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Solvability of some strongly unilateral problems in L^1 without regularity condition on the obstacle

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Abstract. An existence result for the strongly nonlinear unilateral problems associated to the equation,

$$-\operatorname{div}(a(x,u,\nabla u)) + g(x,u,\nabla u) = f \in L^1(\Omega),$$

is proved without any regularity condition on the obstacle.

Keywords: Sobolev spaces, boundary value problems, truncations, unilateral problems.

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§1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$. Let $f \in L^1(\Omega)$. Consider the nonlinear Dirichlet problem associated to the following equation:

$$Au + g(x,u,\nabla u) = f, \tag{1}$$

where A is a Leray-Lions operator acting from $W_0^{1,p}(\Omega)$ into its dual and g is a nonlinearity satisfying a suitable conditions (cf. (H2)). Firstly, the variational case (i.e., where $f \in W^{-1,p'}(\Omega)$) of the unilateral problems associated to the equation (1) is studied in [4] by Bensoussane and al. While the case of L^1-data is treated in [5], but under the restrictions that $g \equiv 0$ and $a \equiv a(x,\nabla u)$. Note that, in all the latter works, the following regularity condition on the obstacle ψ,

$$\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \tag{2}$$

is supposed and plays a principal role to obtain the existence solution. Our purpose in this paper, is then to study the previous nonlinear unilateral problem but without assuming any regularity on the obstacle ψ and any coercivity on the nonlinearity g. To overcome those difficulties, we have introduced some more general coercivity (cf. (6)) and another complicated test function (cf. (18)). It would be interesting at this stage to refer the reader to the works [1, 2] in which, the degenerated case is studied but under the regularity condition (2).

§2. Basic assumptions and main results

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1 < p < \infty$. Given an obstacle (measurable) function $\psi : \Omega \to \overline{\mathbb{R}}$, we consider the convex set

$$K_\psi = \{u \in W_0^{1,p}(\Omega);\ u \geq \psi \ \ a.e. \text{ in } \ \Omega)\}, \tag{3}$$

such that $K_\psi \cap L^\infty(\Omega)$ is not empty. Note that K_ψ is obviously a closed subset of $W_0^{1,p}(\Omega)$. Suppose that $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following:

(H_1) **Growth and monotonicity conditions:**

$$|a_i(x,s,\xi)| \leq [k(x) + |s|^{p-1} + |\xi|^{p-1}], \text{ for } \ i = 1,\dots,N, \tag{4}$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0 \ \text{ for all } \ \xi \neq \eta \in \mathbb{R}^N. \tag{5}$$

There exist $\delta(x)$ in $L^1(\Omega)$ and a strictly positive constant α such that, for some fixed element v_0 in $K_\psi \cap L^\infty(\Omega)$

$$a(x,s,\zeta)(\zeta - \nabla v_0) \geq \alpha|\zeta|^p - \delta(x), \tag{6}$$

for a.e. x in Ω, all $s \in \mathbb{R}$ and all $\zeta \in \mathbb{R}^N$, where $k(x)$ is a positive function in $L^{p'}(\Omega)$.

(H_2) **Sign and natural growth conditions:** g is a Carathéodory function satisfying,

$$g(x,s,\xi).s \geq 0, \tag{7}$$

$$|g(x,s,\xi)| \leq b(|s|)(|\xi|^p + h(x)), \tag{8}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a positive increasing function and $h(x)$ is a positive function in $L^1(\Omega)$. We now introduce the functional spaces which will be used after,

$$\mathcal{T}_0^{1,p}(\Omega) = \left\{u :\ \Omega \to \mathbb{R} \ \text{ measurable}, \ \ T_k(u) \in W_0^{1,p}(\Omega) \ \text{ for all } \ k > 0\right\},$$

where $T_k(s) = \max(-k, \min(k,s))$.

