Monografías Matemáticas *García de Galdeano* **42**, 265–274 (2019)

THE MATROID STRUCTURE OF VECTORS OF THE MORDELL-WEIL LATTICE AND THE TOPOLOGY OF PLANE QUARTICS AND BITANGENT LINES

Ryutaro Sato and Shinzo Bannai

Abstract. In this paper, we introduce the terminology of matroids into the study of Zariski-pairs related to rational elliptic surfaces, aiming to simplify the presentation and arguments involved. As an application, we provide new examples of Zariski N-ples of relatively low degree. Namely we show that a Zariski 102-ple of degree 18 exists.

Keywords: Elliptic Surfaces, Mordell-Weil lattice, Matroids, Zariski-pairs.

AMS classification: 14J27, 14E20, 05B35.

§1. Introduction

In this paper, we study the embedded topology of plane curves. We are interested in the following situation. Let $C_1, C_2 \subset \mathbb{P}^2$ be plane curves. Then $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ form a Zariski-pair if the following conditions are satisfied

1. There exist tubular neighborhoods $T(C_i)$ of C_i $(i = 1, 2)$ such that the pairs $(T(C_1), C_1)$ and $(T(C_2), C_2)$ are homeomorphic as pairs.
2. The pairs $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic as pairs.

The notion of a Zariski-pair was first defined in [1] by E. Artal–Bartolo and has been an object of interest to many mathematicians. The key in studying Zariski pairs is finding a suitable method to distinguish the curves. Many invariants have been used, such as the fundamental groups of the complements $\pi_1(\mathbb{P}^2 \setminus C_i)$, the Alexander polynomials $\Delta_{C_i}(t)$ and the existence/non-existence of certain Galois covers branched along C_i (see [2] for a survey on these topics). More recently, newer types of invariants such as "linking invariants" and "splitting invariants" have been developed in studying reducible plane curves ([3, 7, 12]). However, as the number of irreducible components of C_i increases, these invariants become more increasingly complex, and it becomes hard to grasp the situation clearly. Hence, we are especially interested in formulating a method in order to present the differences in the curves and the classification comprehensively.

An attempt at this was done in [5],[4] where the second author together with colleagues considered invariants of subsets of the set of irreducible components. This approach proved to be effective and was able to produce new examples of Zariski pairs. However the examples produced were relatively simple, maybe too simple, to appreciate the usefulness of the approach fully. In this paper, we introduce the terminology of *matroids* into our setting in order to make the results more accessible to a wider audience and also to present more complex examples to demonstrate the usefulness of considering subarrangements more fully.

We introduce some notation to explain the kind of arrangements that we will study. Let Q be a smooth quartic curve and $z_o \in Q$ be a general point of Q. It is known that a rational elliptic surface S_{Q,z_0} can be associated to Q and z_o as follows (see [14, 5] for details): Let $\tilde{f}_Q : \tilde{S}_Q \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ branched along Q, and let $\mu : S_Q \to \tilde{S}_Q$ be the canonical resolution of singularities. Also, let Λ_{z_o} be the pencil of lines through z_o. Then the inverse image $\overline{\Lambda}_{z_o}$ of Λ_{z_o} in $\overline{S}_Q$ gives rise to a pencil of curves with genus 1. Next, the base points of $\overline{\Lambda}_{z_o}$ can be resolved by two consecutive blow-ups, whose composition is denoted by $\nu_{z_o} : S_{Q,z_o} \to \overline{S}_Q$. The morphism $\phi_{z_o} : S_{Q,z_o} \to \mathbb{P}^1$ induced by $\overline{\Lambda}_{z_o}$ gives a genus 1 fibration, and the exceptional divisor of the second blow-up in μ_{z_o} gives a section denote by O. Hence, we have an elliptic surface $\phi_{z_o} : S_{Q,z_o} \to \mathbb{P}^1$ associated to Q and z_o. Note that the covering transformation of $\widehat{S}_Q$ induces an involution on S_{Q,z_o} which we will denote by σ.

