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MINIMAL ENERGY C^r-SURFACES ON UNIFORM POWELL-SABIN TYPE MESHES

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Abstract. In this paper we present a method to obtain a C^r-surface approximating a Lagragian data set in a polygonal domain and minimizing a certain "energy functional". We give a convergence result and a numerical and graphical example involving C^2-surfaces.

Keywords: α-triangulation, Powell-Sabin element, variational spline, minimal energy, approximation, smoothing.

AMS classification: 41A15, 65D07, 65D10, 65D17.

§1. Introduction

In the context of the fitting and design of curves and surfaces, variational methods based on the minimization of a given functional have received considerable attention due to their efficiency and usefulness. Such functionals typically contain two terms: the first indicates how well the curve or surface approximates a given data set, while the second controls the degree of smoothness or fairness of the curve or surface. For example, discrete smoothing D^m-splines [1, 2] and discrete smoothing variational splines [6] provide specific examples of variational curves and surfaces. In [7] a functional of the above type is minimized in a parametric space of bicubic splines. Moreover, in all cases the obtained splines approximate a Lagrangian or Hermite data set. Other papers related to this matter are [3], [5] and references therein.

In this work we present a method to obtain a C^r-quadratic spline surface $(r \geq 1)$ on a polygonal domain $D \subset \mathbb{R}^2$ which approximates a Lagrangian data set and minimizes an "energy functional" given by a linear combination of the usual semi-norms $|\cdot|_m, m = 1, \ldots, r+1$, on the Sobolev space $H^{r+1}(D)$. The minimization space is a spline space constructed from a Δ^1-type triangulation $\mathscr{T}$ over D and its Powell-Sabin associated subtriangulation $\mathscr{T}_6$ (cf. [8]).

This paper is organized as follows: in Section 2, we recall some preliminary notations and results. Section 3 is devoted to formulate the problem and to present a method to solve it, while in Section 4 a convergence result is proved. In Section 5 we briefly describe the method to obtain the basis functions with local support over the unit reference triangle, and we give a numerical and graphical example for Franke's test function.

§2. Notations and preliminaries

Let $D \subset \mathbb{R}^2$ be a polygonal domain and let us consider the Sobolev space $H^{r+1}(D)$, whose elements are (classes of) functions u defined in D such that u and their partial derivatives (in the distribution sense) $\partial^\beta u$ belong to $L^2(D)$, with $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ and $|\beta| = \beta_1 + \beta_2 \le r+1$. In this space we consider the usual norm

$$\|u\| = \left(\sum_{|\beta| \le r+1} \int_D \partial^\beta u(x)^2 \, dx \right)^{1/2},$$

the semi-norms

$$|u|_m = \left(\sum_{|\beta| = m} \int_D \partial^\beta u(x)^2 \, dx \right)^{1/2}, \quad m = 1, \ldots, r+1,$$

and the corresponding inner semi-products

$$(u, v)_m = \sum_{|\beta| = m} \int_D \partial^\beta u(x) \partial^\beta v(x) dx; \quad m = 1, \ldots, r+1.$$

We will consider a uniform Δ^1-type triangulation $\mathscr{T}$ of D, and the associated Powell-Sabin triangulation $\mathscr{T}_6$ of $\mathscr{T}$, which is obtained by joining the centre Ω_T of the inscribed circle of each interior triangle $T \in \mathscr{T}$ to the vertices of T and to the centres $\Omega_{T'}$ of the inscribed circles of the neighbouring triangles $T' \in \mathscr{T}$. When T has a side that is on the boundary of D, the point Ω_T is joined to the mid-point of this side, to the vertices of T and to the centres $\Omega_{T'}$ of the inscribed circles of the neighbouring triangles $T' \in \mathscr{T}$.

