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A TWO-DIMENSIONAL RUIN PROBLEM ON THE POSITIVE QUADRANT, WITH EXPONENTIAL CLAIMS: FEYNMAN-KAC FORMULA, LAPLACE TRANSFORM AND ITS INVERSION

Florin Avram, Zbigniew Palmowski and Martijn Pistorius

Abstract. We consider an exit problem of a certain two-dimensional process from a cone, inspired by applications in insurance and queueing theory. One motivation is to study the joint ruin problem for two insurance companies (insurance/reinsurance), or for two branches of the same company, which divide between them both claims and premia in some specified proportions, the goal being to split the risk (in particular when the claims are big). Another motivation is to provide an example of a multi-dimensional ruin model admitting analytic solutions. Indeed, we succeed, in the simplest particular case of exponential claims, to derive both the Laplace transform of the perpetual ruin probabilities, and to invert it, obtaining therefore an explicit solution for our model.

Keywords: Classical risk theory, two-dimensional ruin probability.

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§1. A two dimensional ruin problem

In this paper we consider a particular two dimensional risk model in which two companies split the amount they pay out of each claim in proportions δ_1 and δ_2 where $\delta_1 + \delta_2 = 1$, and the premiums according to rates c_1 and c_2. Let U_i denote the risk process of the i'th company

$$U_i(t) := -\delta_i S(t) + c_i t + u_i, \qquad i = 1, 2\ ,$$

where u_i denotes the initial reserve and

$$S(t) = \sum_{i=1}^{N(t)} \sigma_i$$

for $N(t)$ being a Poisson process with intensity λ and the claims σ_i being i.i.d. random variables independent of $N(t)$ with distribution function $F(x)$. We shall denote by μ the reciprocals of the means of σ_i, respectively. We shall assume that the second company, to be called reinsurer, gets smaller profits per amount paid, i.e.:

$$p_1 = \frac{c_1}{\delta_1} > \frac{c_2}{\delta_2} = p_2.$$

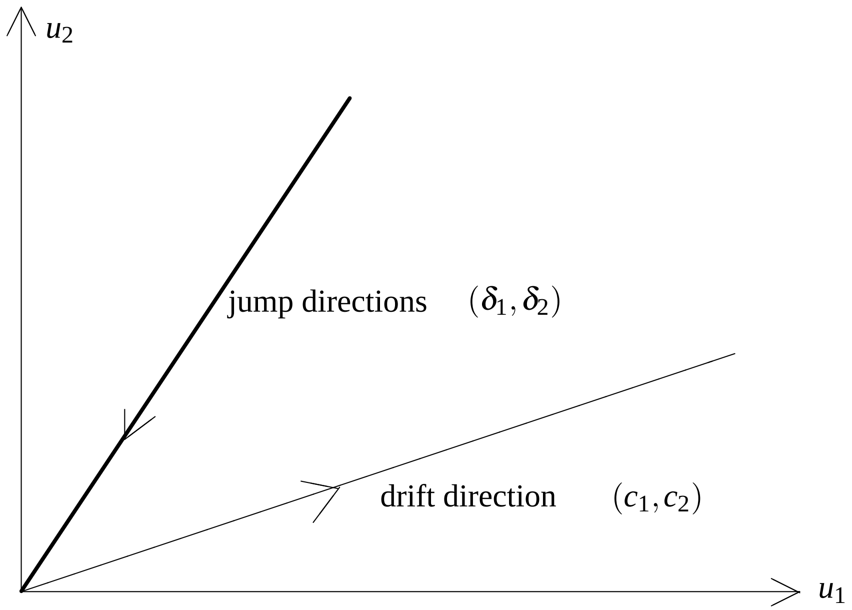

Figure 1: Geometrical considerations

As usual in risk theory, we assume that $p_i > \rho := \frac{\lambda}{\mu}$, which implies that in the absence of ruin, $U_i(t) \to \infty$ as $t \to \infty$ ($i = 1,2$). Ruin happens at the time $\tau = \tau(u_1,u_2)$ when at least one insurance company is ruined:

$$\tau(u_1,u_2) := \inf\{t \geq 0 : U_1(t) < 0 \quad \text{or} \quad U_2(t) < 0\},$$

i.e. at the exit time of $(U_1(t),U_2(t))$ from the positive quadrant. In this paper we will analyse the perpetual or ultimate ruin probability:

$$\psi(u_1,u_2) = P\left[\tau(u_1,u_2) < \infty\right]. \tag{1}$$

Although ruin theory under multi-dimensional models admits rarely analytic solutions, we are able to obtain in our problem a closed form solution for (1) if σ_i are exponentially distributed with intensity μ.