$$\begin{array}{ccccc} \widehat{S}_Q & \xleftarrow{\ \mu\ } & \overline{S}_Q & \xleftarrow{\ \nu_{z_o}\ } & S_{Q,z_o} \\ {\scriptstyle \widehat{f}_Q}\downarrow & & \downarrow{\scriptstyle \overline{f}_Q} & & \downarrow{\scriptstyle \phi_{z_o}} \\ \mathbb{P}^2 & \xleftarrow[\ q\ ]{} & \overline{\mathbb{P}^2} & & \mathbb{P}^1 \end{array}$$

We denote the set of sections of ϕ_{z_o} by $\mathrm{MW}(S_{Q,z_o})$. The sections will be identified with their images and considered as curves on S_{Q,z_o}. It is known that $\mathrm{MW}(S_{Q,z_o})$ can be endowed with an abelian group structure with a pairing $\langle , \rangle : \mathrm{MW}(S_{Q,z_o}) \to \mathbb{Q}$ called the *height pairing* (see [10]). When considering the height pairing, $\mathrm{MW}(S_{Q,z_o})$ is called the Mordell-Weil lattice of S_{Q,z_o}.

Let $f = \widehat{f_Q} \circ \mu \circ \nu_{z_o}$. For a section $s \in \mathrm{MW}(S_{Q,z_o})$, let $C_s = f(s)$, the image of s under f. The curve C_s is a rational curve in $\mathbb{P}^2$ whose local intersection numbers with Q become even. Such curves are called contact curves of Q. Note that $f(s) = f(-s)$ where $-s$ is the negative of s with respect to the group structure of $\mathrm{MW}(S_{Q,z_o})$. The curves C that we will study are reducible curves of the form

$$C = Q + C_{s_1} + \cdots + C_{s_r}$$

for some choice of $s_1, \ldots, s_r \in \mathrm{MW}(S_{Q,z_o})$. The additional data related to $\mathrm{MW}(S_{Q,z_o})$ allows us to distinguish the curves.

Assume for simplicity that $\mathrm{MW}(S_{Q,z_o})$ is torsion free. Let $E_i = \{s^i_1, \ldots, s^i_r\} \subset \mathrm{MW}(S_{Q,z_o})$ $(i = 1, 2)$ be subsets of $\mathrm{MW}(S_{Q,z_o})$ such that $C_{s^i_j} \neq C_{s^i_k}$ for $j \neq k$. We will consider the matroid structure on E_1, E_2 induced by the linear dependence relations in $\mathrm{MW}(S_{Q,z_o}) \otimes \mathbb{Q}$. Let $C_i = Q + C_{s^i_1} + \cdots + C_{s^i_r}$ $(i = 1, 2)$.

Theorem 1. *Under the above settings, if* $\mathrm{MW}(S_{Q,z_o})$ *is torsion free and* E_1, E_2 *have distinct matroid structures, then there exist no homeomorphisms* $h : \mathbb{P}^2 \to \mathbb{P}^2$ *with* $h(C_1) = C_2$ *and* $h(Q) = Q$.

Moreover, if $h(C_1) = C_2$ *implies* $h(Q) = Q$ *necessarily and the combinatorics of* C_1, C_2 *are the same, then* $(\mathbb{P}^2, C_1)$ *and* $(\mathbb{P}^2, C_2)$ *form a Zariski-pair.*

Theorem 1 allows us to distinguish Zariski pairs and Zariski N-ples by simply calculating the matroid structures of the subsets of $\mathrm{MW}(S_{Q,z_o})$. However, to actually construct Zariski pairs, we need to choose the subsets $\{s^i_1, \dots, s^i_r\}$ so that they have the same combinatorics, which is a somewhat delicate matter. Fortunately, we were able to use classical results on smooth quartics and bitangent lines, which can be found in [6], to overcome this difficulty.

In the case where Q is a smooth quartic, it is known that $\mathrm{MW}(S_{Q,z_o}) \cong E_7^*$. The E_7^* lattice has 28 pairs of minimal vectors $\pm l_1, \dots, \pm l_{28}$ of height $\frac{3}{2}$. Furthermore, $L_i = C_{l_i} = C_{-l_i}$ become bitangent lines of Q, and there is a bijection between the set of pairs $\pm l_i$ and the set of bitangent lines L_i. The combinatorics of these bitangent lines are known, as in the following proposition which will be proved later in Section 4.2.

