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ANALYSIS OF A COUPLED PARABOLIC-HYPERBOLIC PROBLEM

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Abstract. This paper deals with the mathematical analysis of a quasilinear parabolic-hyperbolic problem in a multidimensional bounded domain Ω. In a region Ω_p a diffusion-advection-reaction type equation is set while in the complementary $\Omega_h \equiv \Omega \setminus \Omega_p$, only advection-reaction terms are taken into account. To begin we provide the definition of a weak solution through an entropy inequality on the whole domain. Since the interface $\partial\Omega_p \cap \partial\Omega_h$ contains outward characteristics for the first-order operator in Ω_h, the uniqueness proof starts by considering first the hyperbolic zone and then the parabolic one. The existence property uses the vanishing viscosity method and to pass to the limit on the hyperbolic zone, we refer to the notion of process solution.

Keywords: Coupling problem, conservation laws, measure valued solution.

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§1. Introduction

We are interested in a coupling of a quasilinear parabolic equation with an hyperbolic first-order equation in a bounded domain Ω of $\mathbb{R}^n$, $n \geq 1$. This problem arises from several simplified physical models like infiltration processes in a stratified subsoil viewed as two layers with different geological characteristics and such that in the second layer we can neglect the effects of diffusivity (see also [4]). For any positive and finite T, the problem in hand reads as follows: find a measurable and bounded function u on $Q \equiv]0,T[\times\Omega$ such that

$$\partial_t u - \sum_{i=1}^{n} \partial_{x_i}(K_h(u)\partial_{x_i}P) + g_h(t,x,u) = 0 \text{ in } Q_h, \tag{1}$$

$$\partial_t u - \sum_{i=1}^{n} \partial_{x_i}(K_p(u)\partial_{x_i}P) + g_p(t,x,u) = \Delta\phi(u) \text{ in } Q_p, \tag{2}$$

$$u = 0 \text{ on }]0,T[\times\partial\Omega, \quad u(0,.) = u_0 \text{ on } \Omega, \tag{3}$$

with the geometrical configuration $\overline{\Omega} = \overline{\Omega}_h \cup \overline{\Omega}_p$ and $\Omega_h \cap \Omega_p = \emptyset$. Besides, for l in $\{h,p\}$, $Q_l =]0,T[\times\Omega_l$. Of course, suitable conditions across the interface between the two regions Q_p and Q_h are needed. If we refer to the analysis of F. Gastaldi and *al.* in [4], these transmission conditions have to be formally written:

$$-K_h(u)\nabla P.\nu_h = (\nabla\phi(u) + K_p(u)\nabla P).\nu_p \text{ on } \Sigma_{hp}, \tag{4}$$

where $\Sigma_{hp} =]0,T[\times\Gamma_{hp}$ and $\Gamma_{hp} = \Gamma_h \cap \Gamma_p$ and $\Gamma_l = \partial\Omega_l$. Moreover ν_l denotes the outward normal unit vector defined $\mathscr{H}^n$-a.e. on Σ_l and for q in $[0,n+1]$, $\mathscr{H}^q$ is the q-dimensional Hausdorff measure. Lastly $\mathscr{H}^{n-1}(\overline{\Gamma_{hp}} \cap (\overline{\Gamma_l \setminus \Gamma_{hp}})) = 0$.

1.1. Main assumptions on data

The pressure field P belongs to $W^{2,+\infty}(\Omega)$ and fulfills $\Delta P = 0$ which is not restrictive as soon as (1) and (2) involve reaction terms. In addition

$$\Gamma_{hp} \subset \{\bar{\sigma} \in \Gamma_h \,;\, \nabla P(\bar{\sigma}).\nu_h \leq 0\}. \tag{5}$$

For l in $\{h, p\}$, the reaction term g_l belongs to $W^{1,+\infty}(]0,T[\times\Omega_l \times \mathbb{R})$ and we set

$$M'_{g_l} = \operatorname*{ess\,sup}_{(t,x,u)\in]0,T[\times\Omega_l\times\mathbb{R}} |\partial_u g_l(t,x,u)|,$$

and the initial data u_0 belongs to $L^\infty(\Omega)$. Thus we are able to define the nondecreasing time-depending function

$$M : t \in [0,T] \to M(t) = \|u_0\|_{L^\infty(\Omega)} e^{M_1 t} + \frac{M_2}{M_1}(e^{M_1 t} - 1),$$

where $M_1 = M'_{g_h} + M'_{g_p}$ and

$$M_2 = \max_{]0,T[\times\Omega_h} |g_h(t,x,0)| + \max_{]0,T[\times\Omega_p} |g_p(t,x,0)|.$$