Geometrical considerations. The solution of the two-dimensional ruin problem (1) strongly depends on the relative sizes of the proportions $\boldsymbol{\delta} = (\delta_1, \delta_2)$ and premium rates $\mathbf{c} = (c_1, c_2)$ – see Figure 1. If, as assumed throughout, the angle of the vector $\boldsymbol{\delta}$ with the u_1 axis is bigger than that of $\mathbf{c}$, i.e. $\delta_2 c_1 > \delta_1 c_2$, we note that starting with initial capital $(u_1,u_2) \in \mathscr{C}$ in the cone $\mathscr{C} = \{(u_1,u_2) : u_2 \leq (\delta_2/\delta_1)u_1\}$ situated below the line $u_2 = (\delta_2/\delta_1)u_1$, the process (U_1,U_2) ends up hitting at time τ the u_1 axis. Thus, in the domain $\mathscr{C}$ ruin occurs iff there is ruin in the one-dimensional problem corresponding to the risk process U_2 with premium c_2 and claims $\delta_2\,\sigma$.

One dimensional reduction. A key observation is that τ in (1) is also equal to

$$\tau(u_1,u_2) = \inf\{t \geq 0 : S(t) > b(t)\},$$

where $b(t) = \min\{(u_1+c_1t)/\delta_1, (u_2+c_2t)/\delta_2\}$. The two dimensional problem (1) may thus be also viewed as a one dimensional crossing problem over a piecewise linear barrier. Note the relation to asset-liability management models, in which regulatory requirements impose prescribed limits of variation for the difference between the assets $P(t)$ of a company and its liabilities $S(t)$ (see [6]), and which also translate typically into (several) linear barriers.

In the case that the initial reserves u_1 and u_2 are such that $(u_1,u_2) \in \mathscr{C}$, that is, $u_2/\delta_2 \leq u_1/\delta_1$, the barrier b is linear, $b(t) = (u_2+c_2t)/\delta_2$, the ruin happens always for the second company. Thus, as we already observed, the problem (1) reduces in fact to the classical one-dimensional ultimate ruin problem with premium c_2 and claims $\delta_2\sigma$, i.e.

$$\psi(u_1,u_2) = \psi_2(u_2) := P(\tau_2(u_2) < \infty),$$

where $\tau_2(u_2) = \inf\{t \geq 0 : U_2(t) < 0\}$ and $\psi_2(u_2)$ is the ruin probability of U_2, with $U_2(0) = u_2$.

Solution in the lower cone $\mathscr{C}$. By the equation above, the solution in the lower cone $\mathscr{C}$ coincides with the ultimate ruin probability $\psi_2(x_2)$ of the classical risk process $U_2(t)/\delta_2$ with drift p_2, claims σ_i and initial point $x_2 = u_2/\delta_2$.

Let us recall some basics of the theory of one-dimensional ruin – see e.g. [8] or [1]. For phase-type claims $(\boldsymbol{\beta},\boldsymbol{B})$, i.e. with $P[\sigma > x] = \boldsymbol{\beta}\mathrm{e}^{\boldsymbol{B}x}\mathbf{1}$, the ruin probability may be written in a simpler matrix exponential form:

$$\psi_2(x_2) = \boldsymbol{\eta}\mathrm{e}^{(\boldsymbol{B}+\boldsymbol{b}\boldsymbol{\eta})x_2}\mathbf{1},$$

with $\boldsymbol{\eta} = \frac{\lambda}{p_2}\boldsymbol{\beta}(-\boldsymbol{B})^{-1}$ (see for example (4) in [2]), and in the case of exponential claim sizes with intensity μ, it reduces to:

$$\psi_2(x_2) = C_2\mathrm{e}^{-\gamma_2 x_2},$$

where $\gamma_2 = \mu - \lambda/p_2$ and $C_2 = \frac{\lambda}{\mu p_2}$.