Proposition 2. *For a general smooth quartic Q, its bitangent lines $L_1, \dots, L_{28}$ and a fixed value $r = 1, \dots, 28$, the combinatorics of curves of the form*

$$Q + L_{i_1} + \cdots + L_{i_r}$$

are the same for any $\{i_1, \dots, i_r\} \subset \{1, \dots, 28\}$. Namely, all L_{i_k} are true bitangents, i.e. they are tangent to Q at two distinct points, and any three of $L_{i_1}, \dots, L_{i_r}$ are non-concurrent.

For curves C_1, C_2 of the form above, it is immediate that $h(C_1) = C_2$ implies $h(Q) = Q$ necessarily. Now, Proposition 2 together with Theorem 1 gives us the following theorem.

Theorem 3. *Let N_r be the number of distinct matroid structures on subsets of the form $\{l_{i_1}, \dots, l_{i_r}\}$, where l_{i_k} is a representative of the pair $\pm l_{i_k}$. Then there exists a Zariski N_r-ple of curves having the combinatorics as in Proposition 2.*

At present, we have not been able to calculate the exact value of N_r due to a lack of computer skills of the authors. However, we have a lower bound as follows:

Proposition 4. *For $r = 1, \dots, 28$, the value of N_r is greater than or equal to n_r given in the following table.*

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14
n_r	1	1	1	2	2	4	6	11	19	37	52	80	95	102

r	15	16	17	18	19	20	21	22	23	24	25	26	27	28
n_r	100	90	70	54	37	23	16	10	5	3	2	1	1	1

We remark that Zariski-pairs involving smooth quartics and its bitangent lines have already been studied by E. Artal-Bartolo and J. Vallès. They gave an example of a pair consisting of a smooth quartic and three bitangent lines. The results were privately communicated to the authors. Also, the second author together with H. Tokunaga and M. Yamamoto have studied the case of four bitangent lines where a Zariski triple exists. Our approach using matroids fails to detect these examples but we think that our work is still worthwhile as it is easy to increase the number of bitangent lines involved and can be applied to non-smooth quartic curves. It also introduces a new point of view that is possibly relatively easier for a wider audience to access and hopefully will connect to other research areas.

The organization of this paper is as follows. In Section 2, we review the basic terminology of matroids and results concerning elliptic surfaces and dihedral covers, which will give the connection between the matroid structure of sections and the topology of the curves. In Section 3, we will prove Theorem 1. In Section 4, we will discuss the case where Q is a smooth quartic and prove Theorem 3 and also give the proof of Proposition 4.

§2. Preliminaries

2.1. Matroids

As will be seen later, the (in)dependence of elements of $\mathrm{MW}(S_{Q,z_o})$ is deeply related to the (non)existence of certain Galois covers of $\mathbb{P}^2$, hence it is important to understand the structure of (in)dependence. Here, Matroid Theory provides a nice framework as it was precisely designed to study generalizations of the notion of linear independence in vector spaces. In this section we briefly review the basic terminology of matroids. We refer to [9] for more details.

There are many different cryptomorphic definitions of Matroids. In our paper, we are interested in the dependence of elements of $\mathrm{MW}(S_{Q,z_o})$, hence we adopt the definition based on *independent sets*. Let E be a finite set and 2^E be the set of subsets of E.

Definition 1. A matroid structure (or simply a matroid) on E is a pair $(E, \mathcal{I})$, where $\mathcal{I} \subset 2^E$ satisfies

1. $\mathcal{I} \neq \emptyset$. (nontriviality)
2. For any $I_1, I_2 \subset E$, if $I_1 \subset I_2$ and $I_2 \in \mathcal{I}$, then $I_1 \in \mathcal{I}$. (descending)
3. For every $I_1, I_2 \in \mathcal{I}$, if $|I_1| < |I_2|$, then there exists $x \in I_2 - I_1$ such that $I_1 \cup \{x\} \in \mathcal{I}$. (augmentation)

Elements of $\mathcal{I}$ will be called *independent sets* and the other subsets will be said to be *dependent*.