We consider the set

$$S_n^{r, r_1, r_1'}(D, \mathscr{T}_6) = \{ S \in C^r(D) : S|_T \in S_n^{r_1, r_1'}(T) \ \forall T \in \mathscr{T} \},$$

where

$$S_n^{r_1, r_1'}(T) = \{ S \in C^{r_1}(T) : S|_{T'} \in \mathbb{P}_n(T'), \ \forall T' \in \mathscr{T}_6, \ T' \subset T, \\ \text{and } S \text{ is of class } C^{r_1'} \text{ at the vertices of } T \}$$

and $\mathbb{P}_n(T')$ indicates the space of bivariate polynomials of total degree at most n over T'.

Let $n = 2r+1$ for r even and $n = 2r$ for r odd. Let $[x]$ denote the integer part of x. In [8] it is shown that given the values of a function f (defined on D) and all its partial derivatives of order at most $r + [\frac{r}{2}]$ at all the vertices of $\mathscr{T}$, there exists a unique function $S \in V_n^r(D, \mathscr{T}_6) = S_n^{r, [(n-1)/2]+1, r+[r/2]}(D, \mathscr{T}_6)$ such that the values of S and all its partial derivatives of order at most $r + [r/2]$ coincide with those of f.

§3. Formulation of the problem

Let $g \in H^{r+1}(D)$. Given, for any $s \in \mathbb{N}^*$, a finite set of points D^s in D and a set of values $Z^s = \{g(a)\}_{a \in D^s}$, we are looking for a C^r-surface ($r \ge 1$) that approximates the points $\{(a, g(a))\}_{a \in D^s} \subset \mathbb{R}^3$ and minimizes the functional energy that is described below.

Let $k = k(s) = \mathrm{card}(D^s)$ and let us denote $\langle\cdot\rangle_k$ (resp. $\langle\cdot\,,\,\cdot\rangle_k$) the usual Euclidean norm (resp. Euclidean inner product) in $\mathbb{R}^k$. Let ρ^s be the evaluation operator $\rho^s : H^{r+1}(D) \longrightarrow \mathbb{R}^k$ defined by $\rho^s(v) = (v(a))_{a\in D^s}$ and let us suppose that

$$\ker(\rho^s) \cap \mathbb{P}_r(D) = \{0\}. \tag{1}$$

Given $\tau = (\tau_1,\dots,\tau_{r+1})$, where $\tau_i \in [0,+\infty)$ for all $i = 1,\dots,r$ and $\tau_{r+1} \in (0,+\infty)$, let us consider the functional defined on $H^{r+1}(D)$ by

$$J_{\tau,s}(v) = \langle\rho^s(v-g)\rangle_k^2 + \sum_{m=1}^{r+1} \tau_m |v|_m^2.$$

Note that the first term of $J_{\tau,s}$ measures how well v approximates the values $\{g(a)\}_{a\in D^s}$ over the set of points D^s (in the least squares sense), while the second one represents the "minimal energy condition" over the semi-norms $|\cdot|_1,\dots,|\cdot|_{r+1}$ weighted by the parameters $\tau_1,\dots,\tau_{r+1}$, respectively. The minimization problem we want to solve is this:

> *Given a Δ^1-type triangulation $\mathscr{T}$ of D and its associated Powell-Sabin triangulation $\mathscr{T}_6$, we look for an element $\sigma_{\tau,s}^{\mathscr{T}_6} \in V_n^r(D,\mathscr{T}_6)$ such that*

$$J_{\tau,s}(\sigma_{\tau,s}^{\mathscr{T}_6}) \le J_{\tau,s}(v),\ \forall v \in V_n^r(D,\mathscr{T}_6). \tag{2}$$

Theorem 1. *Problem* (2) *has a unique solution that is also the unique solution of the following variational problem:*

$$\begin{cases} \text{Find } \sigma_{\tau,s}^{\mathscr{T}_6} \in V_n^r(D,\mathscr{T}_6) \text{ such that} \\ \langle\rho^s(\sigma_{\tau,s}^{\mathscr{T}_6}),\rho^s(v)\rangle_k + \sum_{m=1}^{r+1} \tau_m(\sigma_{\tau,s}^{\mathscr{T}_6},v)_m = \langle Z^s,\rho^s(v)\rangle_k,\ \forall v \in V_n^r(D,\mathscr{T}_6). \end{cases} \tag{3}$$