In the opposite case $u_2/\delta_2 > u_1/\delta_1$ however, when a barrier composed of two lines is involved, the problem is considerably harder. We will analyze only the case of exponential claim sizes.

As always in the case of phase-type jumps, it is possible to provide a "Feynman-Kac" differential system for the perpetual ruin probabilities (for example by embedding the semi-Markov jump process into a fluid model). The following result was shown in [5], the full version of this paper:

Theorem 1. *The vector $(\psi(u_1,u_2), \phi(u_1,u_2))^T$ containing the perpetual ruin probability as its first coordinate is the solutions of the Feynman-Kac system:*

$$\begin{pmatrix} c_1 & 0 \\ 0 & -\delta_1 \end{pmatrix}\begin{pmatrix} \psi_{u_1} \\ \phi_{u_1} \end{pmatrix} + \begin{pmatrix} c_2 & 0 \\ 0 & -\delta_2 \end{pmatrix}\begin{pmatrix} \psi_{u_2} \\ \phi_{u_2} \end{pmatrix} + \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with the boundary condition:

$$\begin{cases} \psi(u_1, \frac{\delta_2}{\delta_1}u_1) = C_2\mathrm{e}^{-\gamma_2\frac{\delta_2}{\delta_1}u_1}, & \textit{for all } u_1 \geq 0, \\ \phi(0,u_2) = 1, & \textit{for all } u_2 \geq 0. \end{cases}$$

The system above is also equivalent to the hyperbolic "telegraphic type" equation:

$$h_{rw} + \mu\lambda\ h = 0$$

with the boundary condition:

$$\begin{cases} h(r,0) = C_2 \mathrm{e}^{-(\gamma_2-\mu)\, r} & \text{for all } r \geq 0, \\ h_w(r, -\frac{\delta_1 r}{c_1}) = -\lambda\, \mathrm{e}^{\mu r - \lambda \frac{\delta_1 r}{c_1}} & \text{for all } r \geq 0, \end{cases}$$

where

$$h(r,w) := \mathrm{e}^{\mu r}\mathrm{e}^{-\lambda w}\ \psi(r,w).$$

Unfortunately, the solution of neither of these two formulations is obvious. The rescue came finally from obtaining an equation for the double Laplace transform in u_1, u_2 (Theorem 2) and inverting this transform in the complex plane using Bromwich type contours (Theorem 3).

§2. The Laplace transform

First note that process $(U_1(t)/\delta_1, U_2(t)/\delta_2)$ has the same ruin probability as the original two-dimensional process $(U_1(t), U_2(t))$. Thus it suffices to analyze the case when $\delta_1 = \delta_2 = 1$ and $c_1 = p_1$, $c_2 = p_2$. Let $x_i = u_i/\delta_i$. In this section we will write $\psi(x_1, x_2)$ for $\psi(u_1, u_2)$ under the above assumptions. By conditioning at the position of the process at time T (the time of crossing of the two lines forming the barrier $b(t)$):

$$T = T(x_1, x_2) = \frac{x_2 - x_1}{p_1 - p_2} \tag{2}$$

it was shown in [4] that if $x_2 > x_1$, it holds that

$$\overline{\psi}(x_1, x_2) := 1 - \psi(x_1, x_2) = \int_0^\infty \overline{\psi}_2(z)\, \widetilde{P}_{x_1,T}(dz), \tag{3}$$

where

$$\widetilde{P}_{x_1,T}(dx) = P_{x_1}(\inf_{s \leq T} U_1(s) > 0, U_1(T) \in dz)/dz$$

and

$$\overline{\psi}_2(z) := 1 - \psi_2(z) = 1 - C_2 \mathrm{e}^{-\gamma_2 z}. \tag{4}$$