Example 1. Let V be a vector space, and $E = \{v_1, \ldots, v_r\} \subset V$. Let $\mathcal{I} = \{I \subset E \mid I \text{ is linearly independent}\}$. Then $\mathcal{I}$ clearly satisfies the conditions (1), (2), (3) in Definition 1. Hence $(E, \mathcal{I})$ is a matroid structure on E.

Definition 2. Let $(E, \mathcal{I})$ be a matroid. A subset $C \subset E$ is called a *circuit* if $C \notin \mathcal{I}$ and all proper subsets of C are independent sets. Moreover, C is a minimal dependent set.

Example 2. Let $V = \mathbb{R}^3$ and $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Let $E = \{v_1, v_2, v_3, v_3\}$ and consider the matroid structure induced by linear independence. Then E itself forms a circuit.

§3. Proof of Theorem 1

In this section, we use the criterion for the existence of dihedral covers given in Section 3.2 to connect the data of matroids of subsets of $\mathrm{MW}(S_{Q,z_o})$ to the data of the embedded topology of the curves in $\mathbb{P}^2$, and prove Theorem 1.

Let $E_i = \{s^i_1, \ldots, s^i_r\} \subset \mathrm{MW}(S_{Q,z_o})$ $(i = 1, 2)$ be subsets of $\mathrm{MW}(S_{Q,z_o})$ such that $C_{s^i_j} \neq C_{s^i_k}$ for $j \neq k$. Consider the matroid structure $(E_i, \mathcal{I}_i)$ on E_i $(i = 1, 2)$ induced by the linear dependence relation in $\mathrm{MW}(S_{Q,z_o}) \otimes \mathbb{Q}$. Let $C_i = Q + C_{s^i_1} + \cdots + C_{s^i_r}$ $(i = 1, 2)$.

Proposition 5. *If there exists a homeomorphism $h : \mathbb{P}^2 \to \mathbb{P}^2$ such that $h(C_1) = C_2$ and $h(Q) = Q$, then $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ are equivalent as matroids.*

Proof. By the assumption that h is a homeomorphism such that $h(C_1) = C_2$ and $h(Q) = Q$, h induces a bijection $\{C_{s_1^1}, \dots, C_{s_r^1}\} \to \{C_{s_1^2}, \dots, C_{s_r^2}\}$ which in turn induces a bijection $h_* : E_1 \to E_2$. Let $I_1 \in \mathcal{I}_1$ be an independent set. Then by Lemma 10, there exists only a finite number of primes such that a D_{2p} cover branched at $2Q + p(\sum_{s \in J_1} C_s)$ for some subset $J_1 \subset I_1$ exists. Since h is a homeomorphism, the same is true for $h_*(I_1)$ which implies that $h_*(I_1) \in \mathcal{I}_2$, by Lemma 9. The converse is also true so we have $I_1 \in \mathcal{I}_1$ if and only if $h_*(I_1) \in \mathcal{I}_2$. Therefore $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ are equivalent as matroids. □

The contrapositive of Proposition 5 gives Theorem 1.

Remark 1. The statement of Proposition 5 concerns the matroid structure over $\mathbb{Q}$. However, from the proof, it is evident that if we consider the matroid structures of the sections in $\mathrm{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z}$ for all p we would be able to distinguish the arrangements in more detail.

Definition 3. Let $(E_1, \mathcal{I}_1)$, $(E_2, \mathcal{I}_2)$ be matroids. The matroids $(E_1, \mathcal{I}_1)$, $(E_2, \mathcal{I}_2)$ are said to be equivalent as matroids if there exists a bijection $\varphi : E_1 \to E_2$ such that $I_1 \in \mathcal{I}_1$ if and only if $\varphi(I_1) \in \mathcal{I}_2$.

3.1. Elliptic surfaces and the Mordell-Weil lattice

In this subsection, we list the basic facts about quartics, rational elliptic surfaces and the Mordell-Weil lattice. We refer the reader to [10], [8] for more details.

