Monografías Matemáticas *García de Galdeano* **42**, 297–306 (2019)

PERIODIC SOLUTIONS FOR IMPULSIVE DIFFERENTIAL EQUATIONS

José Manuel Uzal

Abstract. In this note we present some results on the existence of periodic solutions for some impulsive differential equations. Two different problems will be considered. First, a first order differential equations with the possible presence of singularities and impulses is studied. The impulses are assumed to happen on the position and at instants of time fixed beforehand. Second, a second order differential equation is considered with state-dependent impulses at both the position and its derivative. This means that the instants of impulsive effects depend on the solutions and they are not fixed beforehand, making the study of this problem more difficult.

Keywords: impulsive differential equations, periodic solutions.

AMS classification: 34B37, 34A37.

§1. Introduction

Some evolutions processes are subject to sudden changes. The mathematical description of these processes leads to impulsive differential equations. The changes are assumed to be instantaneous, since their length is negligible in comparison with the duration of the process. Thus, solutions of impulsive differential equations are, in general, piecewise continuous functions. Furthermore, the existence of impulsive effects could cause complicated phenomena. This type of differential equations can describe population dynamics, biological phenomena or several physical situations [12]. Moreover, impulses can be introduced on a system to generate a particular dynamic (for example periodic motions) or to control a process.

There are two large classes of impulsive differential equations, with impulses at fixed times or with state-dependent impulses. On the first class, the moments of impulsive effect are known beforehand. Techniques and tools used on the classical theory of differential equations can sometimes be generalized and applied to this case rather easily. On the second case, the times of impulsive effect change depending on the solution, making its study much more difficult, because the space of solutions does not have such good properties and some solutions could have unexpected behaviors. We refer the reader to [1, 7, 11, 12] for some results and applications of the impulsive differential equations.

In this note we will study two different problems. First, we consider a first order impulsive problem with impulses at fixed times and singularities. For example, the following problem could be considered

$$\begin{aligned} &x'(t) = -\frac{1}{(x(t))^{\alpha}} + e(t), \quad t \neq t_k; \\ &\Delta x(t_k) = I_k(x(t_k)); \\ &x(0) = x(T). \end{aligned} \tag{1.1}$$

Here $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $x(t_k^-), x(t_k^+)$ denote the limits of x as t approaches t_k from the left and right, respectively. The main difficulty of this problem is the presence of the term $1/x^\alpha$, because it makes it difficult to find a region where the possible solutions could be located. Differential equations with singularities have been studied in recent years because they appear in a lot of physical models [13], and the introduction of impulses makes the number of applications even larger, although its study could become much more difficult.

We present a result on the existence of periodic solutions for a problem much more general that (1.1) and including a large class of nonlinearities.

The second problem considered is a more classical second order differential equation. In this case, the presence of state-dependent impulses is studied. This makes its study much more difficult. Our aim is to guarantee the existence of periodic solutions of

$$\begin{aligned} &x''(t) + g(x(t)) = p(t, x(t), x'(t)), && t \neq \gamma_i(x(t), x'(t)); \\ &x(t^+) = x(t) + I_i(x(t), x'(t)), && t = \gamma_i(x(t), x'(t)); \\ &x'(t^+) = x(t) + J_i(x(t), x'(t)), && t = \gamma_i(x(t), x'(t)); \end{aligned} \tag{1.2}$$

This type of problems is harder because the moments of impulse depend on the solution of the differential equation. For example, the equation $t = \gamma_i(x(t), x'(t))$ could have no solutions, one solution or infinitely many; and the solutions of this equation need not to depend continuously on an initial data.

The rest of this note is organized as follows: in Section 2 we state some general facts about impulsive differential equations. In Section 3 we state our existence result for problem (1.1) and in Section 4 for problem (1.2).

§2. General facts about impulsive differential equations

Let A be a subset of $\mathbb{R}^n$, $f : \mathbb{R} \times A \longrightarrow \mathbb{R}^n$, $\gamma_i : A \longrightarrow \mathbb{R}$ and $\phi_i : A \longrightarrow A$. An impulsive differential equation is an expression of the form

$$\begin{aligned} &x'(t) = f(t, x(t)), && t \neq \gamma_i(x(t)); \\ &x(t^+) = \phi_i(x(t)), && t = \gamma_i(x(t)). \end{aligned} \tag{2.1}$$

There are mainly two large classes of impulsive differential equations:

- Equations with fixed moments of impulsive effect: in this case, γ_i is a constant function, i.e., $\gamma_i(x) = t_i$. The moments of impulsive effect are fixed and they are the same for every solution. Then, (2.1) can be written as

$$\begin{aligned} &x'(t) = f(t, x(t)), && t \neq t_i; \\ &x(t_i^+) = \phi_i(x(t_i)). \end{aligned}$$

  A problem of this type will be considered in Section 3. The solutions of this problem are piecewise continuous functions with (possible) discontinuities at t_i.

