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TOOLS TO PROVE A PARABOLIC LEWY-STAMPACCHIA'S INEQUALITY

Yassine Tahraoui

Abstract. We studied a quasilinear parabolic variational inequality of Lewy-Stampacchia type governed by a pseudomonotone operator of Leray-Lions type in a joint work with O. Guibé, A. Mokrane and G. Vallet [6]. We propose here some tools and techniques used to deal with the difficulties, which appear in the study of the problem.

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§1. Introduction

We are interessted in a nonlinear parabolic problem with contraint and homogeneous Dirichlet boundary conditions. More precisely, we prove the existence of a solution satisfying the following Lewy-Stampacchia's inequality

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot,\cdot,u,\nabla u)] - f \leq g^- = (f - \partial_t\psi + \operatorname{div}[a(\cdot,\cdot,\psi,\nabla\psi)])^-,$$

associated with the following problem

$$\int_0^T \langle \partial_t u, v-u\rangle dt + \int_Q a(t,x,u,\nabla u)\nabla(v-u)dxdt \geq \int_0^T \langle f, v-u\rangle dt, \quad u_0(0) = u_0$$

where $u \mapsto -\operatorname{div}[a(t,x,u,\nabla u)]$ is a pseudomomotone operator under the constraint $u \geq \psi$. We propose to present tools to show the existence of a solution for the above mentioned problem.

After the first results of H. Lewy and G. Stampacchia [8] concerning inequalities in the context of superharmonic problems, many authors have been interested in the so-called Lewy-Stampacchia's inequality associated with obstacle problems. Without exhaustiveness, let us cite the papers of A. Mokrane and F. Murat [10] for pseudo-monotone elliptic problems, A. Mokrane and G.Vallet [11] in the context of Sobolev spaces with variable exponents. The literature on Lewy-Stampacchia's inequality is mainly aimed at elliptic problems, or close to elliptic problems and fewer papers are concerned with other type of problems. Let us cite J. F. Rodrigues [12] for hyperbolic problems, F. Donati [4] for parabolic problems with a monotone operator or L. Mastroeni and M. Matzeu [9] in the case of a double obstacle.

The aim of O. Guibé, A. Mokrane, Y. Tahraoui and G. Vallet [6] was to extend F. Donati's work [4] to pseudo-monotone parabolic problems with a Leray-Lions operator. The authors proposed a result with very general assumptions on the Carathéodory function a, by using a method of penalization of the constraint associated with a suitable perturbation of the operator. As proposed e.g. by [7, p.102], this perturbation is one of the main new point of the proof. Indeed, without it, one is usually only concerned by Lewy-Stampacchia's inequality in the elliptic case, and one needs to assume, as in [10], some additional, now useless, holder-continuity assumptions with respect to u and ∇u. Thus, this perturbation allows us on the one hand to prove Lewy-Stampacchia's inequality in the pseudomonotone parabolic case, and on the other hand to reduce significantly the list of assumptions. Let us mention also that, with this method, one is able to revisit Lewy-Stampacchia's inequality proposed in [10, 11] by assuming only basic assumptions. The second essential result is an extension of the formula of time-integration by parts of Mignot-Bamberger[2] & Alt-Luckhaus[1] to non-classical situations. Some information are also given too about the time-continuity of an element u when u and $\partial_t u$ are not in spaces in duality relation. We propose in this paper to present tools and techniques used by the authors to deal with the difficulties in the study of some terms in [6].

First of all, we need to precise the functional setting and the assumptions on the data. Denote by $D \subset \mathbb{R}^d, d \geq 1$ a Lipshitz bounded domain, $T > 0$, $Q = D\times]0,T[$and $p, p' \in]1,+\infty[$ such that$\frac{1}{p} + \frac{1}{p'} = 1$. $V = W_0^{1,p}(D)$ if $p \geq 2$ and $V = W_0^{1,p}(D) \cap L^2(D)$ with the graph-norm else. Then, $V' = W^{-1,p'}(D)$ if $p \geq 2$ and $V' = W^{-1,p'}(D) + L^2(D)$ else and the Lions-Guelfand triple $V \underset{d}{\hookrightarrow} H \underset{d}{\hookrightarrow} V'$ holds.
$W(0,T) = \{u \in L^p(0,T,V),\ \partial_t u \in L^{p'}(0,T,V')\}$ and $\mathcal{K}(\psi) := \{u \in W(0,T),\ u \geq \psi\}$.