In this paper, an *elliptic surface* is a smooth projective surface S, with a relatively minimal genus 1 fibration $\phi : S \to C$ over a smooth projective curve C having a section $O : C \to S$. We identify O with its image in S. We also assume that S has at least one singular fiber. Let $\mathrm{Sing}(\phi) = \{v \in C \mid \phi^{-1}(v) \text{ is singular }\}$. For $v \in \mathrm{Sing}(\phi)$, we put $F_v = \phi^{-1}(v)$ and denote its irreducible decomposition by $F_v = \Theta_{v,0} + \sum_{i=1}^{m_i - 1} a_{v,i}\theta_{v,i}$, where $m_{v,i}$ is the number of irreducible components and $\Theta_{v,0}$ is the unique irreducible component with $\Theta_{v,0}.O = 1$. The subset of $\mathrm{Sing}(\phi)$ that corresponds to reducible singular fibers will be denoted by R. Let $\mathrm{MW}(S)$ be the set of sections of $\phi : S \to C$.

The set $\mathrm{MW}(S)$ can be endowed with a group structure as follows. Let E_S be the generic fiber of ϕ and $\mathbb{C}(C)$ be the function field of C. It is known that there is a bijection between $\mathbb{C}(C)$ rational points $E_S(\mathbb{C}(C))$ of E_S and $\mathrm{MW}(S)$. Furthermore, since we have $O \in \mathrm{MW}(S)$, $(E(S), O)$ can be considered as an elliptic curve over $\mathbb{C}(C)$ and has a group structure where O acts as the identity element.

Furthermore, under these circumstances, $\mathrm{MW}(S)$ becomes a finitely generated abelian group with a pairing $\langle , \rangle : \mathrm{MW}(S) \to \mathbb{Q}$ called the height pairing ([10]). The explicit formula to calculate the pairing for $s_1, s_2 \in \mathrm{MW}(S)$ is given by

$$\langle s_1, s_2 \rangle = \chi(S) + s_1.O + s_2.O - s_1.s_2 - \sum_{v \in R} \mathrm{contr}_v(s_1, s_2).$$

The formulas for calculating $\mathrm{contr}_v(s_1, s_2)$ can be found in [10]. In the following we will only need the values of $\mathrm{contr}_v(s_1, s_2)$ for singular fibers of type I_2 of the form $F_v = \Theta_{v,0} + \Theta_{v,1}$. In this case we have

$$\mathrm{contr}_v(s_1, s_2) = \begin{cases} 1/2 & (s_1.\Theta_{v,1} = s_2.\Theta_{v,1} = 1) \\ 0 & \text{otherwise} \end{cases}.$$

3.2. Criterion for existence of dihedral covers

Let D_{2n} be the dihedral group of order $2n$. We present a criterion for the existence of certain dihedral covers of $\mathbb{P}^2$ in terms of $\mathrm{MW}(S)$. The existence/non-existence of the dihedral covers will enable us to distinguish the topology of the curves.

Let Q be a quartic plane curve, $z_o \in Q$ be a general point of Q, $s_1, \dots, s_r \in \mathrm{MW}(S_{Q,z_o})$ be sections such that $C_{s_i} \neq C_{s_j}$, where $C_{s_i} = f(s_i)$ as in the Introduction.

Theorem 6 ([5, Corollary 4]). *Let p be an odd prime. Under the above setting, there exists a D_{2p}-cover of $\mathbb{P}^2$ branched at $2Q + p(C_{s_1} + \cdots + C_{s_r})$ if and only if there exists integers $a_i \in \{1, \dots, p-1\}$ for $i = 1, \dots r$ such that $\sum_{i=1}^r a_i s_i \in p\,\mathrm{MW}(S)$.*

Corollary 7. *If there exists a D_{2p} cover branched at $2Q + p(C_{s_1} + \cdots + C_{s_r})$, then the images of $s_1, \dots, s_r$ in $\mathrm{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z}$ become linearly dependent.*

Note that the converse of Corollary 7 is not true, as it is necessary for the images of $s_1, \dots, s_r$ to have a linear dependence relation where all coefficients are non-zero for there to exist a dihedral cover. If there does not exist such linear dependence relation, the branch locus will not be the whole of $2Q + p(C_{s_1} + \cdots + C_{s_r})$. To exclude such cases, the notion of circuits is useful.