- Equations with unfixed moments of the impulsive effect: these equations have the form (2.1) with γ_i non-constant functions. The moments of the impulsive effect occur when the point (t, x) meets a "hypersurface" given by $t = \gamma_i(x)$. The points of discontinuity depend on the solution, and sometimes solutions can not be extended over a large interval, especially if the solution intersects a hypersurface $t = \gamma_i(x)$ more than once. Therefore, it is interesting to impose some hypotheses in order to ensure that solutions intersect each hypersurface only once. For simplicity, in Section 4 we consider the case with just one hypersurface.

We state some general results for equations with unfixed moments of the impulsive effect. Consider

$$\begin{aligned} x'(t) &= f(t, x(t)), \quad t \neq \gamma(x(t)); \\ x(t^+) &= \phi(x(t)), \quad\ \ t = \gamma(x(t)). \end{aligned} \tag{2.2}$$

The following hypotheses will be needed in Section 4:

1. $f : \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function and locally Lipschitz in the second variable and all the solutions of $u' = f(t, u)$ exist for all $t \in \mathbb{R}$;
2. $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous function;
3. $\gamma : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\gamma \in C^1(\mathbb{R}^n, \mathbb{R})$, there exist $\gamma_-, \gamma_+ \in (0, T)$ such that $\gamma_- < \gamma_+$ and $0 < \gamma_- \leq \gamma(x) \leq \gamma_+ < T \quad \forall\, x \in \mathbb{R}^n$;
4. $f(t, x) = f(t + T, x) \quad \forall\, (t, x) \in \mathbb{R}\times\mathbb{R}^n$;
5. $\gamma(x) > \gamma(\phi(x)) \quad \forall\, x \in \mathbb{R}^n$;
6. $D\gamma(x)\cdot f(t, x) < 1 \quad \forall\, (t, x) \in \mathbb{R}\times\mathbb{R}^n$.

Under these hypotheses the following lemmas hold:

Lemma 1. *For any $x_0 \in \mathbb{R}^n$, there is a unique solution $x(\cdot; 0, x_0)$ of* (2.2) *satisfying $x(0) = x_0$.*

Lemma 2. *For any $x_0 \in \mathbb{R}^n$, there is a unique $t_{x_0} \in (0, T)$ such that $t_{x_0} = \gamma(x(t_{x_0}))$.*

Lemma 3. *The map $\Gamma : x_0 \in \mathbb{R}^n \longrightarrow t_{x_0} \in (0, T)$ is continuous.*

Lemma 4. *The map $P : x_0 \in \mathbb{R}^n \longrightarrow x(T; x_0) \in \mathbb{R}^n$ is continuous.*

Proof. Let

$$\begin{aligned}
&f_1 : \zeta \in \mathbb{R}^n \longrightarrow (\zeta, \zeta) \in \mathbb{R}^n\times\mathbb{R}^n;\\
&f_2 : (\zeta, \sigma) \in \mathbb{R}^n\times\mathbb{R}^n \longrightarrow (t_\zeta, \sigma) \in [0, T]\times\mathbb{R}^n;\\
&f_3 : (t, \zeta) \in [0, T]\times\mathbb{R}^n \longrightarrow (t, x(t; 0, \zeta)) \in [0, T]\times\mathbb{R}^n;\\
&f_4 : (t, \zeta) \in [0, T]\times\mathbb{R}^n \longrightarrow (t, \varphi(\zeta)) \in [0, T]\times\mathbb{R}^n;\\
&f_5 : (t, \zeta) \in [0, T]\times\mathbb{R}^n \longrightarrow x(T; t, \zeta) \in \mathbb{R}^n.
\end{aligned}$$

Each of these functions is continuous and $P(x_0) = (f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(x_0)$. □

§3. A first order problem

In this section we study the existence of a T-periodic solution of

$$x'(t) = -\frac{1}{(x(t))^\alpha} + e(t) \tag{3.1}$$

under impulsive effects

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \tag{3.2}$$

with $\alpha > 0$, $T > 0$, $e : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and T-periodic, $I_k : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = T$. We reduce the previous problem to a boundary value problem, so by a T-periodic solution of (3.1)–(3.2) we understand a piecewise continuous function $u : [0, T] \longrightarrow (0, \infty)$, with discontinuities at the points t_k, $u(0) = u(T)$ and satisfying (3.1) and (3.2).

Instead of considering (3.1)–(3.2), we are going to study

$$\begin{aligned} &x'(t) = f(x(t)) + e(t), \quad t \neq t_k;\\ &\Delta x(t_k) = I_k(x(t_k));\\ &x(0) = x(T). \end{aligned} \tag{3.3}$$

In this case, $f : (0, \infty) \longrightarrow (a, b)$ is a continuous function with $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$.

In order to prove the existence of periodic solutions [8] we use a classical result due to Mawhin [5]. We briefly present some definitions and results.