Assume in the sequel the following:

H_1 :

$$A : W^{1,p}(D) \to W^{-1,p'}(D) \quad v \mapsto A(v) = -\operatorname{div}\Big[a(t,x,v,\nabla v)\Big],$$

where

$H_{1,1}$ $a : (t,x,u,\xi) \in Q \times \mathbb{R} \times \mathbb{R}^d \mapsto a(t,x,u,\xi) \in \mathbb{R}^d$ is a Carathéodory function on $Q \times \mathbb{R}^{d+1}$,

$H_{1,2}$ $\forall (t,x) \in Q$ a.e., $u \in \mathbb{R}$, $\forall \xi, \eta \in \mathbb{R}^d$,

$$\xi \neq \eta \Rightarrow [a(t,x,u,\xi) - a(t,x,u,\eta)].(\xi - \eta) > 0.$$

$H_{1,3}$ $\exists \bar{\alpha} > 0, \bar{\beta} > 0$ and $\bar{\gamma} \geq 0$, functions $\bar{h} \in L^1(Q), \bar{k} \in L^p(Q)$ and two exponents $q, r < p$ such that, for a.e. $(t,x) \in Q$, $\forall u \in \mathbb{R}$,$\forall \xi \in \mathbb{R}^d$,

$$\begin{aligned} a(t,x,u,\xi).\xi &\geq \bar{\alpha}|\xi|^p - \Big[\bar{\gamma}|u|^q + |\bar{h}(t,x)|\Big], \\ |a(t,x,u,\xi)| &\leq \bar{\beta}\Big[|\bar{k}(t,x)| + |u|^{r/p} + |\xi|\Big]^{p-1}. \end{aligned}$$

H_2 : $\psi \in L^p(0,T,W^{1,p}(D)) \cap L^p(0,T,L^2(D))$; that $\partial_t \psi$ belongs to $L^{p'}(0,T,V')$ and $\psi \leq 0$ on ∂D.

H_3 : the right hand side f, which is assumed to be such that $g = f - \partial_t \psi - A(\psi) = g^+ - g^-$ belongs to the order dual $L^p(0,T,V)^* = (L^{p'}(0,T,V'))^+ - (L^{p'}(0,T,V'))^+$, i.e. $g^+, g^- \in (L^{p'}(0,T,V'))^+$ the non-negative elements of $L^{p'}(0,T,V')$.

H_4 : $u_0 \in L^2(D)$ satisfies the constraint, *i.e.* $u_0 \geq \psi(0)$.

Let us now recall the main result in [6].

Theorem 1. *Under the above assumptions (H_1)-(H_4), there exists at least $u \in \mathcal{K}(\psi)$ with $u(t=0) = u_0$ and such that, for any $v \in L^p(0,T,V)$, $v \geq \psi$*

$$\int_0^T \langle \partial_t u, v-u\rangle dt + \int_Q a(t,x,u,\nabla u)\nabla(v-u)dxdt \geq \int_0^T \langle f, v-u\rangle dt.$$

Moreover, the following Lewy-Stampacchia's inequality holds

$$0 \leq \partial_t u - div[a(\cdot,\cdot,u,\nabla u)] - f \leq g^- = (f - \partial_t\psi + div[a(\cdot,\cdot,\psi,\nabla\psi)])^-.$$

§2. Strong continuity in $L^2(D)$

Let us denote by $V(D)$ ($V_0(D)$ resp.) the following space $W^{1,p}(D)\cap L^2(D)$ ($W_0^{1,p}(D)\cap L^2(D)$ resp.) and $V'(D) = W^{-1,p'}(D) + L^2(D)$. We have the following result.

Lemma 2. *If $u \in L^p(0,T;V(D))$ and $\partial_t u \in L^{p'}(0,T;V'(D))$ then $u \in C([0,T],L^2(D))$.*

Remark 1. This result is not the usual one since u and $\partial_t u$ are not in spaces being in duality relation and few words are needed concerning the time-derivative. Note that both $V(D)$ and $V_0(D)$ are dense subspaces of the chosen pivot space $L^2(D)$ so that it can be identify to a subspace of $V'(D)$ or $(V(D))'$. Therefore, u, as an element of $L^p(0,T;V(D)) \hookrightarrow L^P(0,T;L^2(D))$, has a time derivative in the sense of $\mathcal{D}'(0,T;L^2(D)) \hookrightarrow \mathcal{D}'(0,T;V'(D))$ and it is assumed to belong to $L^{p'}(0,T;V'(D))$.

Remark 2. Note that Lemma 2 ensures that the obstacle $\psi \in C([0,T],L^2(D))$ and therefore $u_0 \geq \psi(0)$ has a sense as elements of $L^2(D)$.

Sketch of the proof. This result is based on a classical method: first in $\mathbb{R}^d$, then in the half-space $\mathbb{R}^d_+$ and finally in D thanks to an atlas of charts.

For $D = \mathbb{R}^N$, we have $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$, therefore we can identify $V'(\mathbb{R}^N)$ with the dual of $V(\mathbb{R}^N)$. By considering the triple $V(\mathbb{R}^N) \underset{d}{\hookrightarrow} L^2(\mathbb{R}^N) \underset{d}{\hookrightarrow} V'(\mathbb{R}^N)$, thanks to [14] (Prop. 1.2 p. 106), one has $u \in C([0,T],L^2(\mathbb{R}^N))$.