Corollary 8. *If the images of $s_1, \dots, s_r$ in $\mathrm{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z}$ forms a circuit, then there exists a D_{2p}-cover branched at $2Q + p(C_{s_1} + \cdots + C_{s_r})$.*

If $s_1, \dots, s_r$ form a circuit over $\mathbb{Q}$, then their images in $\mathrm{MW}(S) \otimes \mathbb{Z}/p\mathbb{Z}$ form a circuit for infinitely many prime numbers p. Hence we have:

Lemma 9. *If $s_1, \dots, s_r$ are linearly dependent, then there are infinitely many prime numbers p such that there exists a D_{2p}-cover branched at $2Q + p(C_{s_{i_1}} + \cdots + C_{s_{i_t}})$ for some nonempty subset $\{i_1, \dots, i_t\} \subset \{1, \dots, r\}$.*

On the other hand, if $s_1, \dots, s_r$ are independent over $\mathbb{Q}$, then they are independent over $\mathbb{Z}/p\mathbb{Z}$ except for a finite number of primes. This implies the following.

Lemma 10. *If $s_1, \dots, s_r$ are independent over $\mathbb{Q}$, then there are only a finite number of prime numbers p such that there exists a D_{2p}-cover branched at $2Q + p(C_{s_{i_1}} + \cdots + C_{s_{i_t}})$ for some nonempty subset $\{i_1, \dots, i_t\} \subset \{1, \dots, r\}$.*

§4. The smooth case

In this section, we will consider the case where Q is a smooth quartic.

4.1. The bitangents of Q and sections of S_{Q,z_o}

We will use the notation given in the Introduction. Let Q be a smooth plane quartic and $z_o \in Q$ be a general point of Q. Since Q is smooth, $\widehat{S}_Q = \overline{S}_Q$. In this case S_{Q,z_o} has only one reducible singular fiber $F_0 = \Theta_{0,0} + \Theta_{0,1}$ of type I_2. The component $\Theta_{0,0}$ is the exceptional divisor of the first blow up of μ_{z_o} in the introduction, and $\Theta_{0,1}$ is the strict transform of the preimage of the tangent line of Q at z_o. All other singular fibers are irreducible. By [8], we have $\mathrm{MW}(S_{Q,z_o}) \cong E_7^*$ where E_7^* is the dual lattice of the root lattice E_7. It is known that the E_7^* lattice has 56 minimal vectors $\pm l_1, \dots, \pm l_{28}$ of height $\frac{3}{2}$. It is also well known that Q has 28 bitangent lines $L_1, \dots, L_{28}$. The correspondence between the 28 pairs of minimal vectors and the 28 bitangent lines is given in [11], but we describe the relation below for the readers convenience.

Lemma 11. *Let $l \in \mathrm{MW}(S_{Q,z_o})$ be a minimal vector of height $\frac{3}{2}$. Then $L = f(l)$ is a bitangent line of Q, where f is the morphism $f : S_{Q,z_o} \to \mathbb{P}^2$ given in the Introduction.*

Proof. By the explicit formula for the height pairing, and since $\chi(S_{Q,z_o}) = 1$ and $l.l = -1$, we have

$$\langle l, l \rangle = 2 + 2l.O - \mathrm{contr}(l, l) = \frac{3}{2}.$$

Where $\mathrm{contr}(l, l)$ is the contribution from the unique reducible singular fiber F_0. Since the possible values of $\mathrm{contr}(l, l) = 0, \frac{1}{2}$, we have $l.O = 0$ and $\mathrm{contr}(l, l) = \frac{1}{2}$ which implies that $l.\Theta_{0,1} = 1$. This implies that l is disjoint with the exceptional set of ν_{z_o}. Also, if we consider the section $-l = \sigma^*(l)$, the preimage of l under the involution σ, we have

$$\langle l, -l \rangle = 1 + l.O + (-l).O - l.(-l) - \mathrm{contr}(l, -l) = -\frac{3}{2}$$