Definition 1. Let X and Y be two normed vector spaces and consider the linear mapping $L : D(L) \subset X \longrightarrow Y$. L is called a Fredholm mapping of index 0 if $\mathrm{Im}(L)$ is a closed subset of Y and $\dim(\ker(L)) = \mathrm{codim}(\mathrm{Im}(L)) < \infty$.

If L is a Fredholm mapping of index 0, there exist two projectors $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ such that $\mathrm{Im}(P) = \ker(L)$ and $\mathrm{Im}(L) = \ker(Q) = \mathrm{Im}(I - Q)$. This implies that $L_{|D(L)\cap\ker(P)} : (I - P)X \longrightarrow \mathrm{Im}(L)$ is an invertible map, and its inverse will be denoted by K_P.

Definition 2. Let $N : X \longrightarrow Y$ be a continuous map between two normed spaces and Ω an open bounded subset of X. We say that N is L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \longrightarrow X$ is a compact map.

Theorem 5. *Let X and Y be two Banach spaces, $L : D(L) \subset X \longrightarrow Y$ a Fredholm mapping of index* 0*, Ω an open bounded subset of X and $N : \overline{\Omega} \subset X \longrightarrow Y$ L-compact on $\overline{\Omega}$. Suppose that*

1. $Lx \neq \lambda Nx \quad \forall\, x \in \partial\Omega \cap D(L), \quad \forall\, \lambda \in (0, 1)$*;*
2. $QNx \neq 0 \quad \forall\, x \in \partial\Omega \cap \ker(L)$*;*
3. $\deg(JQN, \Omega \cap \ker(L), 0) \neq 0$*, where $J : \mathrm{Im}(Q) \longrightarrow \ker(L)$ is an isomorphism and* deg *represents the Brouwer's degree.*

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

We introduce the following hypotheses:

(H1) $\lim\limits_{s\to 0^+} f(s) = a^+$, $\lim\limits_{s\to\infty} f(s) = b^-$.

(H2) There exist $m_k, M_k \in \mathbb{R}$ such that $m_k \leq I_k(s) \leq M_k \quad \forall\, s > 0$.

(H3) If $c_1 = \dfrac{-m_1 - \cdots - m_q}{T} - \dfrac{1}{T}\displaystyle\int_0^T e(t)\,dt$ and $c_2 = \dfrac{-M_1 - \cdots - M_q}{T} - \dfrac{1}{T}\displaystyle\int_0^T e(t)\,dt$, then $c_1, c_2 \in (a, b)$.

(H4) For $\widetilde{M}_k = \max\{|M_k|, |m_k|\}$ and $r_2 = \inf\{s > 0 : f(s) \geq c_2\}$, it holds that

$$r_2 - 2\left(\widetilde{M}_1 + \cdots + \widetilde{M}_q\right) - \int_0^T e(t) + |e(t)|\, dt > 0.$$

We define

$$\begin{aligned} X = \{&x : [0,T] \longrightarrow \mathbb{R} \mid x(0) = x(T), x \text{ continuous except at } t_k, \\ &\text{there exist } x(t_k^-), x(t_k^+) \text{ and } x(t_k) = x(t_k^-)\}; \\ \|x\| &= \sup\{|x(t)| : t \in [0,T]\} \\ Y &= X \times \mathbb{R}^q; \\ Lx &= (g_1, \Delta x(t_1), \cdots, \Delta x(t_q)), \text{ with } g_1(t) = x'(t); \\ Nx &= (g_2, I_1(x(t_1)), \cdots, I_q(x(t_q))), \text{ with } g_2(t) = f(x(t)) + e(t). \end{aligned}$$

Lemma 6. *Suppose that hypotheses (H1)–(H4) are satisfied. Then there exist two positive constants A_1 and A_2 such that $A_2 \leq x(t) \leq A_1$ for all $t \in [0,T]$, and for x any solution of the equation $Lx = \lambda Nx$, $\lambda \in (0,1]$. The constants A_1 and A_2 are independent of λ.*

Proof. Let $x \in X$ with $\min\{x(t) : t \in [0,T]\} > 0$ such that there exists $\lambda \in (0,1)$ with

$$\begin{cases} x'(t) = \lambda f(x(t)) + \lambda e(t), \ t \in [0,T], \ t \neq t_k; \\ \Delta x(t_k) = \lambda I_k(x(t_k)), \ k \in \{1, \ldots, q\}. \end{cases}$$

Integrating over $[0,T]$ we obtain

$$\int_0^T x'(t)\, dt = \lambda \int_0^T f(x(t))\, dt + \lambda \int_0^T e(t)\, dt.$$

The first integral is equal to

$$\int_0^T x'(t)\, dt = \sum_{k=1}^{q+1} \int_{t_{k-1}^+}^{t_k^-} x'(t)\, dt = -x(0) - \sum_{k=1}^{q} (x(t_k^+) - x(t_k^-)) + x(T) = -\lambda \sum_{k=1}^{q} I_k(x(t_k)).$$

