{N}} \in \Pi_{n\in\mathbb{N}} X_n,\ \exists (y_n)_{n\in\mathbb{N}} \subset X_\infty : d(x_n,y_n) \to 0 \text{ and } d(T_n x_n, T_\infty y_n) \to 0.$$

We will say that $T_\infty$ is a $(H)$-limit of $(T_n)_{n\in\mathbb{N}}$ when property $(H)$ is satisfied by the family $(T_n)_{n\in\overline{\mathbb{N}}}$.

The following proposition discloses a relationship between the two notions of convergence $(G)$ and $(H)$.

**Proposition 9.** *Let $(X_n)_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of a metric space $(X,d)$ such that $X_\infty \subset \liminf X_n$. Let $(T_n : X_n \to X)_{n\in\overline{\mathbb{N}}}$ be a family of mappings such that $T_\infty$ is continuous on $X_\infty$. If $T_\infty$ is a $(H)$-limit of $(T_n)_{n\in\mathbb{N}}$ then $T_\infty$ is a $(G)$-limit of $(T_n)_{n\in\mathbb{N}}$.*

*Proof.* Let $x \in X_\infty$; by the inclusion $X_\infty \subset \liminf X_n$ there exists a sequence $(x_n)$ in $X$ such that $x_n \in X_n$ and $x_n \to x$. By property $(H)$ we can find a sequence $(y_n)$ in $X_\infty$ satisfying $d(x_n,y_n) \to 0$ and $d(T_n x_n, T_\infty y_n) \to 0$. Thus $y_n \to x$ and (by continuity of $T_\infty$) we get $T_\infty y_n \to T_\infty x$. We conclude that $T_n x_n \to T_\infty x$ and then property $(G)$ holds. □

It is easy to see that a $(G)$-limit is not necessarily a $(H)$-limit of the sequence: consider the family of mappings $(T_n : \mathbb{R}_+ \to \mathbb{R})_{n\in\overline{\mathbb{N}}}$ defined by $T_n x := \frac{nx}{1+nx}$ and $T_\infty x := 1$ for any $x \in \mathbb{R}_+$. We know that $T_\infty$ is a $(G)$-limit of $(T_n)$. But property $(H)$ is not satisfied: for the null sequence $(x_n)$ we get $|T_n 0 - T_\infty y_n| = 1$ for any sequence $(y_n)$ converging to 0.

When all the subsets are equal to the whole space, we obtain the following comparison with uniform convergence.

**Proposition 10.** *Let $(T_n : M \to X)_{n\in\overline{\mathbb{N}}}$ be a family of mappings where $M$ is a nonempty subset of a metric space $(X,d)$.*

*(a) If $(T_n)_{n\in\mathbb{N}}$ converges uniformly to $T_\infty$ on $M$ then $T_\infty$ is a $(H)$-limit of $(T_n)_{n\in\mathbb{N}}$.*

*(b) The converse holds when $T_\infty$ is uniformly continuous on $M$.*

*Proof.* Property (a) is obvious. To prove the second one we assume that the limit map is uniformly continuous on $M$ and that the convergence of $(T_n)$ to $T_\infty$ is not uniform. Thus, there exists a sequence $(x_n)$ in $M$ such that $(d(T_n x_n, T_\infty x_n))$ does not converge to 0. If property $(H)$ holds we can find a sequence $(y_n)$ in $M$ satisfying $d(x_n,y_n) \to 0$ and $d(T_n x_n, T_\infty y_n) \to 0$. By uniform continuity of the limit map $T_\infty$, we get $d(T_\infty y_n, T_\infty x_n) \to 0$ and then $d(T_n x_n, T_\infty x_n) \to 0$ which leads to a contradiction. This completes the proof. □

To show that the converse is not true in the general case, we can consider the space $X := (0,+\infty)$ and the sequence $(T_n : X \to X)_{n\in\mathbb{N}^*}$ defined by $T_n x := \frac{n}{1+nx}$. Then $(T_n)$ converges in the sense of $(H)$ to $T_\infty : X \to X$ defined by $T_\infty x := \frac{1}{x}$ $(\forall x > 0)$. Indeed, for any sequence $(x_n)$ in $X$ and for the sequence $(y_n) := (x_n + \frac{1}{n})_{n\in\mathbb{N}^*}$ we have $|x_n - y_n| \to 0$ and $|T_n x_n - T_\infty y_n| \to 0$. But this convergence is not uniform because $\sup_{x\in X} |T_n x - T_\infty x| = \sup_{x>0} \frac{1}{x(1+nx)} = +\infty$.

*Remark* 2. (i) The uniqueness of a $(H)$-limit is not a direct consequence of its existence. Let us consider the sequence $(T_n)_{n\in\mathbb{N}^*}$ given by $T_n x := x/n$ and the null map $T_\infty$ defined on $[0,1]$. It is obvious that $(T_n)$ converges uniformly to $T_\infty$ on $[0,1]$ and then property $(H)$ is satisfied. Consider another map $T'_\infty$ defined on $[0,1]$ by $T'_\infty x := 0$ if $x \in [0,1[$, $T'_\infty 1 := 1$. Then $T'_\infty$ is also a $(H)$-limit of $(T_n)$: for any sequence $(x_n)$ in $[0,1]$, there exists a sequence $(y_n) := (x_n - x_n/n) \subset [0,1[$ such that $|x_n - y_n| \to 0$ and $|T_n x_n - T'_\infty y_n| = x_n/n \to 0$.

(ii) If $(T_n)$ is a sequence of k-Lipschitz maps and if $X_\infty \subset \liminf X_n$ then $(T_n)$ has at most one continuous $(H)$-limit. This is a consequence of propositions 1 and 9.

We now give our second result of stability.

**Theorem 11.** *Let $(X,d)$ be a metric space, $(X_n)_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of $X$ and let $(T_n : X_n \to X)_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property $(H)$ and such that $T_\infty$ is a $k_\infty$-contraction. If, for any $n \in \overline{\mathbb{N}}$, $x_n$ is a fixed point of $T_n$ then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_\infty$.*

*Proof.* By property $(H)$, there exists a sequence $(y_n)$ in $X_\infty$ such that $d(x_n,y_n) \to 0$ and $d(T_n x_n, T_\infty y_n) \to 0$. From the following inequalities:

$$\begin{aligned} d(x_n,x_\infty) &\le d(T_n x_n, T_\infty y_n) + d(T_\infty y_n, T_\infty x_\infty) \\ &\le d(T_n x_n, T_\infty y_n) + k_\infty d(y_n, x_\infty) \end{aligned}$$

we get

$$d(x_n,x_\infty) \le (1-k_\infty)^{-1}(d(T_n x_n, T_\infty y_n) + k_\infty d(y_n,x_n)).$$

We immediately deduce the convergence of $(x_n)$ to $x_\infty$. □

When all the domains are equal to the whole space $X$, Nadler's theorem is a direct consequence (cf. [4, Theorem 1]):

**Corollary 12.** *Let $(X,d)$ be a metric space, $(T_n : X \to X)_{n\in\mathbb{N}}$ a sequence of mappings which converges uniformly to a contraction mapping $T_\infty : X \to X$. If, for any $n \in \overline{\mathbb{N}}$, $x_n$ is a fixed point of $T_n$ then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x_\infty$.*

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