Proof. Condition (1) allows us to be sure that $v \mapsto [\![v]\!] = \left(\langle\rho^s(v)\rangle_k^2 + \sum_{m=1}^{r+1}\tau_m|v|_m^2\right)^{1/2}$ is a norm on $V_n^r(D,\mathscr{T}_6)$ equivalent to $\|\cdot\|$. As a consequence, the symmetric and continuous bilinear form $a : V_n^r(D,\mathscr{T}_6) \times V_n^r(D,\mathscr{T}_6) \to \mathbb{R}$, defined by $a(u,v) = \langle\rho^s(u),\rho^s(v)\rangle_k + \sum_{m=1}^{r+1}\tau_m(u,v)_m$, is $V_n^r(D,\tau_6)$-elliptic. Besides, the mapping $\varphi : V_n^r(D,\mathscr{T}_6) \to \mathbb{R}$, defined by $\varphi(v) = \langle Z^s,\rho^s(v)\rangle_k$, is a linear and continuous form. We obtain the result by applying the Lax-Milgram Lemma. □

Let us denote $N = \dim\left(V_n^r(D,\mathscr{T}_6)\right)$. If $\{v_1,\dots,v_N\}$ is a basis of the finite element space $V_n^r(D,\mathscr{T}_6)$ whose elements have local support, and $\sigma_{\tau,s}^{\mathscr{T}_6} = \sum_{i=1}^N \alpha_i v_i$, then Problem (3) gives rise to the linear system

$$CX = B, \tag{4}$$

where

$$B = \left(\left(\langle Z^s,\rho^s(v_i)\rangle_k\right)_{i=1}^N\right)^t, \quad X = \left((\alpha_i)_{i=1}^N\right)^t,$$

$$C = \left(\langle\rho^s(v_i),\rho^s(v_j)\rangle_k + \sum_{m=1}^{r+1}\tau_m(v_i,v_j)_m\right)_{i,j=1}^N.$$

Remark 1. It can be shown that C is a symmetric, positive definite and banded matrix.

§4. Convergence

Let $\Delta = (r+1)(r+2)/2$ and $A^0 = \{a_1^0, a_2^0, \ldots, a_\Delta^0\}$ be a $\mathbb{P}_r$-unisolvent subset of D and let us suppose that

$$\sup_{x\in D} \min_{a\in D^s} \langle x-a\rangle_2 = \mathrm{O}(1/s), \quad s \to +\infty. \tag{5}$$

Then, there exist $C>0$ and $s_1 \in \mathbb{N}^*$ such that, for all $s \geq s_1$, there exists $\{a_1^s, \ldots, a_\Delta^s\} \subset D^s$ verifying

$$\langle a_i^0 - a_i^s\rangle_2 \leq \frac{C}{s}, \quad \forall\, i = 1, \ldots, \Delta,\ \forall s \geq s_1. \tag{6}$$

Lemma 2. *Let us suppose that (5) is verified and let $A^s = \{a_1^s, \ldots, a_\Delta^s\} \subset D^s$ be any subset verifying (6). Then, there exists $s^* \in \mathbb{N}$ such that, for each $s \geq s^*$, the application $[\![\cdot]\!]^s$ defined by*

$$[\![v]\!]^s = \left(\sum_{i=1}^{\Delta} v\big(a_i^s\big)^2 + |v|_{r+1}^2\right)^{1/2}$$

is a norm on $H^{r+1}(D)$ uniformly equivalent with respect to s to the norm $\|\cdot\|$.

Proof. Let $s_0 \in \mathbb{N}$ be such that A^s is a $\mathbb{P}_r$-unisolvent subset of D for all $s \geq s_0$. Then $[\![\cdot]\!]^s$ is a norm on $H^{r+1}(D)$ for all $s \geq s_0$. From the continuous injection of $H^{r+1}(D)$ into $C^0(\overline{D})$, there exists $C_1 > 0$ such that $[\![v]\!]^s \leq C_1\, \|v\|$ for all $s \geq s_0$ and $v \in H^{r+1}(D)$.