We can deduce

$$(m_1 + \cdots + m_q) + \int_0^T e(t)\, dt \leq \int_0^T -f(x(t))\, dt \leq (M_1 + \ldots + M_q) + \int_0^T e(t)\, dt.$$

by using hypothesis (H2). We obtain that there exist $\xi, \eta \in [0,T] \setminus \{t_1, \ldots, t_q\}$ such that

$$-Tf(x(\xi)) \leq (M_1 + \cdots + M_q) + \int_0^T e(t)\, dt, \qquad -Tf(x(\eta)) \geq (m_1 + \cdots + m_q) + \int_0^T e(t)\, dt$$

by using the mean value theorem for definite integrals. Then $f(x(\xi)) \geq c_2$ and $f(x(\eta)) \leq c_1$, so there exist $r_1, r_2 > 0$ such that $x(\xi) \geq r_2$ and $x(\eta) \leq r_1$ (r_2 as defined by (H4) and r_1 analogously). It can be checked that

$$x(t) \leq r_1 + \left(\widetilde{M}_1 + \cdots + \widetilde{M}_q\right) + \int_0^T |x'(u)|\, du, \quad x(t) \geq r_2 - \left(\widetilde{M}_1 + \cdots + \widetilde{M}_q\right) - \int_0^T |x'(u)|\, du$$

Furthermore,

$$\int_0^T |x'(t)|\,dt \le \widetilde{M}_1 + \cdots + \widetilde{M}_q + \int_0^T e(t)\,dt + \int_0^T |e(t)|\,dt$$

This implies that

$$A_2 := r_2 - 2\sum_{k=1}^{q} \widetilde{M}_k - \int_0^T e(t) + |e(t)|\,dt \le x(t) \le A_1 := r_1 + 2\sum_{k=1}^{q} \widetilde{M}_k + \int_0^T e(t) + |e(t)|\,dt$$

and $A_2 > 0$ by hypothesis (H4). Therefore the lemma is proved. □

Theorem 7. *Suppose that hypotheses (H1)–(H4) are satisfied. Then problem* (3.3) *has at least one solution.*

Proof. We define

$$\Omega = \{x \in X : \min\{x(t) : t \in [0,T]\} > A_2 - \sigma_2,\ A_2 - \sigma_2 < \|x\| < A_1 + \sigma_1\},$$

with $0 < \sigma_2 < A_2$ and $\sigma_1 > 0$ two constants. The set Ω is bounded and open, $QN(\overline{\Omega})$ is bounded and $(K_P(I-Q)N)(\overline{\Omega})$ is relatively compact. Furthermore, for each $\lambda \in (0,1)$

$$Lx = \lambda Nx \Longrightarrow A_2 \le x(t) \le A_1 \quad \forall\, t \in [0,T] \Longrightarrow x \notin \partial\Omega.$$

Define $J : (b, 0, \ldots, 0) \in \mathrm{Im}(Q) \longrightarrow b \in \ker(L)$ an isomorphism. We must prove that $QNx \neq 0$ for every $x \in \partial\Omega \cap \ker(L)$ and $\deg(JQN, \Omega \cap \ker(L), 0)$ is not equal to 0.

Let $x \in \ker(L)$ with $QNx = 0$. We must check that $x \notin \partial\Omega$.

$$QNx = 0 \Longrightarrow \frac{1}{T}\int_0^T \big[f(x(t)) + e(t)\big]\,dt + \frac{1}{T}\sum_{k=1}^{q} I_k(x(t_k)) = 0,$$

$$x \in \ker(L) \Longrightarrow x \text{ constant} \Longrightarrow x(t) = x(0) \quad \forall\, t \in [0,T].$$

We obtain from the previous equations that

$$-f(x(0)) = \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}\sum_{k=1}^{q} I_k(x(0)).$$

This implies that $c_2 \le f(x(0)) \le c_1$ by hypothesis (H3). Then we can conclude the following: $A_2 - \sigma_2 < A_2 \le r_2 \le x(t) \le r_1 \le A_1 < A_1 + \sigma_1$, which implies that $x \notin \partial\Omega$.