To simplify we write $M = M(T)$ and we assume that the following local hypotheses are fulfilled:

i) K_p is a Lipschitzian function on $[-M,M]$ while K_h is a *nondecreasing* Lipschitzian function on $[-M,M]$ with constants respectively K'_p and K'_h . Besides, thanks to a translation argument, it is not a restriction to suppose that $K_h(0) = K_p(0) = 0$.

ii) ϕ is a nondecreasing Lipschitzian function on $[-M,M]$ such that ϕ^{-1} exists, this including the case when $\mathscr{L}(\{x \in [-M,M], \phi'(x) = 0\}) = 0$, where $\mathscr{L}$ refers to the Lebesgue measure on $\mathbb{R}$. Furthermore $\phi(0) = 0$.

Remark 1. The monotonicity of K_h and (5) show that on the transmission zone $K'_h \nabla P.\nu_h \leq 0$. That way Σ_{hp} is really included in the set of outward characteristics for the first-order operator in the hyperbolic domain and along the interface the information is leaving the hyperbolic domain. This essential property will guide us for the statement of uniqueness by first considering the behavior of a solution in the hyperbolic area and then in the parabolic one.

1.2. Notations and functional spaces

In the sequel, σ (resp. $\bar{\sigma}$) is variable of Σ_i (resp. Γ_i), $i \in \{h, hp, p\}$. This way, $\sigma = (t, \bar{\sigma})$ for any t of $[0,T]$. It will be referred to the Hilbert space

$$W(0,T) \equiv \{v \in L^2(0,T;H^1_0(\Omega))\,;\, \partial_t v \in L^2(0,T;H^{-1}(\Omega))\},$$

used with the graph's norm. We denote $\langle .,. \rangle$ the pairing between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$. Besides, we also need to consider the Hilbert space

$$V = \{v \in H^1(\Omega_p)\,;\, v = 0 \text{ a.e. on } \Gamma_p \setminus \Gamma_{hp}\},$$

used with $\|v\|_V = \|\nabla v\|_{L^2(\Omega_p)^n}$. We denote $\langle\langle .,.\rangle\rangle$ the pairing between V and V'.

Lastly, to simplify the writing,

$$\begin{aligned}
F(x,u,k) &= \operatorname{sign}(u-k)\big((K_p(u)-K_p(k))\,\mathbb{I}_{\Omega_p}(x) + (K_h(u)-K_h(k))\,\mathbb{I}_{\Omega_h}(x)\big)\\
&\equiv F_p(u,k)\,\mathbb{I}_{\Omega_p}(x) + |K_h(u)-K_h(k)|\,\mathbb{I}_{\Omega_h}(x),\\
G(t,x,u,k) &= \operatorname{sign}(u-k)g_p(t,x,u)\,\mathbb{I}_{\Omega_p}(x) + \operatorname{sign}(u-k)g_h(t,x,u)\,\mathbb{I}_{\Omega_h}(x)\\
&\equiv G_p(u,k)\,\mathbb{I}_{\Omega_p}(x) + G_h(u,k)\,\mathbb{I}_{\Omega_h}(x),\\
\mathscr{F}_h(a,b,c) &= \frac{1}{2}\{|K_h(a)-K_h(b)| - |K_h(c)-K_h(b)| + |K_h(a)-K_h(c)|\}.
\end{aligned}$$

In this framework, we set:

Definition 1. A function u is a weak solution to the coupling Problem (1)–(4) if and only if u belongs to $L^\infty(Q)$, $\phi(u)$ to $L^2(0,T;V)$ and:

i) for all $\varphi \in \mathscr{D}(]-\infty,T[\times\Omega)$, $\varphi \geq 0$, and for any real k,

$$\begin{aligned}
\int_Q |u-k|\,\partial_t\varphi\,dx\,dt - \int_{Q_p} \nabla\,|\phi(u)-\phi(k)|\,.\nabla\varphi\,dx\,dt + \int_\Omega |u_0-k|\,\varphi(0,.)\,dx\\
- \int_Q F(x,u,k)\nabla P.\nabla\varphi\,dx\,dt - \int_Q G(t,x,u,k)\varphi\,dx\,dt \geq 0,
\end{aligned} \tag{6}$$

ii) for all $\zeta \in L^1(\Sigma_h \setminus \Sigma_{hp})$, $\zeta \geq 0$, and for any real k,

$$\operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_h\setminus\Sigma_{hp}} \mathscr{F}_h(u(\sigma+\tau\nu_h),0,k)\nabla P(\bar{\sigma}).\nu_h\zeta\,d\mathscr{H}^n \leq 0. \tag{7}$$