Suprun [7] (see also Bertoin [3, Lem. 1]) gives the resolvent of a spectrally negative Lévy process killed as it enters the nonpositive half-line as

$$\begin{aligned} &\frac{1}{q} P_{x_1}(\inf_{s \leq e_q} U_1(s) > 0, U_1(e_q) \in dz)/dz \\ &\quad = \exp\{-q^+(q) z\} W^{(q)}(x_1) - \mathbf{1}_{\{x_1 \geq z\}} W^{(q)}(x_1 - z), \end{aligned} \tag{5}$$

where $q^+(q)$ largest root of $\kappa(\alpha) = q$ where the characteristic exponent κ is given in this case by

$$\kappa(\alpha) = p_1 \alpha - \frac{\lambda \alpha}{\mu + \alpha}$$

and $W^{(q)}:[0,\infty)\to[0,\infty)$, called the *q-scale function,* is continuous and increasing function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\alpha y}W^{(q)}(y)\,dy=(\kappa(\alpha)-q)^{-1},\qquad\qquad \alpha>q^+(q).$$

Now we obtain the Laplace transform of the non-ruin probability with respect to the initial reserves:

$$\tilde{\psi}(p,q)=\int_0^\infty\int_0^\infty \mathrm{e}^{-px_1}\mathrm{e}^{-qx_2}\,\overline{\psi}(x_1,x_2)\,dx_1\,dx_2.$$

Note that

$$\tilde{\psi}(p,q)=\int_0^\infty\int_0^{x_1}\mathrm{e}^{-px_1}\mathrm{e}^{-qx_2}\overline{\psi}(x_1,x_2)\,dx_2\,dx_1+\int_0^\infty\int_{x_1}^\infty \mathrm{e}^{-px_1}\mathrm{e}^{-qx_2}\overline{\psi}(x_1,x_2)\,dx_2\,dx_1.$$

The first Laplace transform is given by

$$\int_0^\infty\int_0^{x_1}\mathrm{e}^{-px_1}\mathrm{e}^{-qx_2}[1-C_2\mathrm{e}^{-\gamma_2x_2}]\,dx_2\,dx_1=\frac{1}{p}\,\frac{[(1-C_2)(p+q)+\gamma_2]}{(p+q)(p+q+\gamma_2)}:=A.$$

Writing $s=p+q$ and $r=(p_1-p_2)q$ we see from (3) and (5) that the second Laplace transform is given by

$$\begin{aligned}
&\int_0^\infty\int_{x_1}^\infty \mathrm{e}^{-px_1}\mathrm{e}^{-qx_2}\overline{\psi}(x_1,x_2)\,dx_2\,dx_1\\
&\quad=(p_1-p_2)\int_0^\infty\int_0^\infty \mathrm{e}^{-sx_1}\left[1-C_2\mathrm{e}^{-\gamma_2 z}\right]\left[\mathrm{e}^{-q^+(r)z}W^{(r)}(x_1)-\mathbf{1}_{\{z\le x_1\}}W^{(r)}(x_1-z)\right]dz\,dx_1\\
&\quad=\frac{p_1-p_2}{\kappa(s)-r}\left[\frac{(1-C_2)q^+(r)+\gamma_2}{q^+(r)(\gamma_2+q^+(r))}-\frac{1}{s}+\frac{C_2}{\gamma_2+s}\right]\\
&\quad=\frac{p_1-p_2}{\kappa(s)-r}\left[\frac{\gamma_2(q^+(r)/\mu+1)}{q^+(r)(\gamma_2+q^+(r))}-\frac{(\mu+p+q)(1-C_2)}{(p+q)(\gamma_2+p+q)}\right]\\
&\quad:=C-B.
\end{aligned}$$

Note that $A-B$ is equal to

$$\frac{(1-C_2)p_2}{p(\kappa(s)-r)}=\frac{(\mu+p+q)(p_2-\rho)}{pp_1(z_1(q)-p)(z_2(q)-p)},$$

where

$$z_1(q)=\frac{-(p_2q+p_1(q+\gamma_1))-\sqrt{(p_2q+p_1(q+\gamma_1))^2-4p_1qp_2(q+\gamma_2)}}{2p_1},$$

$$z_2(q)=\frac{-(p_2q+p_1(q+\gamma_1))+\sqrt{(p_2q+p_1(q+\gamma_1))^2-4p_1qp_2(q+\gamma_2)}}{2p_1}.$$