We identify $\ker(L) \cap \Omega$ with the interval $(A_2 - \sigma_2, A_1 + \sigma_1)$ of $\mathbb{R}$. Then the degree of JQN in $\Omega \cap \ker(L)$ with respect to 0 is $\deg(\varphi, (p,q), 0)$, where $(p,q) = (A_2 - \sigma_2, A_1 + \sigma_1)$ and the function $\varphi : [p,q] \longrightarrow \mathbb{R}$ is given by

$$\varphi(x) = f(x) + \frac{1}{T}\int_0^T e(t)\,dt + \frac{1}{T}\sum_{k=1}^{q} I_k(x).$$

It can be proved that $\varphi(p) < 0 < \varphi(q)$. Then $\deg(\varphi, (p,q), 0) \neq 0$ using properties of the Brouwer's degree. Therefore we can use Theorem 5, so there exists $x \in D(L) \cap \overline{\Omega}$ such that $Lx = Nx$, which implies that the impulsive boundary value problem (3.3) has one positive T-periodic solution. □

§4. A second order problem

In this section we study the following second order differential equation

$$x''(t) + g(x(t)) = p(t, x(t), x'(t)) \tag{4.1}$$

with $g : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and $p : \mathbb{R}^3 \longrightarrow \mathbb{R}$ bounded, continuous and T-periodic on the first variable. We consider state-dependent impulses

$$\begin{aligned} x(t^+) &= x(t) + I_i(x(t), x'(t)) \\ x'(t^+) &= x(t) + J_i(x(t), x'(t)) \end{aligned} \tag{4.2}$$

when $t = \gamma_i(x(t), x'(t))$, $i \in \{1, \ldots, q\}$. Here $I_i, J_i, \gamma_i \in C(\mathbb{R}^2, \mathbb{R})$. For simplicity we consider $q = 1$.

There are few existence results for periodic problems with state-dependent impulses, some examples include [2, 4, 10, 11].

In order to prove the existence of periodic solutions, the idea is to reduce (4.1)–(4.2) to a first order planar system and to consider the map P defined in Lemma 4. Then a T-periodic solution would be a fixed point of P.

The first idea was to use Poincaré-Birkhoff fixed point theorem, which states that every area-preserving, orientation-preserving homeomorphism of an annulus that rotates the boundaries in opposite directions has at least two fixed points. There are some extensions of this result. It has been applied to second-order problems and to second order problems with impulses at fixed times (see [3, 6, 9]). We were not able to apply it to our problem other than in some trivial cases.

Consider the following simplification of a partial extension stated in [9]:

Theorem 8. *Let* Γ_- *and* Γ_+ *be two closed and convex curves surrounding the origin,* $\operatorname{int}(\Gamma_+)$ *the interior of* Γ_+ *in the sense of Jordan curve theorem,* $\mathcal{A}$ *the annulus bounded by* Γ_- *and* Γ_+ *and* $F : \overline{\operatorname{int}(\Gamma_+)} \longrightarrow \mathbb{R}^2$ *a continuous map. We denote*

$$E = \{z \in \mathcal{A} : |F(z)| \leq |z|\},$$

with $U(O)$ *a neighborhood of the origin and L a real orthogonal matrix with* $\det(L) = 1$.

If $\gamma : [a, b] \longrightarrow \mathbb{R}^2$ *is a curve connecting* Γ_- *and* Γ_+ *and* $\gamma([a, b]) \cap (J \cup E)$ *is nonempty, then F has at least one fixed point.*

The proof of this result is based on Brouwer's degree and some of its properties. We apply this result to our problem. First, we define a very important family of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$, which will be fundamental in the proof of our result.

Definition 3. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a continuous map. We say that the map F has the property of partial boundedness if there is a bounded set $D \subset \mathbb{R}^2$, a convex cone and a curve $\Gamma : \lambda \in [0, \infty) \longrightarrow (x(\lambda), y(\lambda)) \in \mathbb{R}^2$, contained in the cone, such that

$$\lim_{\lambda \to \infty} (|x(\lambda)| + |y(\lambda)|) = +\infty \quad \text{and} \quad (F \circ \Gamma)([0, \infty)) \subset D.$$

The associated first-order planar system of (4.1) is

$$\begin{aligned} x' &= y \\ y' &= -g(x) + p(t,x,y). \end{aligned} \tag{4.3}$$

This system has been widely studied. Suppose that g and p are locally Lipschitz and

$$\lim_{x\to+\infty} g(x) = +\infty; \quad \lim_{x\to-\infty} g(x) = -\infty. \tag{g_0}$$

Consider the autonomous differential equation $x'' + g(x) = 0$ and define

$$G(x) = \int_0^x g(s)\,ds.$$

We have that G is bounded from below, has an absolute minimum and $G(x)$ goes to ∞ as $|x| \longrightarrow \infty$.

It can be checked that if $\zeta > 0$ is large enough and $x(0) = \zeta$, $y(0) = 0$, then the solutions are periodic and the least period of this solution, $\tau(\zeta)$, is given by the expression

$$\tau(\zeta) = \sqrt{2}\int_{h(\zeta)}^{\zeta} \frac{1}{\sqrt{c - G(y)}}\,dy,$$

where $h(\zeta)$ is a negative number such that $G(h(\zeta)) = G(\zeta)$ and $c = G(\zeta)$. Assume the following hypothesis:

$$\lim_{\zeta\to+\infty} \tau(\zeta) = 0. \tag{τ_0}$$

We can use polar coordinates on (4.3) for sufficiently large r, so

$$\begin{cases} \theta' = -\sin^2\theta - \dfrac{(g(r\cos\theta) - p(t, r\cos\theta, r\sin\theta))\cos\theta}{r}, \\ r' = r\cos\theta\sin\theta - (g(r\cos\theta) - p(t, r\cos\theta, r\sin\theta))\sin\theta. \end{cases} \tag{4.4}$$

Given an initial condition (r_0, θ_0), with r_0 sufficiently large, let $(r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))$ be the solution of (4.4) verifying the initial condition (r_0, θ_0) at time $t = 0$.