On the other hand, for every $s \in \mathbb{N}$ we have

$$\frac{1}{2}\sum_{i=1}^{\Delta} v\left(a_i^0\right)^2 \leq \sum_{i=1}^{\Delta}\left(v\left(a_i^0\right) - v\left(a_i^s\right)\right)^2 + \sum_{i=1}^{\Delta} v\left(a_i^s\right)^2,$$

and from Sobolev's Hölder Imbedding Theorem for the space $H^{r+1}(D)$ into $C^0(\overline{D})$, we obtain

$$\frac{1}{2}\sum_{i=1}^{\Delta} v\left(a_i^0\right)^2 + |v|_{r+1}^2 \leq \sum_{i=1}^{\Delta}\langle a_i^0 - a_i^s\rangle_2^2 \|v\|^2 + \sum_{i=1}^{\Delta} v\left(a_i^s\right)^2 + |v|_{r+1}^2,\ \forall s \in \mathbb{N}^*.$$

Since A^0 is P_r-unisolvent, it follows that the application $v \mapsto \left(\frac{1}{2}\sum_{i=1}^{\Delta} v(a_i^0)^2 + |v|_{r+1}^2\right)^{1/2}$ is a norm on $H^{r+1}(D)$ that, besides, is equivalent to $\|\cdot\|$ (see Proposition I-2.2 of [2]). Hence, there exists $C_2 > 0$ such that

$$C_2\|v\|^2 \leq \sum_{i=1}^{\Delta}\langle a_i^0 - a_i^s\rangle_2^2 \|v\|^2 + \sum_{i=1}^{\Delta} v\left(a_i^s\right)^2 + |v|_{r+1}^2,\ \forall s \in \mathbb{N}^*.$$

Moreover, by (6) we have

$$C_2\|v\|^2 \leq \frac{\Delta C^2}{s^2}\|v\|^2 + \sum_{i=1}^{\Delta} v\left(a_i^s\right)^2 + |v|_{r+1}^2,\ \forall s \geq s_1.$$

Let $s_2 \geq s_1$ be such that $\Delta C^2/s_2^2 < C_2$. Then, $C_3 = C_2 - \Delta C^2/s_2^2$ satisfies

$$C_3\|v\|^2 \leq \left(C_2 - \frac{\Delta C^2}{s^2}\right)\|v\|^2 \leq ([\![v]\!]^s)^2,\ \forall s \geq s_2.$$

Consequently, $C_3^{1/2}\|v\| \leq [\![v]\!]^s$ for all $s \geq s_2$. Thus, it suffices to take $s^* = \max\{s_0, s_2\}$. □

Let $g \in C^{n+1}(D)$ and let $\mathscr{H} \subset \mathbb{R}_*^+$ be a subset that admits 0 as an accumulation point. Let, for each $h \in \mathscr{H}$, $\mathscr{T}$ be a uniform Δ^1-type triangulation of D such that h is the diameter of the triangles of $\mathscr{T}$. Let $\mathscr{T}_6$ be the associated Powell-Sabin triangulation of $\mathscr{T}$ and $s_h \in V_n^r(D, \mathscr{T}_6)$ be the unique function that interpolates the values of g and its partial derivatives of order at most $r + [r/2]$ at all the vertices of $\mathscr{T}$.

Lemma 3. *There exists $C > 0$, depending only on r and g, such that, for all $h \in \mathscr{H}$, we have*

$$\max_{x\in\overline{D}} |(s_h - g)(x)| \leq C h^{n+1} \tag{7}$$

and

$$|s_h - g|_m \leq C h^{n+1-m}, \ \forall m = 1, \dots, r+1. \tag{8}$$

Proof. The result is analogous to Theorem 2 in [8], taking into account that there exists $C > 0$ such that $|s_h - g|_m \leq C \max_{x\in\overline{D}} \max_{|\beta|=m} |\partial^\beta (s_h - g)(x)|$ holds for all $m = 1, \dots, r+1$. □