If $D = \mathbb{R}^N_{+/resp.-} = \{(x',x_d) \in \mathbb{R}^d;\ \ x_d > 0$ (resp. $x_d < 0\}$, the method is based on a suitable extension of u to $\mathbb{R}^d$. Following a recommendation of F. Murat, we consider the following extension, used *e.g* in [5]

$$\tilde{u}(t,x',x_d) = \begin{cases} u(t,x',x_d); & x_d > 0 \\ -3u(t,x',-x_d) + 4u(t,x',-2x_d); & x_d < 0. \end{cases}$$

Note that $\tilde{u} \in L^p(0,T;V(\mathbb{R}^d))$ and, thanks to a change of variables, that for any $\varphi \in C_c^\infty(]0,T[\times\mathbb{R}^d)$ one gets

$$\int_0^T \int_{\mathbb{R}^d} \tilde{u}(t,x)\partial_t\varphi(t,x)\,dx\,dt = \int_0^T \int_{\mathbb{R}^d_-} (-3u(t,x',-x_d) + 4u(t,x',-2x_d))\partial_t\varphi(t,x',x_d)\,dx\,dt$$
$$+ \int_0^T \int_{\mathbb{R}^d_+} u(t,x)\partial_t\varphi(t,x)\,dx\,dt.$$

Then

$$\int_0^T \int_{\mathbb{R}^d} \tilde{u}(t,x)\partial_t\varphi(t,x)\,dx\,dt$$
$$= \int_0^T \int_{\mathbb{R}^d_+} (\partial_t(\varphi(t,x',x_d) - 3\varphi(t,x',-x_d) + 2\varphi(t,x',-\frac{x_d}{2}))u(t,x,x_d)\,dx\,dt.$$

Remark that $\psi(t,x) = \varphi(t,x',x_d) - 3\varphi(t,x',-x_d) + 2\varphi(t,x',-\frac{x_d}{2}) = 0 \quad if \quad x_d = 0$ and $\partial_t\psi(t,x) = 0$ if $x_d = 0$, which implies $\psi \in W^{1,\infty}(0,T;V_0(\mathbb{R}^d_+))$.
Note that $\|\psi\|_{L^p(0,T;V_0(\mathbb{R}^d_+))} \leq 8\|\varphi\|_{L^p(0,T;V(\mathbb{R}^d))}$. Therefore,

$$|\int_0^T \langle\partial_t\tilde{u},\varphi\rangle dt| = |\int_0^T \int_{\mathbb{R}^d_+} u\partial_t\psi\,dx\,dt| \leq \|\partial_t u\|_{L^{p'}(0,T;V'(\mathbb{R}^d_+))}\|\psi\|_{L^p(0,T;V_0(\mathbb{R}^d_+))} \leq C\|\varphi\|_{L^p(0,T;V(\mathbb{R}^d))}$$

Thus $\partial_t\tilde{u} \in L^{p'}(0,T;V'(\mathbb{R}^d))$.Then, one concludes that $\tilde{u} \in C([0,T],L^2(\mathbb{R}^d))$ i.e $u \in C([0,T],L^2(\mathbb{R}^d_+))$. Finally, the result holds in the general case by considering an atlas of charts as proposed *e.g* in [5]. □

§3. Penalization and perturbation of the operator

Denote by $\tilde{q} = \min(p,2)$ and let us define the function Θ

$$\Theta : \mathbb{R} \to \mathbb{R}, \quad x \mapsto -[x^-]^{\tilde{q}-1},$$

and the perturbed operator

$$\tilde{a}(t,x,u,\xi) : Q \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \quad (x,t,u,\xi) \mapsto \tilde{a}(t,x,u,\xi) = a(t,x,\max(u,\psi(t,x)),\xi). \quad (3.1)$$

Remark 3. We wish to draw the reader's attention to the fact that with the proposed perturbation: $\tilde{a}(t,x,u,\xi) = a(t,x,\max(u,\psi),\xi)$, the idea is to make formally the operator monotone and not pseudomonotone any more on the free-set where the constraint is violated.

We define $\mathcal{A} : L^p(0,T;V) \to L^{p'}(0,T;V')$ such that $[\mathcal{A}(u)](t) := \tilde{A}(u(t)) = -\operatorname{div}[\tilde{a}(t,x,u,\nabla u)]$ and note that, the above assumption H_1 still holds.
For any positive ε, a cosmetic modification of [13, Section 8.4] yields the following result.