The proof of the following lemma is a consequence of results that can be found on [3].

Lemma 9. *Suppose* (g_0) *is satisfied. Then there exists* $d > 0$ *sufficiently large such that*

$$r_0 > d \Longrightarrow \frac{d}{dt}\theta(t; r_0, \theta_0) < 0 \quad \forall\, \theta_0 \in \mathbb{R}.$$

Furthermore, there exists a continuous and non-decreasing function β *from* $[d, \infty)$ *to* $(0, \infty)$ *such that*

$$r_0 > d \Longrightarrow \frac{d}{dt}\theta(t; r_0, \theta_0) \geq -\beta(r_0) \quad \forall\, \theta \in \mathbb{R}.$$

Define $n_*(r, t)$ and $n^*(r, t)$ as the two non-negative integers such that for any solution of (4.3) with initial values $\sqrt{x(0)^2 + y(0)^2} = r$, the solution makes at least $n_*(r, t)$ and at most $n^*(r, t)$ turns around the origin on the interval $[0, t]$.

Lemma 10. *Let $t \in (0, T]$. If (g_0) and (τ_0) are satisfied, then*

$$\lim_{r \to \infty} n_*(r, t) = +\infty.$$

Lemma 11. *Suppose (g_0) and (τ_0) are satisfied and $t \in (0, T]$. Then*

$$\forall\, N \in \mathbb{N},\ \exists\, \rho > 0 : r_0 > \rho \Longrightarrow \theta(t; r_0, \theta_0) - \theta_0 < -2N\pi \quad \forall\, \theta_0 \in \mathbb{R}.$$

Next, we state and prove the existence of T-periodic solutions for problem (4.1)–(4.2).

Theorem 12. *Suppose (g_0), (τ_0) and the six hypotheses in Section 2 are satisfied and let*

$$\phi : (x, y) \in \mathbb{R}^2 \longrightarrow (x + I_1(x, y), y + J_1(x, y)) \in \mathbb{R}^2.$$

If ϕ has the property of partial boundedness, then P has at least one fixed point, that is, there exists at least one T-periodic solution of

$$\begin{aligned} x''(t) + g(x(t)) &= p(t, x(t), x'(t)), & t &\neq \gamma(x(t), x'(t)); \\ x(t^+) &= x(t) + I_1(x(t), x'(t)), & t &= \gamma(x(t), x'(t)); \\ x'(t^+) &= x(t) + J_1(x(t), x'(t)), & t &= \gamma(x(t), x'(t)). \end{aligned} \tag{4.5}$$

Proof. There exist D a compact subset $\mathbb{R}^2$, a convex cone and a curve Γ starting at the origin, $\Gamma : \lambda \in [0, \infty) \longrightarrow (x(\lambda), y(\lambda)) \in \mathbb{R}^2$ contained in the cone such that

$$\lim_{\lambda \to \infty} (|x(\lambda)| + |y(\lambda)|) = +\infty \qquad \text{and} \qquad (\phi \circ \Gamma)([0, \infty)) \subset D.$$

The function f_5 as defined in the proof of Lemma 4 is also continuous. There exists $M_D > 0$ such that $|f_5(t, x)| \leq M_D$ for all $(t, x) \in [\gamma_-, \gamma_+] \times D$.

Take R_1 and R_2 sufficiently large with $R_2 > R_1 > M_D$ and satisfying

$$\theta(\gamma_+; \theta_0, R_1) - \theta(0; \theta_0, R_1) > -a, \qquad \theta(\gamma_-; \theta_0, R_2) - \theta(0; \theta_0, R_2) < -a - 4\pi.$$

for some $a > 0$. We can restrict ourselves to $\theta \in [0, 2\pi]$. Then we have that

$$\theta(t_{(\theta_0, R_2)}; \theta_0, R_2) - \theta(t_{(\theta_1, R_1)}; \theta_1, R_1) < -2\pi \tag{4.6}$$

for $\theta_0, \theta_1 \in [0, 2\pi)$, with $t_{(\theta_i, R_j)}$ the unique impulsive point given by Lemma 2. Take the curves $C_i = \{z \in \mathbb{R}^2 : |z| = R_i\}$, $i \in \{1, 2\}$. In order to use Theorem 8, let $\beta : I \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ a curve connecting C_1 and C_2, with z_1 and z_2 its initial and final points. The associated curve $\tilde{\beta}(t) = (P_2 \circ f_3 \circ f_2 \circ f_1 \circ \beta)(t)$ makes at least one turn around the origin because of (4.6), where $P_2 : [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ denotes the projection. So the curves $\tilde{\beta}$ and Γ intersect at least in one point. Let z be that point. Then $\phi(z) \in D$ and furthermore $|P(z)| \leq M_D < |z|$. This implies that the map P satisfies the hypotheses of Theorem 8, so $P : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ has at least one fixed point, which implies that there exists a T-periodic solution of (4.5). □