Theorem 1. *Assume that (H_1) and (H_2) hold and $f \in L^1(\Omega)$. Then there exists at least one solution of the following unilateral problem,*

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega),\ u \geq \psi \ a.e. \ \text{ in } \ \Omega,\ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad \displaystyle\leq \int_\Omega f T_k(u-v)\,dx,\ \forall \ \ v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{cases} \tag{9}$$

Remark 1. We obtain the same result if we assume only that the sign condition (7) is verified at infinity, and also if the data is of the form $f - \mathrm{div}F$, with $f \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega))^N$. Then, the previous result holds true when the datum is a measure which does not charges a zero p-capacity set.

Proof of Theorem 1.
Step 1: Approximate problem. Let us approximate the nonlinearity by

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$$

and consider the approximate unilateral problems:

$$\begin{cases} u_n \in K_\psi, \\ \langle Au_n, u_n - v\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)(u_n - v)\,dx \le \int_\Omega f_n(u_n - v)\,dx, \ \ \forall v \in K_\psi, \end{cases} \tag{10}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$. We define the operators G_n and A in $W_0^{1,p}(\Omega)$ by,

$$\langle G_n u, v\rangle = \int_\Omega g_n(x,u,\nabla u)v\,dx \ \text{ and } \ \langle Au, v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v\,dx.$$

Thanks to a Hölder's inequality and growth conditions (4), we can easily show that A and G_n are bounded.

Proposition 2. *The operator $B_n = A + G_n$ from K_ψ into $W^{-1,p'}(\Omega)$ is pseudomonotone. Moreover, B_n is coercive in the following sense:*

$$\frac{< B_n v, v - v_0 >}{\|v\|} \longrightarrow +\infty \ \textit{ if } \ \|v\| \longrightarrow +\infty, \ v \in K_\psi.$$

This proposition will be proved below. In view of Proposition 2, the problem (10) has a solution by the classical result (cf. Theorem 8.2 in Chapter 2 of [8]).
Step 2: A priori estimates. In this step, we will prove an uniform estimate for the truncated solution $T_k(u_n)$. Let $k \ge \|v_0\|_\infty$ and let $\varphi_k(s) = se^{\gamma s^2}$, where $\gamma = (b(k)/\alpha)^2$. It is well known that,

$$\varphi_k'(s) - \frac{b(k)}{\alpha}|\varphi_k(s)| \ge \frac{1}{2}, \ \ \forall s \in \mathbb{R}. \tag{11}$$

Taking $u_n - \eta\varphi_k(T_l(u_n - v_0))$ ($\eta = e^{-\gamma l^2}$) as test function in (10), where $l = k + \|v_0\|_\infty$, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_l(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx \\ + \int_\Omega g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n - v_0))\,dx \le \int_\Omega f_n\varphi_k(T_l(u_n - v_0))\,dx.$$

Since $g_n(x,u_n,\nabla u_n)\varphi_k(T_l(u_n - v_0)) \ge 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$, then

$$\int_{\{|u_n - v_0| \le l\}} a(x,u_n,\nabla u_n)\nabla(u_n - v_0)\varphi_k'(T_l(u_n - v_0))\,dx \\ \le \int_{\{|u_n| \le k\}} |g_n(x,u_n,\nabla u_n)||\varphi_k(T_l(u_n - v_0))|\,dx + \int_\Omega f_n\varphi_k(T_l(u_n - v_0))\,dx.$$

By using (6), (8) and the fact that $\{x\in\Omega, |u_n(x)|\le k\}\subseteq\{x\in\Omega : |u_n-v_0|\le l\}$, we get

$$\int_\Omega |\nabla T_k(u_n)|^p\,dx \le 2C_k, \tag{12}$$

where C_k is a positive constant depending on k.
Step 3: Convergence in measure of u_n. Let $k_0\ge\|v_0\|_\infty$ and fix $k>k_0$. Taking $v=u_n-T_k(u_n-v_0)$ as a test function in (10), we get

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n-v_0)\,dx\\ \le\int_\Omega f_nT_k(u_n-v_0)\,dx.\end{aligned} \tag{13}$$