Theorem 3. *There exists $u_\epsilon \in W(0,T)$ such that $u_\epsilon(t=0) = u_0$ and*

$$\partial_t u_\epsilon - \operatorname{div}\left[\tilde{a}(t,x,u_\epsilon,\nabla u_\epsilon)\right] + \frac{1}{\varepsilon}\Theta(u_\epsilon - \psi) = f. \quad (3.2)$$

§4. From regular to general case

To prove the main result. On the one hand, we need some estimate for the penalization term. For that we impose an additional regularity on some data to get the desired estimate which permits to prove that the solution satisfies the constraint. On the other hand, we need some additional regularity to use an integration by part formula given in Section 5 to prove Lewy-Stampacchia's inequality. Then, we obtain the general case thanks the following density lemma.

Lemma 4. *The positive cone of $L^p(0,T;V)\cap L^2(Q)$ is dense in the positive cone of $\mathcal{V}'$, the dual set of $\mathcal{V}=L^p(0,T,V)$.*

Note that by truncation argument, the same result holds for the positive cone of $L^p(0,T;V)\cap L^{p'}(Q)$ when $p<2$. This result is given in [4, Lemma p.593]. We propose in [6] a sketch of a proof following the idea of [10].

§5. Mignot-Bamberger / Alt -Luckhaus integration by part formula

Note that $\mu_\varepsilon := \partial_t u_\varepsilon - \operatorname{div}[\tilde a(\cdot,\cdot,u_\varepsilon,\nabla u_\varepsilon)] - f = \frac{1}{\varepsilon}[(u_\varepsilon-\psi)^-]^{\tilde q-1} \geq 0$, so that the limit $\mu := \partial_t u - \operatorname{div}[\tilde a(\cdot,\cdot,u,\nabla u)] - f$ is a non-negative Radon measure which is also an element of $L^{p'}(0,T;V')$.

Using an idea from A. Mokrane and F. Murat [10], denote by $z_\varepsilon := g^- - \frac{1}{\varepsilon}[(u_\varepsilon-\psi)^-]^{\tilde q-1}$, we have

$$\partial_t u_\varepsilon + A(u_\varepsilon) + z_\varepsilon = g^+ + \partial_t\psi + A(\psi) \quad i.e. \quad \partial_t(u_\varepsilon-\psi) + A(u_\varepsilon) - A(\psi) + z_\varepsilon = g^+.$$

Observing that

$$\partial_t u_\varepsilon + A(u_\varepsilon) - f = -z_\varepsilon + g^-.$$

as in [10] in the elliptic case and under more restrictive assumptions on the operator a, proving that z_ε^- converges to 0 in an appropriate space leads to the Lewy-Stampacchia's inequality. Due to the time variable and the weak assumption on a we have to face to additional difficulties. For technical reasons, we will assume only that, on top of $g^-\in L^{p'}(Q)\cap L^p(0,T;V)$, $g^-\geq 0$, that $\partial_t g^-\in L^{\tilde q'}(Q)$. Roughly speaking it allows one to use a test function depending on g^- and together with Lemma 5 to perform an integration by part formula and then the convergence analysis of z_ε^-.

Lemma 5. *Consider $u\in L^p(0,T,W^{1,p}(D))\cap L^p(0,T,L^2(D))$ such that $\partial_t u\in L^{p'}(0,T,V')$. Let $\Psi : Q\times\mathbb{R}\to\mathbb{R}$ be a function such that $(t,x)\mapsto\Psi(t,x,\lambda)$ is measurable, $\lambda\mapsto\Psi(t,x,\lambda)$ is non-decreasing (càdlàg[1], or càglàd[2]) and denote by $\Lambda : Q\times\mathbb{R}\to\mathbb{R}, (t,x,\lambda)\mapsto\int_a^\lambda\Psi(t,x,\tau)d\tau$ where a is any arbitrary real number. Assume moreover that $|\Psi(t=0)|\leq h+|\lambda|^\alpha$ and that $\partial_t\Psi$ exists with $|\Psi(\lambda=0)|+|\partial_t\Psi|\leq h$ where $h\in L^2(Q)$ and $\alpha\in[0,1]$. If $\Psi(t,x,u)\in L^p(0,T,V)$, then, for any $\beta\in W^{1,\infty}(0,T)$ and any $0\leq s<t\leq T$,*

$$\begin{aligned}\int_s^t <\partial_t u,\Psi(\sigma,x,u)>\beta d\sigma &= \int_D \Lambda(t,x,u(t))\beta(t)dx - \int_D\Lambda(s,x,u(s))\beta(s)dx\\ &\quad -\int_s^t\int_D\Lambda(\sigma,x,u)\beta' dxd\sigma - \int_s^t\int_D\partial_t\Lambda(\sigma,x,u)\beta dxd\sigma.\end{aligned}$$