References

[1] BAINOV, D., AND SIMEONOV, P. *Impulsive Differential Equations: Periodic Solutions and Applications*. CRC Press, 1993.

[2] BAJO, I., AND LIZ, E. Periodic boundary value problem for first order differential equations with impulses at variable times. *J. Math. Anal. Appl. 204*, 1 (1996), 65–73. doi:10.1006/jmaa.1996.0424.

[3] DING, T. R., AND ZANOLIN, F. Periodic solutions of Duffing's equations with superquadratic potential. *J. Differ. Equ. 97*, 2 (1992), 328–378. doi:10.1016/0022-0396(92)90076-Y.

[4] FRIGON, M., AND O'REGAN, D. First order impulsive initial and periodic problems with variable moments. *J. Math. Anal. Appl. 233*, 2 (1999), 730–739. doi:10.1006/jmaa.1999.6336.

[5] GAINES, R. E., AND MAWHIN, J. L. *Coincidence Degree and Nonlinear Differential Equations*. Springer, 1977. doi:10.1007/BFb0089537.

[6] JACOBOWITZ, H. Periodic solutions of $x'' + f(x,t) = 0$ via the Poincaré-Birkhoff theorem. *Journal of Differential Equations 20*, 1 (1976), 37–52. doi:10.1016/0022-0396(76)90094-2.

[7] LAKSHMIKANTHAM, V., BAINOV, D. D., AND SIMEONOV, P. S. *Theory of Impulsive Differential Equations*, vol. 6. World scientific, 1989. doi:10.1142/0906.

[8] NIETO, J. J., AND UZAL, J. M. Pulse positive periodic solutions for some classes of singular nonlinearities. *Appl. Math. Lett. 86* (2018), 134–140. doi:10.1016/j.aml.2018.06.025.

[9] QIAN, D., CHEN, L., AND SUN, X. Periodic solutions of superlinear impulsive differential equations: a geometric approach. *J. Differ. Equ. 258*, 9 (2015), 3088–3106. doi:10.1016/j.jde.2015.01.003.

[10] RACHŮNKOVÁ, I., AND TOMEČEK, J. *State-Dependent Impulses: Boundary Value Problems on Compact Interval*, vol. 6. Springer, 2015.

[11] SAMOILENKO, A. M., AND PERESTYUK, N. A. *Impulsive Differential Equations*, vol. 14 of *World Scientific Series on Nonlinear Science.* World Scientific Publishing, 1995. doi:10.1142/9789812798664.

[12] STAMOVA, I., AND STAMOV, G. *Applied Impulsive Mathematical Models*. CMS Books in Mathematics. Springer International Publishing, 2016. doi:10.1007/978-3-319-28061-5.

[13] TORRES, P. J. *Mathematical models with singularities*, vol. 1 of *Atlantis Briefs in Differential Equations*. Atlantis Press, Paris, 2015.

J. M. Uzal
Departamento de Estatística, Análise Matemática e Optimización
Universidade de Santiago de Compostela
josemanuel.uzal@rai.usc.es

MONOGRAFÍAS DEL SEMINARIO MATEMÁTICO GARCÍA DE GALDEANO

Desde 2001, el Seminario ha retomado la publicación de la serie *Monografías* en un formato nuevo y con un espíritu más ambicioso. El propósito es que en ella se publiquen tesis doctorales dirigidas o elaboradas por miembros del Seminario, actas de congresos en cuya organización participe o colabore el Seminario, y monografías en general. En todos los casos, se someten al sistema habitual de arbitraje anónimo.

Los manuscritos o propuestas de publicaciones en esta serie deben remitirse a alguno de los miembros del Comité editorial. Los trabajos pueden estar redactados en español, francés o inglés.

Las monografías son recensionadas en *Mathematical Reviews* y en *Zentralblatt MATH*.

Últimos volúmenes de la serie:

21. A. Elipe y L. Floría (eds.): *III Jornadas de Mecánica Celeste*, 2001, ii + 202 pp., ISBN: 84-95480-21-2.

22. S. Serrano Pastor: *Modelos analíticos para órbitas de satélites artificiales de tipo quasi-spot*, 2001, vi + 76 pp., ISBN: 84-95480-35-2.