Theorem 4. *Let us suppose that, in addition to hypothesis (5), the following hypotheses are verified:*

$$\text{There exist } C > 0 \text{ and } s_0 \in \mathbb{N}^* \text{ such that } k(s) \leq C s^2 \text{ for all } s \geq s_0; \tag{9}$$

$$\tau_{r+1} = o(s^2), \quad s \to +\infty; \tag{10}$$

$$\tau_m = o(\tau_{r+1}), \quad s \to +\infty, \quad \forall m = 1, \dots, r; \tag{11}$$

$$\frac{s^2 h^{2s+2}}{\tau_{r+1}} = o(1), \quad s \to +\infty. \tag{12}$$

Let $\sigma_{\tau,s}^h$ be the unique solution of Problem (2) for the triangulation $\mathscr{T}_6$ fixed before. Then,

$$\lim_{s\to+\infty} \left\| g - \sigma_{\tau,s}^h \right\| = 0.$$

Proof. Since $\sigma_{\tau,s}^h$ is the solution of Problem(2), we have $J_{\tau,s}(\sigma_{\tau,s}^h) \leq J_{\tau,s}(s_h)$, that is to say:

$$\langle \rho^s (\sigma_{\tau,s}^h - g) \rangle_k^2 + \sum_{m=1}^{r+1} \tau_m |\sigma_{\tau,s}^h|_m^2 \leq \langle \rho^s (s_h - g) \rangle_k^2 + \sum_{m=1}^{r+1} \tau_m |s_h|_m^2. \tag{13}$$

By (7) we know that there exists $C > 0$ such that $(s_h(a) - g(a))^2 \leq C h^{2n+2}$ for all $a \in D^s$ and, by using (8) and (9), we can be sure that there exists $C_1 > 0$ such that

$$\begin{aligned} &\frac{1}{\tau_{r+1}} \langle \rho^s (s_h - g) \rangle_k^2 + \sum_{m=1}^{r} \frac{\tau_m}{\tau_{r+1}} |s_h|_m^2 + |s_h|_{r+1}^2 \\ &\quad \leq \frac{C_1 s^2 h^{2n+2}}{\tau_{r+1}} + \sum_{m=1}^{r} \frac{\tau_m}{\tau_{r+1}} \left(C_1 h^{n+1-m} + |g|_m \right)^2 + \left(C_1 h^{n-r} + |g|_{r+1} \right)^2 \end{aligned} \tag{14}$$

for all $s \geq s_0$. Moreover, from (10) and (12) we obtain

$$h = o(1), \quad s \to +\infty,$$

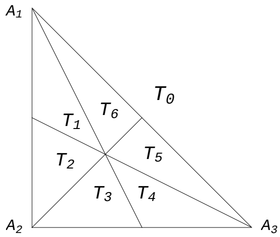

Figure 1: Powell-Sabin triangulation of T_0.

and taking into account (11) and (12) we can be sure that there exists $C > 0$ and $s_1 \in \mathbb{N}^*$ such that

$$\frac{C_1 s^2 h^{2n+2}}{\tau_{r+1}} + \sum_{m=1}^{r} \frac{\tau_m}{\tau_{r+1}} \left(C_1 h^{n+1-m} + |g|_m\right)^2 + \left(C_1 h^{n-r} + |g|_{r+1}\right)^2 \leq C \tag{15}$$

for all $s \geq s_1$. By using (13), (14) and (15) we obtain

$$|\sigma^h_{\tau,s}|^2_{r+1} \leq C, \quad \langle \rho^s(\sigma^h_{\tau,s} - g)\rangle^2_k \leq C\tau_{r+1}, \ \forall\, s \geq s_1.$$

The remainder of the proof is analogous to the proof of Theorem VI-3.2 in [2] from step 2), with s, D^s, Δ and r instead of d, A^d, $\mathfrak{M}$ and $m-1$, respectively. □