Hence we obtain $l.(-l) = 2$. Let $\widehat{l} = \nu_{z_o}(l)$ and $\widehat{-l} = \nu_{z_o}(-l)$. The above implies that $\widehat{l}.\widehat{-l} = \widehat{l}.\widehat{Q} = 2$, where $\widehat{Q}$ is the ramification locus of $\widehat{f_Q}$. Now since $(\widehat{f_Q})^*(L) = \widehat{l} + \widehat{-l}$ we have $2L.L = (\widehat{l} + \widehat{-l}).(\widehat{l} + \widehat{-l})$. Hence we obtain $L.L = 1$ which implies that L is a line in $\mathbb{P}^2$. Also, the local intersection numbers of L and Q must be even by construction, hence L is a bitangent line. □

Remark 2. Note that the two points of tangency may coincide to give a line L intersecting Q at a single point with multiplicity 4, which we will still consider to be a bitangent line.

Lemma 12. *Let L be a bitangent line of Q and let $f^*(L) = l + l'$. Then l, l' become minimal sections with height $\frac{3}{2}$ and $l' = \sigma^* l = -l$.*

Proof. By following through the proof of Lemma 11 backwards, we have the desired result. □

The above two lemmas give us the following propositon.

Proposition 13. *There is a bijection between the set of 28 bitangent lines of Q and the set of 28 pairs of minimal vectors of the E_7^* lattice.*

4.2. The configuration of bitangents

In this subsection we explain the proof of Proposition 2. In [6], an explicit set of equations for computing the equations of the 28 bitangents, called Riemann's Equations, is given. Using these equations, it is possible to calculate the equations of all 28 bitangents provided that one has the data of seven bitangent lines $L_1, \dots, L_7$ of Q, which form an Aronhold set (i.e. a septuple of bitangents such that, for any subset of three bitangents the six points of tangency do not lie on a conic.). We can assume that $L_1, \ldots, L_7$ are given by the following equations for a suitable choice of coordinates where $[t_0, t_1, t_2]$ are homogeneous coordinates of $\mathbb{P}^2$:

$$L_1 = V(t_0), L_2 = V(t_1), L_3 = V(t_2), L_4 = V(t_0 + t_1 + t_2)$$
$$L_{4+i} = V(a_{0i}t_0 + a_{1i}t_1 + a_{2i}t_2), (i = 1, 2, 3)$$

Reimann's Equations gives us the explicit equations of the bitangent lines in terms of the coefficients a_{ij}. It also allows us to recover the equation of Q similarly. Once we have explicit equations it is possible to calculate the combinatorics of the quartic and bitangent lines. We used the open-source mathematics software system SageMath [13] for the actual calculations.

Lemma 14. *Let $L_1, \ldots, L_7$ be lines defined as above. Then, for a general choice of a_{0i}, a_{1i}, a_{2i} $(i = 1, 2, 3)$ the following hold:*

1. *There exists a smooth quartic Q having $L_1, \ldots, L_7$ as an Aronhold set of bitangents.*
2. *Any three bitangent lines of Q are non-concurrent.*
3. *Every bitangent line of Q is a true bitangent, i.e. it is tangent to Q at two distinct points.*

Proof. The equations of Q, and its bitangents $L_1, \ldots, L_{28}$ are given in terms of a_{ij} by Riemann's equation as in [6]. Since all three condition in the statements are closed conditions on a_{0i}, a_{1i}, a_{2i} $(i = 1, 2, 3)$, it is enough to find one example where the statements hold. Almost any choice will serve our purpose. We omit the details of the equations and calculations do to lack of space. □

Lemma 14 immediately implies Proposition 2.

4.3. The proof of Proposition 4

In this subsection, we describe the method we used to distinguish the matroid structures of minimal vectors of the E_7^* lattice in order to calculate n_r. We used SageMath [13] for the actual calculations.

The object that we want to classify are the matroid structures on the sets of the form $\{l_{i_1}, \ldots, l_{i_r}\}$ where l_{i_r} are representatives of pairs $\pm l_{i_r}$ of minimal vectors of height $\frac{3}{2}$. It is known that the E_7^* lattice can be representation in $\mathbb{Q}^8$ in a way so that the minimal vectors are of the form

$$\pm\frac{1}{4}(1, 1, 1, 1, 1, 1, -3, -3)$$

and its permutations. We use this representation in our calculations.