Similarly, C can be written as

$$\begin{aligned}
&\frac{p_1-p_2}{\kappa(s)-r}\,\frac{\gamma_2(q^+(r)/\mu+1)}{q^+(r)(\gamma_2+q^+(r))}\\
&\qquad\qquad=\frac{(\mu+p+q)(p_2-\rho)}{pp_1(z_1(q)-p)(z_2(q)-p)}\left[p\frac{(p_1-p_2)(q^+(q(p_1-p_2))+\mu)}{p_2(q^+(q(p_1-p_2))(\gamma_2+q^+(q(p_1-p_2))}\right].
\end{aligned}$$

Theorem 2. *The Laplace transform of the perpetual ruin probability $\psi(u_1,u_2)$ is given by*

$$\tilde{\psi}(p,q)=\frac{(\mu+p+q)(p_2-\rho)(1+ph(q))}{pp_1(p-z_1(q))(p-z_2(q))}, \tag{6}$$

where

$$h(q)=\frac{(p_1-p_2)(q^+(q(p_1-p_2))+\mu)}{p_2(q^+(q(p_1-p_2))(\gamma_2+q^+(q(p_1-p_2))}.$$

Noting that $q^+(q(p_1-p_2))$ is the largest root of $\kappa(\alpha)=q(p_1-p_2)$ and $z_2(q)$ is the largest root of $\kappa(v+q)=q(p_1-p_2)$ we identify

$$q^+(q(p_1-p_2))=z_2(q)+q.$$

Hence

$$h(q)=\frac{(p_1-p_2)(z_2(q)+q+\mu)}{p_2(q+z_2(q))(\gamma_2+q+z_2(q))}. \tag{7}$$

Let $h_\pm(q)$ be a $h(q)$ when we put $z^\pm$ instead of $z_2(q)$ and

$$z^-(q)=a(q)-ib(q), \qquad z^+(q)=a(q)+ib(q)$$

for

$$a(q)=\frac{-(p_1\mu-\lambda+p_2q+p_1q)}{2p_1},$$

$$b(q)=\frac{\sqrt{4p_1(p_2q\mu+p_2q^2-\lambda q)-(p_1\mu-\lambda+p_2q+p_1q)^2}}{2p_1}.$$

An explicit inversion is possible here - see full version of this paper [5]:

Theorem 3. *Let $x_2>x_1$. If $\rho\le\frac{p_2^2}{p_1}$ holds, then*

$$\overline{\psi}(x_1,x_2)=1-C_1\mathrm{e}^{-\gamma_1x_1}+\omega(x_1,x_2),$$

where

$$\omega(x_1,x_2)=\frac{1}{4\pi}\int_{q_+}^{q_-}(h^--h^+)(q)\left(f(z^+(q),q)-f(z^-(q),q)\right)\,dq$$

for

$$q_+=-\frac{1}{p_1-p_2}(\sqrt{\lambda}+\sqrt{p_1\mu})^2, \qquad q_-=-\frac{1}{p_1-p_2}(\sqrt{\lambda}-\sqrt{p_1\mu})^2$$

and

$$f(p,q)=\frac{(\mu+p+q)(p_2-\rho)}{p_1b(q)}\mathrm{e}^{px_1}\mathrm{e}^{qx_2}.$$

If $\rho>\frac{p_2^2}{p_1}$ holds, then

$$\overline{\psi}(x_1,x_2)=1-C_1\mathrm{e}^{-\gamma_1x_1}-C_2\mathrm{e}^{-\gamma_2x_1}+\frac{p_2}{p_1}\mathrm{e}^{-\gamma_3x_1}\mathrm{e}^{-\gamma_2x_2}+\omega(x_1,x_2),$$

where $\gamma_3=\frac{\mu}{p_2}\left(\rho-\frac{p_2^2}{p_1}\right)$.

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Florin Avram
Laboratoire de Mathématiques Appliquées
CNRS UMR 5142 - Université de Pau et des Pays de l'Adour
IPRA, B.P. 1155, 64013 Pau Cedex, France
Florin.Avram@univ-Pau.fr

Zbigniew Palmowski
University of Wroclaw,
pl. Grunwaldzki 2/4,
50-384 Wroclaw, Poland,
zpalma@math.uni.wroc.pl

Martijn Pistorius
Department of Mathematics,
King's College London,
Strand, London WC2R 2LS, UK,
pistoriu@mth.kcl.ac.uk