23. M. V. Sebastián Guerrero: *Dinámica no lineal de registros electrofisiológicos*, 2001, viii + 251 pp., ISBN: 84-95480-43-3.

24. Pedro J. Miana: *Cálculo funcional fraccionario asociado al problema de Cauchy*, 2002, 171 pp., ISBN: 84-95480-57-3.

25. Miguel Romance del Río: *Problemas sobre Análisis geométrico convexo*, 2002, xvii + 214 pp., ISBN: 84-95480-76-X.

26. Renato Álvarez-Nodarse: *Polinomios hipergeométricos y q-polinomios*, 2003, vi + 341 pp., ISBN: 84-7733-637-7.

27. M. Madaune-Tort, D. Trujillo, M. C. López de Silanes, M. Palacios y G. Sanz (eds.): *VII Jornadas Zaragoza-Pau de Matemática Aplicada y Estadística*, 2003, xxvi + 523 pp., ISBN: 84-96214-04-4.

28. Sergio Serrano Pastor: *Teorías analíticas del movimiento de un satélite artificial alrededor de un planeta. Ordenación asintótica del potencial en el espacio fásico*, 2003, 164 pp., ISBN: 84-7733-667-9.

29. Pilar Bolea Catalán: *El proceso de algebrización de organizaciones matemáticas escolares*, 2003, 260 pp., ISBN: 84-7733-674-1.

30. Natalia Boal Sánchez: *Algoritmos de reducción de potencial para el modelo posinomial de programación geométrica*, 2003, 232 pp., ISBN: 84-7733-667-9.

31. M. C. López de Silanes, M. Palacios, G. Sanz, J. J. Torrens, M. Madaune-Tort y D. Trujillo (eds.): *VIII Journées Zaragoza-Pau de Mathématiques Appliquées et de Statistiques*, 2004, xxvi + 578 pp., ISBN: 84-7733-720-9.

32. Carmen Godés Blanco: *Configuraciones de nodos en interpolación polinómica bivariada*, 2006, xii + 163 pp., ISBN: 84-7733-841-9.

33. M. Madaune-Tort, D. Trujillo, M. C. López de Silanes, M. Palacios, G. Sanz y J. J. Torrens (eds.): *Ninth International Conference Zaragoza-Pau on Applied Mathematics and Statistics*, 2006, xxxii + 440 pp., ISBN: 84-7733-871-X.

34. B. Lacruz, F. J. López, P. Mateo, C. Paroissin, A. Pérez-Palomares y G. Sanz (eds.): *Pyrenees International Workshop on Statistics, Probability and Operations Research, SPO 2007*, 2008, 205 pp., ISBN: 978-84-92521-18-0.

35. M. C. López de Silanes, M. Palacios, G. Sanz, J. J. Torrens, M. Madaune-Tort, C. Paroissin y D. Trujillo, (eds.): *Tenth International Conference Zaragoza-Pau on Applied Mathematics and Statistics*, 2010, xxx + 302 pp., ISBN: 978-84-15031-53-6.

36. L. M. Esteban, B. Lacruz, F. J. López, P. M. Mateo, A. Pérez-Palomares, G. Sanz y C. Paroissin, (eds.): *The Pyrenees International Workshop on Statistics, Probability and Operations Research: SPO 2009*, 2011, 164 pp., ISBN: 978-84-15031-92-5.

37. J. Giacomoni, M. Madaune-Tort, C. Paroissin, G. Vallet, M. C. López de Silanes, M. Palacios, G. Sanz, J. J. Torrens, (eds.): *Eleventh International Conference Zaragoza-Pau on Applied Mathematics and Statistics*, 2012, xxvi+208 pp., ISBN: 978-84-15538-15-8.

38. L. M. Esteban, B. Lacruz, F. J. López, P. M. Mateo, A. Pérez-Palomares, G. Sanz, C. Paroissin, (eds.): *The Pyrenees International Workshop and Summer School on Statistics, Probability and Operations Research SPO 2011*, 2013, 132 pp., ISBN: 978-84-15770-81-7.

39. M. C. López de Silanes, M. Palacios, G. Sanz, É. Ahusborde y C. Amrouche, (eds.): *Twelfth International Conference Zaragoza-Pau on Mathematics*, 2014, xxx + 224 pp., ISBN: 978-84-16028-35-1.

40. É. Ahusborde, C. Amrouche, G. Warnault, M. C. López de Silanes, M. Palacios y G. Sanz, (eds.): *Thirteenth International Conference Zaragoza-Pau on Mathematics and its Applications*, 2016, xxvi + 134 pp., ISBN: 978-84-16515-68-4.

41. M. C. López de Silanes, M. Palacios, É. Ahusborde, C. Amrouche y G. Carbou, (eds.): *Fourteenth International Conference Zaragoza-Pau on Mathematics and its Applications*, 2017, xxiii + 222 pp., ISBN: 978-84-17358-00-6.