§2. The uniqueness property

The proofs of all what follows may be found in [1]

2.1. Study in the hyperbolic zone

We derive from (6) and (7) and by using (5) an entropy inequality on the hyperbolic domain that will be the starting point to establish a time-Lipschitzian dependence in $L^1(\Omega_h)$ of a weak solution to (1)-(4) with respect to the corresponding initial data. To begin let us state:

Proposition 1. *Let u be a bounded and measurable function on Q satisfying (6) and (7). Then for any real k and any φ of $\mathscr{D}(]0,T[\times\mathbb{R}^n)$, $\varphi \geq 0$,*

$$\begin{aligned}
-\int_{Q_h} \big(|u-k|\,\partial_t\varphi - |K_h(u)-K_h(k)|\,\nabla P.\nabla\varphi - G_h(u,k)\varphi\big)\,dx\,dt\\
\leq \int_{\Sigma_h\setminus\Sigma_{hp}} |K_h(k)|\,\nabla P(\bar{\sigma}).\nu_h\varphi(\sigma)\,d\mathscr{H}^n\\
- \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_h\setminus\Sigma_{hp}} |K_h(u(\sigma+\tau\nu_h))|\,\nabla P(\bar{\sigma}).\nu_h\varphi(\sigma)\,d\mathscr{H}^n.
\end{aligned} \tag{8}$$

From Proposition 1 and thanks to the method of doubling variables, we derive:

Theorem 2. *Let u_1 and u_2 be two bounded and measurable functions on Q_h satisfying (6) and (7) respectively for initial data $u_{0,1}$ and $u_{0,2}$. Then*

$$\text{for a.e. } t \text{ in } \left]0,T\right[,\ \int_{\Omega_h} |u_1(t,.) - u_2(t,.)|\,dx \leq e^{M'_{g_h} t} \int_{\Omega_h} |u_{0,1} - u_{0,2}|\,dx.$$

2.2. Study in the parabolic zone

By restricting (6) on the parabolic zone, we give first some information on the regularity for $\partial_t u$. Then we characterize u on Q_p through a variational equality including an advection term that corresponds to entering data from the hyperbolic zone. Indeed:

Proposition 3. *Let u be a bounded function on Q such that $\nabla\phi(u)$ belongs to $L^2(Q_p)^n$ and satisfying (6). Then $\partial_t u$ belongs to $L^2(0,T;V')$. Furthermore, for any φ in $L^2(0,T;V)$,*

$$\begin{aligned} &\int_0^T \langle\langle \partial_t u, \varphi \rangle\rangle\,dt + \int_{Q_p} \nabla\phi(u).\nabla\varphi\,dx\,dt + \int_{Q_p} K_p(u)\nabla P.\nabla\varphi\,dx\,dt \\ &\quad + \int_{Q_p} g_p(t,x,u)\varphi\,dx\,dt + \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} K_h(u(\sigma+\tau\nu_h))\nabla P(\bar{\sigma}).\nu_h \varphi\,d\mathscr{H}^n = 0. \end{aligned} \tag{9}$$

2.3. The uniqueness theorem

Theorem 2 ensures a uniqueness property on the hyperbolic zone. On the parabolic one, the lack of regularity of the time partial derivative of a weak solution to (1)–(4) requires a doubling the time variable. Furthermore, to deal with the convective term, we assume that there exists a real θ in $]1/2,+\infty[$ and a positive constant $\mathscr{C}$ such that

$$\forall (x,y) \in [-M,M]^2,\ \left|(K_p\circ\phi^{-1})(x) - (K_p\circ\phi^{-1})(y)\right| \leq \mathscr{C}\,|x-y|^\theta. \tag{10}$$

Then we have:

Theorem 4. *Assume that (10) holds. Then (1)–(4) admits at most one weak solution.*