§5. Numerical and graphical examples

We have considered, for different values of k, sets D^s consisting of k points arbitrarily distributed over the domain $D = [0,1] \times [0,1]$. We have taken, for different values of q, uniform partitions $\{t_i = i/q\}_{i=0}^q$ of the interval $[0,1]$ into q subintervals, from which we obtain uniform partitions of D whose elements are $\{[t_i,t_{i+1}] \times [t_j,t_{j+1}]\}_{i,j=0}^{q-1}$. By dividing each rectangle $\{[t_i,t_{i+1}] \times [t_j,t_{j+1}]\}$ by the diagonal that joins the points (t_i,t_{j+1}) and (t_{i+1},t_j), we obtain a Δ^1-type triangulation $\mathscr{T}$ of D formed by $(q+1)^2$ vertices, from which we consider its associated Powell-Sabin's triangulation $\mathscr{T}_6$.

In the examples presented in this work, we look for C^2-surfaces, hence, the finite element vector space considered is $V^2_5(D,\mathscr{T}_6) = S^{2,3,3}_5(D,\mathscr{T}_6)$. To construct a basis of such space whose elements have local support we have considered the reference triangle T_0 with vertices $A_1 = (0,1)$, $A_2 = (0,0)$ and $A_3 = (1,0)$ and the linear functionals given by $L_i(f) = f(A_i)$ for $i = 1,2,3$; $L_i(f) = \partial_x f(A_{i-3})$ for $i = 4,5,6$; $L_i(f) = \partial_y f(A_{i-6})$ for $i = 7,8,9$; $L_i(f) = \partial_{\{x,2\}} f(A_{i-9})$ for $i = 10,11,12$; $L_i(f) = \partial_{x,y} f(A_{i-12})$ for $i = 13,14,15$; $L_i(f) = \partial_{\{y,2\}} f(A_{i-15})$ for $i = 16,17,18$; $L_i(f) = \partial_{\{x,3\}} f(A_{i-18})$ for $i = 19,20,21$; $L_i(f) = \partial_{x,\{y,2\}} f(A_{i-21})$ for $i = 22,23,24$; $L_i(f) = \partial_{\{x,2\},y} f(A_{i-24})$ for $i = 25,26,27$; and $L_i(f) = \partial_{\{y,3\}} f(A_{i-27})$ for $i = 28,29,30$.

To compute the solution of the linear system (4), we have considered the basis functions $\{w_1,\ldots,w_{30}\}$ over T_0 that verify $L_i(w_j) = \delta_{ij}$. To this end, let $\{T_1,\ldots,T_6\}$ be the microtriangles of the Powell-Sabin triangulation of T_0 (see Figure 1). Over each triangle T_d every polynomial p of total degree five can be expressed as $p(x) = \sum_{\substack{i,j,k=0,\ldots,5 \\ i+j+k=5}} c^d_{ijk} \lambda_1^i \lambda_2^j \lambda_3^k$, where

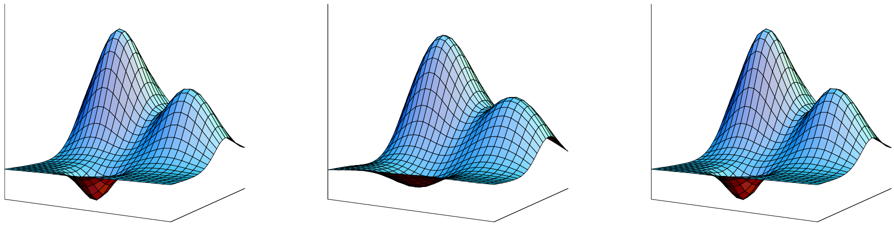

Figure 2: Franke's function and two approximating surfaces.