Since $g_n(x,u_n,\nabla u_n)T_k(u_n-v_0)\ge 0$ on the subset $\{x\in\Omega, |u_n(x)|>k_0\}$, then the previous inequality (13) and the natural growth (8) imply that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx\le kb(k_0)\left[\int_\Omega |h(x)|\,dx+\int_\Omega|\nabla T_{k_0}(u_n)|^p\,dx\right]+kC. \tag{14}$$

Therefore (12) and (14) allow to have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx\le k[C_{k_0}+C].$$

Thanks to (6) and since $\{x\in\Omega, |u_n(x)|\le k\}\subseteq\{x\in\Omega : |u_n-v_0|\le k+\|v_0\|_\infty\}$ and since k is arbitrary, we can deduce that

$$\int_\Omega |\nabla T_k(u_n)|^p\,dx\le kC_2. \tag{15}$$

Now, we aim to prove that u_n converges to some function u in measure. To do this, we use (15) and we follow the same argument as in [3]. Hence, we conclude that

$$\begin{aligned} &T_k(u_n)\rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega),\\ &T_k(u_n)\to T_k(u) \quad \text{strongly in } L^p(\Omega) \text{ and } a.e. \text{ in } \Omega.\end{aligned} \tag{16}$$

Indeed since $T_k(u_n)$ converge weakly for some $v_k\in W_0^{1,p}(\Omega)$ and since $u_n\to u$ $a.e.$ in Ω, then $T_k(u_n)\to T_k(u)$ $a.e.$ in Ω finally by applying the lemma 1.3 in [8, chap 1], we deduce (16). This yields, by using (4) that there exists a function $h_k\in(L^{p'}(\Omega))^N$, such that (for a subsequence denoted again u_n)

$$a(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup h_k \text{ weakly in } (L^{p'}(\Omega))^N \text{ as } n\to\infty. \tag{17}$$

Step 4: Almost everywhere convergence of the gradient. We fix $k>\|v_0\|_\infty$, and let $w_{n,h}=T_{2k}(u_n-v_0-T_h(u_n-v_0)+T_k(u_n)-T_k(u))$ and $w_h=T_{2k}(u-v_0-T_h(u-v_0))$, with $h>2k$. For $\eta=\exp(-4\gamma k^2)$, we define the following function as

$$v_{n,h}=u_n-\eta\,\varphi_k(w_{n,h}) \tag{18}$$

(which is an admissible function of (10) since T_k and φ_k opere in the space $W_0^{1,p}(\Omega)$ (cf [7, pp. 151-152]) and $v_{n,h} \geq \psi$). Now, taking $v_{n,h}$ as test function in (10), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,h}\varphi_k'(w_{n,h}) + g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx \leq \int_\Omega f_n\varphi_k(w_{n,h})\,dx. \tag{19}$$

Note that, $\nabla w_{n,h} = 0$ on the set where $|u_n| > h+5k$, therefore, setting $m = 5k+h$, and denoting by $\varepsilon_h^1(n), \varepsilon_h^2(n), \ldots$ various sequences of real numbers which converge to zero as n tends to infinity for any fixed value of h. By virtue of (19), since $u_n \to u$ *a.e.* in Ω and the fact that $g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h}) \geq 0$ on the set $\{x \in \Omega, |u_n(x)| > k\}$, we can write

$$\begin{aligned}\int_\Omega & a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx \\ &+ \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx \leq \int_\Omega f\varphi_k(w_h)\,dx + \varepsilon_h^1(n).\end{aligned} \tag{20}$$

Splitting the first integral on the left hand side of (20) where $|u_n| \leq k$ and $|u_n| > k$, we have

$$\begin{aligned}\int_\Omega & a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h})\varphi_k'(w_{n,h})\,dx \\ &= \int_{\{|u_n|\leq k\}} a(x,T_m(u_n),\nabla T_m(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi_k'(w_{n,h})\,dx \\ &\quad + \int_{\{|u_n|>k\}} a(x,T_m(u_n),\nabla T_m(u_n))\nabla w_{n,h}\varphi_k'(w_{n,h})\,dx.\end{aligned} \tag{21}$$