§3. Existence Property

3.1. The Viscous Problem

We obtain an existence result for (1)–(4) through the vanishing viscosity method that consists here in introducing, for any positive ε, $\phi_\varepsilon = \phi + \varepsilon\mathbb{I}_\mathbb{R}$ and the next formal problem : find a bounded and measurable function u_ε on Q such that

$$\partial_t u_\varepsilon - \sum_{i=1}^n \partial_{x_i}(K_h(u_\varepsilon)\partial_{x_i}P) + g_h(t,x,u_\varepsilon) = \varepsilon\Delta\phi_\varepsilon(u_\varepsilon) \ \text{ in } \ Q_h, \tag{11}$$

$$\partial_t u_\varepsilon - \sum_{i=1}^n \partial_{x_i}(K_p(u_\varepsilon)\partial_{x_i}P) + g_p(t,x,u_\varepsilon) = \Delta\phi_\varepsilon(u_\varepsilon) \ \text{ in } \ Q_p, \tag{12}$$

$$u_\varepsilon = 0 \ \text{ on } \ \Sigma, \quad u_\varepsilon(0,.) = u_0 \ \text{ in } \ \Omega, \tag{13}$$

and to have a well-posed problem, we express the transmission conditions across Σ_{hp}:

$$-\big(\varepsilon\nabla\phi_\varepsilon(u_\varepsilon)+K_h(u_\varepsilon)\nabla P\big).\nu_h = \big(\nabla\phi_\varepsilon(u_\varepsilon)+K_p(u_\varepsilon)\nabla P\big).\nu_p \ \text{ on } \ \Sigma_{hp},$$
$$u_\varepsilon|_{Q_h} = u_\varepsilon|_{Q_p} \ \text{ on } \ \Sigma_{hp}.$$

Our objective is first to establish that for a fixed ε, Problem (11)–(13) has a unique weak solution u_ε. Secondly, we look for *a priori* estimates for $(u_\varepsilon)_{\varepsilon>0}$ that are sufficient to study its limit when ε goes to 0^+. To begin we state:

Theorem 5. *For any positive ε, there exists a unique weak solution u_ε to the regularized problem (11)–(13) in $W(0,T)\cap L^\infty(Q)$. This solution fulfills*

$$u_\varepsilon(0,.) = u_0 \ \textit{ a.e. in } \ \Omega, \tag{14}$$

and the variational equality, for any v in $H_0^1(\Omega)$ and for a.e. t in $]0,T[$,

$$\langle \partial_t u_\varepsilon, v\rangle + \int_\Omega \big(\lambda_\varepsilon(x)\nabla\phi_\varepsilon(u_\varepsilon).\nabla v + K(x,u_\varepsilon)\nabla P.\nabla v + g(t,x,u_\varepsilon)v\big)\,dx = 0, \tag{15}$$

where to simplify the writing:

$$K(x,u) = K_h(u)\,\mathbb{I}_{\Omega_h}(x) + K_p(u)\,\mathbb{I}_{\Omega_p}(x), \quad \lambda_\varepsilon(x) = \varepsilon\,\mathbb{I}_{\Omega_h}(x) + \mathbb{I}_{\Omega_p}(x),$$
$$g(t,x,u) = g_h(t,x,u)\,\mathbb{I}_{\Omega_h}(x) + g_p(t,x,u)\,\mathbb{I}_{\Omega_p}(x).$$

In addition, $\sqrt{t}\,\partial_t u_\varepsilon$ is an element of $L^2(Q)$.

Proof. Some commentaries: the existence property for (14)–(15) uses the Schauder-Tychonoff fixed point theorem and the uniqueness statement is based on an Holmgren-type duality method. This duality method is also used to derive the additional local smoothness property for the time-partial derivative of u_ε. □

Let us now mention the *a priori* estimates satisfied by the sequence of viscous solutions $(u_\varepsilon)_{\varepsilon>0}$ that are the starting point to study its limit when ε goes to 0^+. The lack of regularity of the initial data but also the fact that the diffusive term depends on the space variable through λ_ε only allow us to establish:

Proposition 6. *There exists a constant C independent from ε such that*

$$\|u_\varepsilon\|_{L^\infty(Q)} \le M \big\|(\lambda_\varepsilon)^{1/2}\nabla\phi_\varepsilon(u_\varepsilon)\big\|_{L^2(Q)^n} \le C, \quad \|\partial_t u_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \le C.$$