Number of triangles	τ_1	τ_2	τ_3	Errors for 1000 points	Errors for 2500 points	Errors for 5000 points
162	10^{-2}	10^{-2}	10^{-5}	$4.95 \cdot 10^{-2}$	$2.7 \cdot 10^{-2}$	$1.54 \cdot 10^{-2}$
162	10^{-5}	10^{-5}	10^{-8}	$4.15 \cdot 10^{-3}$	$2.51 \cdot 10^{-3}$	$2.51 \cdot 10^{-3}$
162	0	0	10^{-5}	$1.67 \cdot 10^{-2}$	$1.06 \cdot 10^{-2}$	$8.12 \cdot 10^{-3}$
722	10^{-2}	10^{-2}	10^{-5}	$4.98 \cdot 10^{-2}$	$2.68 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$
722	10^{-5}	10^{-5}	10^{-8}	$1.3 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$9.51 \cdot 10^{-4}$
722	0	0	10^{-5}	$2.2 \cdot 10^{-2}$	$1.48 \cdot 10^{-2}$	$8.04 \cdot 10^{-3}$
722	0	0	10^{-7}	$1.51 \cdot 10^{-3}$	$1.09 \cdot 10^{-3}$	$8.02 \cdot 10^{-4}$

Table 1: Table of errors for different values of the parameters τ, q and k.

$(\lambda_1, \lambda_2, \lambda_3)$ is the vector of barycentric coordinates of x with respect to T_d, for all $x \in T_d$. By applying the relations (see [4]) that must verify the B-coefficients of a given function f in order to be of class C^2, we determine the B-coefficients of the basis functions $\{w_i\}_{i=1}^{30}$.

Figure 2 shows the graphic of Franke's function (on the left) and of approximating surfaces for $q = 4$, $k = 1500$, $\tau_1 = \tau_2 = 10^{-2}$, $\tau_3 = 10^{-4}$ (in the middle) and $q = 14$, $k = 1600$, $\tau_1 = \tau_2 = 10^{-5}$, $\tau_3 = 10^{-8}$ (on the right).

The error estimations have been computed by using the relative error formula

$$E = \left(\frac{\sum_{\upsilon=1}^{2500} (f - \sigma)(a_\upsilon)^2}{\sum_{\upsilon=1}^{2500} f(a_\upsilon)^2} \right)^{1/2},$$

where $\{a_1, \ldots, a_{2500}\}$ are arbitrary points in $[0,1]^2$, f is Franke's function and $\sigma = \sigma_{\tau,s}^{\mathscr{T}_6}$. Table 1 shows the errors committed for different values of τ, q and k.

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References

[1] ARCANGÉLI, R. Some applications of discrete D^m-splines. In *Mathematical methods in Computer Aided Geometric Design*, T. Lyche and L. L. Schumaker, Eds. Academic Press, 1998, pp. 35–44.

[2] ARCANGÉLI, R., LÓPEZ DE SILANES, M. C., AND TORRENS, J. J. *Multidimensional Minimizing Splines*. Kluwer Academic Publisher, 2004.

[3] DUCHON, J. Interpolation des fonctions de deux variables suivant le principle de la flexion des plaques minces. *RAIRO 10*, 12 (1976), 5–12.

[4] FARIN, G. Bézier polynomials over triangles and the construction of piecewise C^r-polynomials. Technical Report TR/91, Brunel University, Uxbridge, England, 1980.

[5] GREINER, G. Surface construction based on variational principles. In *Wavelets, Images and Surface Fitting*, P. J. Laurent, A. Le Méhauté and L. L. Schumaker, Eds. Wellesley, 1994, pp. 277–286.

[6] KOUIBIA, A., AND PASADAS, M. Approximation by discrete variational splines. *J. Comput. Appl. Math. 116* (2000), 145–156.

[7] KOUIBIA, A., AND PASADAS, M. Approximation of surfaces by fairness bicubic splines. *Adv. Comput. Math. 20* (2004), 87–103.

[8] LAGHCHIM-LAHLOU, M., AND SABLONNIÈRE, P. C^r-finite elements of Powell-Sabin type on the three direction mesh. *Adv. Comput. Math. 6* (1996), 191–206.

[9] SABLONNIÈRE, P. Error bounds for Hermite interpolation by quadratic splines on an α-triangulation. *IMA Journal of Numerical Analysis 7*, 4 (1987), 495–508.

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