Proof. We propose here to present the proof introduced in [6].Thanks to the assumptions, Ψ is a measurable function on $Q\times\mathbb{R}$ and Λ is a Carathéodory function on $Q\times\mathbb{R}$. Moreover,

$$\begin{aligned}|\Psi(t,x,\lambda)| &\leq |\Psi(t=0)| + \int_0^t|\partial_t\Psi(s,x,\lambda)|ds \leq (T+1).h(t,x)+|\lambda|^\alpha,\\ |\Lambda(t,x,\lambda)| &\leq |\lambda-a|\Big[(T+1).h(t,x)+|\lambda|^\alpha\Big] \leq C(T,a)\Big[|\lambda|^2+h^2(t,x)+h(t,x)+1\Big]\end{aligned}$$

[1] right continuous with left limit
[2] left continuous with right limit

so that $\Lambda, \Psi \in L^2_{loc}(\mathbb{R}, L^2(Q))$ and the Nemitskii operator associated with Λ is continuous from $L^2(Q)$ to $L^1(Q)$. Concerning the time-derivation of Λ, for any $\varphi \in D(Q \times \mathbb{R})$, Fubini's theorem yields

$$\begin{aligned} -\int_{Q\times\mathbb{R}} \Lambda(t,x,\lambda)\partial_t\varphi(t,x,\lambda)dtdxd\lambda &= -\int_{Q\times\mathbb{R}}\int_a^\lambda \Psi(t,x,\tau)d\tau\partial_t\varphi(t,x,\lambda)dtdxd\lambda \\ &= \int_{Q\times\mathbb{R}}\int_a^\lambda \partial_t\Psi(t,x,\tau)d\tau\varphi(t,x,\lambda)dtdxd\lambda. \end{aligned}$$

As a consequence,

$$\partial_t\Lambda(t,x,\lambda) = \int_a^\lambda \partial_t\Psi(t,x,\tau)d\tau, \quad \left|\partial_t\Lambda(t,x,\lambda)\right| \le |\lambda - a|h(t,x) \le |\lambda|^2 + h^2(t,x)/4 + |a|h(t,x)$$

so that the Nemitskii operator associated with $\partial_t\Lambda$ is continuous from $L^2(Q)$ to $L^1(Q)$.

Thanks to the assumptions, $u \in C([0,T], L^2(D))$ and one extends u to $\bar{u}$ in $\mathbb{R}$ by $\bar{u}(t) = u_0$ if $t < 0$ and $\bar{u}(t) = u(T)$ si $t > T$. Therefore, if $I_1 := (-1, T+1)$, $\bar{u} \in L^p(I_1, W^{1,p}(D)) \cap L^\infty(I_1, L^2(D)) \cap C(\bar{I}_1, L^2(D))$ such that $\partial_t\bar{u} \in L^{p'}(I_1, V')$ with $\partial_t\bar{u} = 0$ when $t < 0$ or $t > T$. Similarly to u, denote by $\bar{\Psi}$ the extension to I_1 of Ψ in the same way and by $\bar{\Lambda}$ the corresponding integral as introduced in the Lemma.

For any fixed $0 < h << 1$, let us denote by

$$v_h : t \mapsto \frac{\bar{u}(t+h) - \bar{u}(t)}{h}, \quad w_h : t \mapsto \frac{\bar{u}(t) - \bar{u}(t-h)}{h}.$$

Consider $\beta \in \mathcal{D}(I_1)$ and h, small enought so that $\text{supp}\beta + [-h, h] \subset I_1$. Then,

$$\begin{aligned} &\int_{I_1} v_h(t)\beta(t)dt = \frac{1}{h}\int_{I_1}[\bar{u}(t+h) - \bar{u}(t)]\beta(t)dt \\ &= \frac{1}{h}\int_{I_1}\bar{u}(t)\beta(t-h)dt - \frac{1}{h}\int_{I_1}\bar{u}(t)\beta(t)dt = \frac{1}{h}\int_{I_1}\bar{u}(t)[\beta(t-h) - \beta(t)]dt \\ &\longrightarrow -\int_{-1}^{T+1}\bar{u}(t)\beta'(t)dt = -\int_0^T u(t)\beta'(t)dt + u(T)\beta(T) - u_0\beta(0) \quad \text{in } L^2(D); \end{aligned}$$

similarly,

$$\begin{aligned} &\int_{I_1} w_h(t)\beta(t)dt = \frac{1}{h}\int_{I_1}[\bar{u}(t) - \bar{u}(t-h)]\beta(t)dt \\ &= \frac{1}{h}\int_{I_1}\bar{u}(t)\beta(t)dt - \frac{1}{h}\int_{I_1}\bar{u}(t)\beta(t+h)dt = \frac{1}{h}\int_{I_1}\bar{u}(t)[\beta(t) - \beta(t+h)]dt \\ &\longrightarrow -\int_{-1}^{T+1}\bar{u}(t)\beta'(t)dt = -\int_0^T u(t)\beta'(t)dt + u(T)\beta(T) - u_0\beta(0) \quad \text{in } L^2(D), \end{aligned}$$