Following the same argument as in [1], we get

$$\begin{aligned}\int_\Omega & a(x,T_m(u_n),\nabla T_m(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\varphi_k'(w_{n,h})\,dx \\ &\geq \int_\Omega \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big]\big[\nabla T_k(u_n) - \nabla T_k(u)\big]\varphi_k'(w_{n,h})\,dx \\ &\quad - \varphi_k'(2k)\int_{\{|u-v_0|>h\}} \delta(x)\,dx + \varepsilon_h^6(n).\end{aligned} \tag{22}$$

We now turn to the second term of the left hand side of (20), using (6), we have

$$\begin{aligned}\left|\int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx\right| &\leq b(k)\int_\Omega (c(x) + |\nabla T_k(u_n)|^p|\varphi_k(w_{n,h})|\,dx \\ &\leq b(k)\int_\Omega c(x)|\varphi_k(w_{n,h})|\,dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_{n,h})| \\ &\quad + \tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi_k(w_n)|\,dx \\ &\quad - \tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla v_0|\varphi_k(w_{n,h})|\,dx.\end{aligned}$$

Following again the same techniques as in [1], we can prove that

$$\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)\varphi_k(w_{n,h})\,dx\right| \\
&\quad\le \int_\Omega \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big][\nabla T_k(u_n)-\nabla T_k(u)]|\varphi_k(w_{n,h})|\,dx \\
&\quad + b(k)\int_\Omega c(x)|\varphi_k(w_h|\,dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_h)|\,dx \\
&\quad - \tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx + \varepsilon_h^8(n).
\end{aligned}\tag{23}$$

Combining (20), (22) and (23), we obtain

$$\begin{aligned}
&\int_\Omega \big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\big] \\
&\qquad\times\big[\nabla T_k(u_n)-\nabla T_k(u)\big]\left(\varphi_k'(w_{n,h}) - \tfrac{b(k)}{\alpha}|\varphi_k(w_{n,h})|\right)dx \\
&\quad\le b(k)\int_\Omega c(x)|\varphi_k(w_h|\,dx + \tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_h)|\,dx \\
&\qquad - \tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx + \int_\Omega f(x)\varphi_k(w_h)\,dx + \varepsilon_h^9(n).
\end{aligned}\tag{24}$$

By (11) we have $\left(\varphi_k'(w_{n,h}) - \frac{b(k)}{\alpha}|\varphi_k(w_{n,h})|\right) \ge \frac{1}{2}$, hence the monotonicity (5) gives

$$\begin{aligned}
&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\times[\nabla T_k(u_n)-\nabla T_k(u)]\,dx \\
&\quad\le 2b(k)\int_\Omega c(x)|\varphi_k(w_h|\,dx + 2\tfrac{b(k)}{\alpha}\int_\Omega \delta(x)|\varphi_k(w_h)|\,dx \\
&\qquad - 2\tfrac{b(k)}{\alpha}\int_\Omega h_k\nabla v_0|\varphi_k(w_h)|\,dx + 2\int_\Omega f(x)\varphi_k(w_h)\,dx + \varepsilon_h^{10}(n).
\end{aligned}\tag{25}$$

Hence, passing to the limit over n and h and invoking lemma 5 of [6], we deduce that

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega). \tag{26}$$

For $\delta > 0$ we can write

$$\begin{aligned}
\text{meas}\{|\nabla u_n - \nabla u| > \delta\} \le\ & \text{meas}\{|u_n| > k\} + \text{meas}\{|u| > k\} \\
& + \text{meas}\{|\nabla T_k(u_n) - \nabla T_k(u)| > \delta\}.
\end{aligned}$$

Reasonings as in [3], we can deduce that each term on the right hand side of the least inequality is less than $\frac{\varepsilon}{3}$. Thus $\nabla u_n \to \nabla u$ in measure, which give again, for a subsequence,