3.2. The viscous limit

A priori estimates collected in Proposition 6 are not sufficient to derive a compactness argument proper to study the behavior of $(u_\varepsilon)_{\varepsilon>0}$ and characterize the corresponding limit. That is why we assume that

$$\phi^{-1} \text{ is Hölder continuous with an exponent } \theta \text{ in }]0,1[. \tag{16}$$

With this assumption and by referring to the arguments put forward in [3, Chapter 2], we state:

Proposition 7. *Under (16) there exists a measurable function u of* $L^\infty(Q)$ *with* $\phi(u)$ *in* $L^2(0,T;V)$ *such that, up to a subsequence when* ε *goes to* 0^+, *the sequence* $(u_\varepsilon)_{\varepsilon>0}$ *converges towards u in* $L^\infty(Q)$ *weak* $*$, *in* $L^q(Q_p)$ *for any finite q and a.e. on* Q_p. *Besides we also have*

$$\nabla\phi_\varepsilon(u_\varepsilon) \rightharpoonup \nabla\phi(u) \ \textit{ weakly in } \ L^2(Q_p)^n, \quad \varepsilon\nabla\phi_\varepsilon(u_\varepsilon) \to 0^+ \ \textit{ strongly in } \ L^2(Q_h)^n.$$

We characterize u on the hyperbolic zone by taking advantage of the maximum principle for u_ε (see Proposition 6) and of

Claim 8. *(see [2].) Let* $\mathscr{O}$ *be an open bounded subset of* $\mathbb{R}^q$ $(q \geq 1)$ *and* $(u_n)_{n>0}$ *a sequence of measurable functions on* $\mathscr{O}$ *such that:*

$$\exists M > 0,\ \forall n > 0,\ \|u_n\|_{L^\infty(\mathscr{O})} \leq M.$$

Then, there exist a subsequence $(u_{\varphi(n)})_{n>0}$ *and a measurable function* π *in* $L^\infty(]0,1[\times\mathscr{O})$ *such that, for all continuous and bounded functions* f *on* $\mathscr{O}\times]-M,M[$,

$$\forall\xi \in L^1(\mathscr{O}),\ \lim_{n\to+\infty}\int_{\mathscr{O}} f(x,u_{\varphi(n)})\xi\,dx = \int_{]0,1[\times\mathscr{O}} f(x,\pi(\alpha,w))\,d\alpha\,\xi\,dx.$$

Here by considering the sequence of solutions to viscous problems (11)–(13) we prove that

Theorem 9. *The coupled parabolic-hyperbolic problem (1)–(4) has at least a weak solution u that is the limit in* $L^q(Q)$, $1 \leq q < +\infty$, *and a.e. on Q of the whole sequence of solutions to viscous Problems (11)–(13) when* ε *goes to* 0^+.

Proof. We consider the function u highlighted in Proposition 7. Since $(u_\varepsilon|_{\Omega_h})_{\varepsilon>0}$ is uniformly bounded, there exist a subsequence – still labelled $(u_\varepsilon|_{\Omega_h})_{\varepsilon>0}$ – and a measurable and bounded function π – called a *process* – on $]0,1[\times Q_h$ such that, for any continuous bounded function ψ on $Q_h\times]-M,M[$ and for any ξ of $L^1(Q_h)$,

$$\lim_{\varepsilon\to0^+}\int_{Q_h}\psi(t,x,u_\varepsilon)\xi\,dx\,dt = \int_{]0,1[\times Q_h}\psi(t,x,\pi(\alpha,t,x))\xi\,d\alpha\,dx\,dt. \tag{17}$$

Our aim is first to establish that, on the hyperbolic zone, the *process* π is reduced to $u|_{\Omega_h}$, independently from α in $]0,1[$, and secondly to prove that u is a weak solution to (1)–(4) for initial data u_0. We reach these two objectives by considering a family of boundary entropy-entropy flux pair $(H_i,Q_{i,l})$, $i\in\{1,2\}$, $l\in\{h,p\}$, defined for any m in $\mathbb{N}^*$ any real k through

$$H_1(z,k) = \left((z-k)^2 + \left(\frac{1}{m}\right)^2\right)^{1/2} - \frac{1}{m},$$

$$H_2(z,k) = \left(\left(\operatorname{dist}(z,\mathscr{I}(0,k))\right)^2 + \left(\frac{1}{m}\right)^2\right)^{1/2} - \frac{1}{m},$$