so that v_h and w_h converge to $\partial_t\bar{u}$ in $\mathcal{D}'[I_1, L^2(D)]$, thus in $\mathcal{D}'[I_1, V']$; and to $\partial_t u$ in $\mathcal{D}'[0,T,L^2(D)]$ and $\mathcal{D}'[0,T,V']$. Moreover, by [3, Corollary A.2 p.145], the properties of Bochner integral and since $\partial_t\bar{u} = 0$ outside $(0, T)$,

$$\int_{I_1} \|v_h(t)\|^{p'}_{V'}dt = \int_{I_1} \frac{1}{h^{p'}}\| \int_t^{t+h} \partial_t\bar{u}(s)ds\|^{p'}_{V'}dt \le \int_{I_1} \frac{1}{h}\int_t^{t+h} \|\partial_t\bar{u}(s)\|^{p'}_{V'}dsdt$$
$$\le \frac{1}{h}\int_{I_1}\int_{-1}^{t+h} \|\partial_t\bar{u}(s)\|^{p'}_{V'}dsdt - \frac{1}{h}\int_{I_1}\int_{-1}^{t} \|\partial_t\bar{u}(s)\|^{p'}_{V'}dsdt = \int_0^T \|\partial_t u(s)\|^{p'}_{V'}ds.$$

Since v_h already converges in the sense of Distributions, as a consequence of the above estimate, one may conclude that v_h converges weakly to $\partial_t\bar{u}$ in $L^{p'}[I_1, V']$ and to $\partial_t u$ in $L^{p'}[0, T, V']$. Similarly, w_h converges weakly to $\partial_t\bar{u}$ in $L^{p'}[I_1, V']$ and to $\partial_t u$ in $L^{p'}[0, T, V']$.

For any $\beta \in D(I_1)$, one has that $\Psi(\cdot, \bar{u})\beta \in L^p(I_1, V)$, since $L^2(D)$ is identified with its dual, one gets that

$$\int_{I_1\times D} v_h\bar{\Psi}(\cdot, u(t))\beta\, dx\, dt = \int_{I_1} < v_h, \bar{\Psi}(\cdot, \bar{u}(t)) > \beta dt \to \int_{I_1} < \partial_t\bar{u}, \bar{\Psi}(\cdot, \bar{u}) > \beta dt,$$
$$\int_{I_1\times D} w_h\bar{\Psi}(\cdot, \bar{u}(t))\beta\, dx\, dt = \int_{I_1} < w_h, \bar{\Psi}(\cdot, \bar{u}(t)) > \beta dt \to \int_{I_1} < \partial_t\bar{u}, \bar{\Psi}(\cdot, \bar{u}) > \beta dt.$$

Let us recall that a is a given real and $\bar{\Lambda}(t, x, \lambda) = \int_a^\lambda \bar{\Psi}(t, x, \tau)d\tau$. Since $\bar{\Psi}$ is a non-decreasing function of its third variable, for any real numbers u and v, one has

$$(v - u)\bar{\Psi}(t, x, u) \le \bar{\Lambda}(t, x, v) - \bar{\Lambda}(t, x, u) = \int_u^v \bar{\Psi}(t, x, \tau)d\tau \le (v - u)\bar{\Psi}(t, x, v).$$

Thus, assuming moreover that β is non-negative,

$$[\bar{u}(t + h, x) - \bar{u}(t, x)]\bar{\Psi}(t, x, \bar{u}(t))\beta \le [\bar{\Lambda}(t, x, \bar{u}(t + h)) - \bar{\Lambda}(t, x, \bar{u}(t))]\beta$$
$$\le [\bar{u}(t + h, x) - \bar{u}(t, x)]\bar{\Psi}(t, x, \bar{u}(t + h))\beta,$$
$$[\bar{u}(t, x) - \bar{u}(t - h, x)]\bar{\Psi}(t, x, \bar{u}(t - h))\beta \le [\bar{\Lambda}(t, x, \bar{u}(t)) - \bar{\Lambda}(t, x, \bar{u}(t - h))]\beta$$
$$\le [\bar{u}(t, x) - \bar{u}(t - h, x)]\bar{\Psi}(t, x, \bar{u}(t))\beta.$$

and, for h small enough to have $\operatorname{supp}\beta + [-h, h] \subset I_1$,