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega, \tag{27}$$

which yields

$$\begin{cases} a(x,u_n,\nabla u_n) \to a(x,u,\nabla u) \text{ a.e. in } \Omega, \\ g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \text{ a.e. in } \Omega. \end{cases} \tag{28}$$

Step 4: Equi-integrability of the nonlinearity. We need to prove that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega). \tag{29}$$

For that, we take $u_n - T_1(u_n - v_0 - T_h(u_n - v_0))$ (with h large enough) as test function in (10) and we conclude by Vitali's theorem.
Step 5: Passage to the limit. Let $v \in K_\psi \cap L^\infty(\Omega)$. Taking $u_n - T_k(u_n - v)$ as test function in (10) and using Fatou's Lemma, we can pass to the limit. This proves Theorem 1. $\square$

Proof of Proposition 2.
The coercivity follows from $(6),(7)$ and the fact that G_n is bounded. It remains to show that B_n is pseudo-monotone. Let a sequence $(u_k) \in W_0^{1,p}(\Omega)$ such that

$$u_k \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad \limsup_{k\to+\infty}\langle B_n u_k, u_k - u\rangle \leq 0. \tag{30}$$

Let $v \in W_0^{1,p}(\Omega)$. We will prove that

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k - v\rangle \geq \langle B_n u, u - v\rangle.$$

Since (u_k) is a bounded sequence in $W_0^{1,p}(\Omega)$, we deduce that $(a(x,u_k,\nabla u_k))_k$ (resp. $(g_n(x,u_k,\nabla u_k))_k$) is bounded in $(L^{p'}(\Omega))^N$ (resp. $(L^{p'}(\Omega))$), then there exists a function $h \in (L^{p'}(\Omega))^N$ (resp. $\rho_n \in L^{p'}(\Omega)$) such that (for a subsequence denoted again (u_k)),

$$\begin{cases} u_k \longrightarrow u \text{ strongly in } L^p(\Omega), \\ a(x,u_k,\nabla u_k) \rightharpoonup h \text{ weakly in } (L^{p'}(\Omega))^N, \\ g_n(x,u_k,\nabla u_k) \rightharpoonup \rho_n \text{ weakly in } L^{p'}(\Omega). \end{cases} \tag{31}$$

The monotonicity condition (5) and (31) allow to get

$$\begin{aligned} \liminf_{k\to+\infty}\langle B_n u_k, u_k - v\rangle &= \liminf_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u_k\,dx - \langle h,\nabla v\rangle + \langle \rho_n, u - v\rangle \\ &\geq -\liminf_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u)\nabla u\,dx + \liminf_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla u\,dx \\ &\quad + \liminf_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u)\nabla u_k\,dx - \langle h,\nabla v\rangle + \langle \rho_n, u - v\rangle. \end{aligned} \tag{32}$$

Hence, after using the convergence (31) we can write

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k - v\rangle \geq \langle \rho_n, u - v\rangle + \langle h, \nabla(u - v)\rangle \tag{33}$$

(for a subsequence, which holds for the all sequence u_k by applying a standard contradiction argument). Now, since v is arbitrary and $\lim\limits_{k\to+\infty}\langle G_n u_k, u_k - u\rangle = 0$, we have by using (30) and (33)

$$\lim_{k\to+\infty}\int_\Omega a(x,u_k,\nabla u_k)\nabla(u_k - u)\,dx = 0.$$

We deduce that

$$\lim_{k\to+\infty}\int_\Omega (a(x,u_k,\nabla u_k)-a(x,u_k,\nabla u))\nabla(u_k-u)\,dx=0.$$

In view of Lemma 5 of [6], we have $\nabla u_k \to \nabla u$ a.e. in Ω, and applying Lemma 1.3 [8, Chap. 1] which with (33) yields

$$\liminf_{k\to+\infty}\langle B_n u_k, u_k-v\rangle \geq \langle B_n u, u-v\rangle.$$

This completes the proof of the proposition. □

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