$$Q_{i,l}(z,k) = \int_k^z \partial_1 H_i(\tau,k)K_l'(\tau)\,d\tau.$$

We take the scalar product in $L^2(]0,T[\times\Omega)$ between

$$\partial_t u_\varepsilon - \operatorname{div}\big(\lambda_\varepsilon(x)\nabla\phi_\varepsilon(u_\varepsilon) + K(x,u_\varepsilon)\nabla P\big) + g(t,x,u_\varepsilon) = 0 \text{ a.e. on } Q,$$

and the function $\partial_1 H_i(u_\varepsilon,k)\zeta_i$, where ζ_1 belongs to $\mathscr{D}(]-\infty,T[\times\Omega)$, $\zeta_1 \geq 0$ while ζ_2 is in $\mathscr{D}(]-\infty,T[\times\mathbb{R}^n)$, $\zeta_2 \geq 0$. As for any i in $\{1,2\}$, $\partial_1 H_i(u_\varepsilon,k)\zeta_i = 0$ on $\partial\Omega$. We take the ε-limit separately on the parabolic and hyperbolic zones by using (17) and Proposition 7. For $i=1$ and ζ_1 in $\mathscr{D}(]-\infty,T[\times\Omega_h)$, the limit with m provides ($dq \equiv d\alpha\, dx\, dt$):

$$-\int_{]0,1[\times Q_h}\big(|\pi-k|\,\partial_t\zeta_1 - F_h(x,\pi,k)\nabla P.\nabla\zeta_1 - G_h(\pi,k)\zeta_1\big)\,dq \leq \int_{\Omega_h}|u_0-k|\,\zeta_1(0,.)\,dx.$$

For $i=2$ and ζ_2 in $\mathscr{D}(Q_h)$, it comes:

$$-\int_{]0,1[\times Q_h}\big(H_2(\pi,k)\partial_t\zeta_2 - Q_{2,h}(\pi,k)\nabla P.\nabla\zeta_2 - G_{2,h}(\pi,k)\zeta_2\big)\,dq \leq 0,$$

where, for l in $\{h,p\}$,

$$G_{i,l}(\pi,k) = g_l(t,x,\pi)\partial_1 H_i(\pi,k).$$

At this point we adapt F. Otto's reasoning providing that

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{]0,1[\times\Sigma_h\setminus\Sigma_{hp}} Q_{2,h}(\pi(\alpha,\sigma+\tau\nu),k)\nabla P(\bar{\sigma}).\nu\zeta\,d\alpha\,d\mathscr{H}^n \leq 0,$$

for any ζ of $L^1_+(\Sigma_h\setminus\Sigma_{hp})$. Condition (7) for π follows by observing that $(Q_{2,h})_{l\in\mathbb{N}^*}$ converges uniformly to $\mathscr{F}_h(z,0,k)$ as m goes to $+\infty$.

Eventually, the process π fulfills (7) and (8), where the integrations with respect to the Lebesgue measure on $\Sigma_h\setminus\Sigma_{hp}$, Ω_h and Q_h are respectively turned into integrations with respect to the Lebesgue measure on $]0,1[\times\Sigma_h\setminus\Sigma_{hp}$, $]0,1[\times\Omega_h$ and $]0,1[\times Q_h$. This way, as a consequence of Theorem 2, if π_1 and π_2 are two process solutions for initial data $u_{0,1}$ and $u_{0,2}$ respectively, then for a.e. t in $]0,T[$,

$$\int_{]0,1[\times\Omega_h}|\pi_1(\alpha,t,x)-\pi_2(\beta,t,x)|\,d\alpha\,d\beta\,dx\,dt \leq \int_{\Omega_h}|u_{0,1}-u_{0,2}|\,dx\,e^{M'_{g_h}t}.$$

Classically we first deduce that, when $u_{0,1}=u_{0,2}$ on Ω_h, there exists a function u_h on Q_h such that, a.e. on Q_h, $u_h(.) = \pi_1(\alpha,.) = \pi_2(\beta,.)$ for a.e. α and β in $]0,1[$. Another consequence of the uniqueness property is that the whole sequence $(u_\varepsilon)_{\varepsilon>0}$ strongly converges to u_h in $L^q(Q_h)$, $1\leq q<+\infty$. Thus $u_h = u|_{\Omega_h}$ a.e. on Q_h and u fulfills (6)–(7). □

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