$$\int_{I_1\times D} v_h\beta\bar{\Psi}(\cdot, u(t))\, dx\, dt \le \int_{I_1\times D} \frac{\bar{\Lambda}(\cdot, \bar{u}(t + h)) - \bar{\Lambda}(\cdot, \bar{u}(t))}{h}\beta\, dx\, dt$$
$$\le \int_{I_1\times D} v_h\beta\bar{\Psi}(\cdot, \bar{u}(t + h))\, dx\, dt,$$
$$\int_{I_1\times D} w_h\beta\bar{\Psi}(\cdot, \bar{u}(t - h))\, dx\, dt \le \int_{I_1\times D} \frac{\bar{\Lambda}(\cdot, \bar{u}(t)) - \bar{\Lambda}(\cdot, \bar{u}(t - h))}{h}\beta\, dx\, dt$$
$$\le \int_{I_1\times D} w_h\beta\bar{\Psi}(\cdot, \bar{u}(t))\, dx\, dt,$$

so that

$$\liminf \int_{I_1\times D} \frac{\bar\Lambda(\cdot,\bar u(t+h))-\bar\Lambda(\cdot,\bar u(t))}{h}\beta\,dx\,dt \geq \int_{I_1} <\partial_t\bar u, \bar\Psi(\cdot,\bar u)> \beta dt$$
$$= \int_0^T <\partial_t u, \Psi(\cdot,u)> \beta dt,$$
$$\limsup \int_{I_1\times D} \frac{\bar\Lambda(\cdot,\bar u(t))-\bar\Lambda(\cdot,\bar u(t-h))}{h}\beta\,dx\,dt \leq \int_{I_1} <\partial_t\bar u, \bar\Psi(\cdot,\bar u)> \beta dt$$
$$= \int_0^T <\partial_t u, \Psi(\cdot,u)> \beta dt.$$

Moreover,

$$\int_{I_1\times D} \frac{\bar\Lambda(t,x,\bar u(t+h))-\bar\Lambda(t,x,\bar u(t))}{h}\beta(t)\,dx\,dt$$
$$= \frac{1}{h}\int_{I_1\times D} \bar\Lambda(t-h,x,\bar u(t))\beta(t-h)\,dx\,dt - \frac{1}{h}\int_{I_1\times D} \bar\Lambda(t,x,\bar u(t))\beta(t)\,dx\,dt$$
$$= \int_{I_1\times D} \frac{\bar\Lambda(t-h,x,\bar u(t))-\bar\Lambda(t,x,\bar u(t))}{h}\beta(t-h)\,dx\,dt + \int_{I_1\times D} \frac{\beta(t-h)-\beta(t)}{h}\bar\Lambda(t,x,\bar u(t))\,dx\,dt$$

and

$$\int_{I_1\times D} \frac{\bar\Lambda(t,x,\bar u(t))-\bar\Lambda(t,x,\bar u(t-h))}{h}\beta(t)\,dx\,dt$$
$$= \int_{I_1\times D} \frac{\bar\Lambda(t,x,\bar u(t))-\bar\Lambda(t+h,x,\bar u(t))}{h}\beta(t+h)\,dx\,dt + \int_{I_1\times D} \frac{\beta(t)-\beta(t+h)}{h}\bar\Lambda(t,x,\bar u(t))\,dx\,dt$$

one gets, by passing to the limit, and thanks to the time-extension procedure,

$$\liminf \int_{I_1\times D} \frac{\bar\Lambda(t-h,x,\bar u(t))-\bar\Lambda(t,x,\bar u(t))}{h}\beta(t-h)\,dx\,dt$$
$$\geq \int_0^T <\partial_t u, \Psi(\cdot,u)> \beta dt + \int_{I_1\times D} \bar\Lambda(\cdot,\bar u)\beta'\,dt$$
$$\geq \limsup \int_{I_1\times D} \frac{\bar\Lambda(t,x,\bar u(t))-\bar\Lambda(t+h,x,\bar u(t))}{h}\beta(t+h)\,dx\,dt$$

Note that

$$\int_{I_1\times D} \frac{\bar\Lambda(t-h,x,\bar u(t))-\bar\Lambda(t,x,\bar u(t))}{h}\beta(t-h)\,dx\,dt$$
$$= -\int_{I_1\times D} \frac{1}{h}\int_{t-h}^{t} \partial_t\bar\Lambda(s,x,\bar u(t))\beta(t-h)ds\,dx\,dt.$$

Since, $|\partial_t \bar{\Lambda}(s, x, \bar{u}(t))\beta(t-h)| \leq \|\beta\|_\infty |\bar{u}(t, x) - a| h(s, x)$ is an integrable function, the properties of the point of Lebesgue (steklov average) yields

$$\int_{I_1\times D} \frac{\bar{\Lambda}(t-h, x, \bar{u}(t)) - \bar{\Lambda}(t, x, \bar{u}(t))}{h}\beta(t-h)\,dx\,dt \to -\int_{I_1\times D} \partial_t \bar{\Lambda}(t, x, \bar{u}(t))\beta(t)\,dx\,dt$$
$$= -\int_Q \partial_t \Lambda(t, x, u(t))\beta(t)\,dx\,dt.$$

Since the same holds for $\limsup \displaystyle\int_{I_1\times D} \frac{\bar{\Lambda}(t, x, \bar{u}(t)) - \bar{\Lambda}(t+h, x, \bar{u}(t))}{h}\beta(t+h)\,dx\,dt$, and if β is regular and non negative, one gets that, for all $\beta \in D^+([0, T])$,

$$\int_0^T < \partial_t u, \Psi(\cdot, u) > \beta dt = \int_D \Lambda(T, x, u(T))\beta(T)dx - \int_D \Lambda(0, x, u_0)\beta(0)dx$$
$$- \int_Q \Lambda(\cdot, u)\beta' dt - \int_Q \partial_t \Lambda(t, x, u(t))\beta(t)\,dx\,dt.$$

Since β is involved in linear integral terms, a classical argument of regularisation yields the result for any non-negative elements of $W^{1,\infty}(0, T)$, then for any elements of $W^{1,\infty}(0, T)$.

Since T is arbitrary, the result holds for any t and $s = 0$, then for any t and s by subtracting the integral from 0 to s to the one from 0 to t. □

A priori, following Lemma's 5 notations, one should denote by $\Psi(t, x, \lambda) = -(g^- - \frac{1}{\varepsilon}[\lambda^-]^{\tilde{q}-1})^-$ and $\Lambda(t, x, \lambda) = \int_0^\lambda \Psi(t, x, \sigma)d\sigma$. For that, we need $\Psi(t, x, u)$ to be a test-function. Since $x \mapsto [x^-]^{\tilde{q}-1}$ is not *a priori* a Lipschitz-continuous function (*e.g.* if $p < 2$[3]), therefore, for any positive k, we will denote by

$\eta_k(x) = (\tilde{q} - 1)\int_0^{x^+} \min(k, s^{\tilde{q}-2})ds$, $\Psi_k(t, x, \lambda) = -(g^- - \frac{1}{\varepsilon}\eta_k(\lambda^-))^-$ and $\Lambda_k(t, x, \lambda) = \int_0^\lambda \Psi_k(t, x, \sigma)d\sigma$. Note that $\Psi_k(t, x, 0) = 0$ and $\partial_t \Psi_k(t, x, \lambda) = \partial_t g^- 1_{\{g^- - \frac{1}{\varepsilon}\eta_k(\lambda^-)<0\}}$ so that, since $\Psi_k(t, x, u)$ is a test-function, by Lemma 5, for any t,

$$-\int_0^t \int_D \partial_t \Lambda_k(s, x, u_\varepsilon - \psi)dxds + \int_D \Lambda_k(t, x, u_\varepsilon(t) - \psi(t))dx - \int_D \Lambda_k(0, x, u_\varepsilon(0) - \psi(0))dx$$
$$- \int_0^t \langle A(u_\varepsilon) - A(\psi), (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-])^- \rangle ds - \int_Q z_\varepsilon (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-])^- dxds$$
$$= -\int_0^t \langle g^+, (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-])^- \rangle ds \leq 0.$$

Remark 4. Note that the perturbation of the operator will play a main role in the study of the

[3] $\tilde{q} = \min(2, p)$

principal term. Indeed, denote by E the set $\{g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-] < 0\}$

$$\begin{aligned}
&- \int_0^T \langle A(u_\varepsilon) - A(\psi), (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-])^- \rangle dt \\
&\quad = \int_Q 1_E \Big[\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)\Big] \nabla [g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-]] \, dx \, dt \\
&\quad = \int_Q 1_E \Big[\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)\Big] \nabla [g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-]] \, dx \, dt,
\end{aligned}$$

therefore,

$$\begin{aligned}
&- \int_0^T \langle A(u_\varepsilon) - A(\psi), (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-])^- \rangle dt \\
&\qquad \geq - \int_Q \Big|\tilde{a}(t, x, \psi, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla \psi)\Big| |\nabla g^-| 1_{\{u_\varepsilon < \psi\}} \, dx \, dt.
\end{aligned}$$

We prove that the last term goes to zero and by analysing the other terms, we obtain Lewy-Stampacchia inequality with regular data.

Finally, we present remark concerning the uniqueness of the solution.

Remark 5. Note that the pseudomonotone assumption of the operator doesn't ensure the uniqueness of the solution. Observe that under additional assumptions on the operator a, namely a local Lipschitz continuity with respect to the third variable, standard arguments allow one to prove the uniqueness of the solution obtained in Theorem 1.

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Y. Tahraoui
Laboratoire de Mathématiques et leurs Applications
UMR CNRS 5142, BP 1155
64013 Pau cedex, France
tahraouiyacine@yahoo.fr