We used an inductive argument on the number of vectors r. For each subset $E \subset \{l_1, \ldots, l_{28}\}$ having r-elements, we assign an $(n_{r-1} + 1)$-ple of integers inductively as follows. The values of n_r will also be determined inductively along the way.

- **Step** (1)

  For every subset with a single element, we assign the pair $\alpha_{1,1} = (1; 1)$.

- **Step** $(k + 1)$

  Suppose that every subset having k elements has been assigned an $(n_{k-1} + 1)$-ple of integers. We set n_k to be the number of distinct $(n_{k-1} + 1)$-ples that have been assigned and label them by $\alpha_{k,1}, \ldots, \alpha_{k,n_k}$. Next, to each subset $E \subset \{l_1, \ldots, l_{28}\}$ having $k + 1$ elements, we assign an $(n_k + 1)$-ple as follows:

  (i) Consider the linear dependence/independence of E. Put $i = 0$ if it is dependent and $i = 1$ if it is independent.

  (ii) Let m^k_j be the number of subsets of E of k elements that have the $(n_{k-1} + 1)$-ple $\alpha_{k,j}$ assigned.

  (iii) Assign the $(n_k + 1)$-ple $(i; m^k_1, \ldots, m^k_{n_k})$ to E.

Lemma 15. *Let E_1, E_2 be subsets of $\{l_1, \ldots, l_{28}\}$ and $|E_1| = |E_2| = r$. If E_1 and E_2 have the same matroid structure, then the $(n_{r-1} + 1)$-ples of integers assigned above are equivalent.*

Proof. We use induction on r to prove this lemma. The case for $r = 1$ is trivial as every subset having a single element has the same pair assigned and has the same matroid structure.

Assume the statement holds for $r = k$. If $|E_1| = |E_2| = k + 1$ and E_1, E_2 have equivalent matroid structure, there exists a bijection $\varphi : E_1 \to E_2$ that preserves independent sets. Hence E_1 is independent if and only if E_2 is independent and the value of i must be equal. Also, φ induces a bijection from $\{E \subset E_1 \mid |E| = k\}$ to $\{E \subset E_2 \mid |E| = k\}$ and an equivalence of matroid structures among the corresponding subsets. Hence the values of m^k_j must be equal do to the hypothesis of induction, and the assigned $(n_k + 1)$-ple are equivalent. □

Lemma 15 and calculations done by computer using SageMath gives Proposition 4.

Acknowledgements

The second author is partially supported by Grant-in-Aid for Scientific Research C (18K03263).

References

[1] Artal Bartolo, E. Sur les couples de Zariski. *J. Algebraic Geom. 3*, 2 (1994), 223–247.

[2] Artal Bartolo, E., Cogolludo, J. I., and Tokunaga, H.-o. A survey on Zariski pairs. In *Algebraic geometry in East Asia—Hanoi 2005*, vol. 50 of *Adv. Stud. Pure Math.* Math. Soc. Japan, Tokyo, 2008, pp. 1–100.

[3] Bannai, S. A note on splitting curves of plane quartics and multi-sections of rational elliptic surfaces. *Topology Appl. 202* (2016), 428–439.

[4] Bannai, S., Guerville-Ballé, B., Shirane, T., and Tokunaga, H.-o. On the topology of arrangements of a cubic and its inflectional tangents. *Proc. Japan Acad. Ser. A Math. Sci. 93*, 6 (2017), 50–53.

[5] Bannai, S., and Tokunaga, H.-o. Geometry of bisections of elliptic surfaces and Zariski N-plets for conic arrangements. *Geom. Dedicata 178* (2015), 219–237.

[6] Dolgachev, I. V. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012. A modern view.

[7] Guerville-Ballé, B., and Meilhan, J.-B. A linking invariant for algebraic curves. arXiv:1602.04916.

[8] Oguiso, K., and Shioda, T. The Mordell-Weil lattice of a rational elliptic surface. *Comment. Math. Univ. St. Paul. 40*, 1 (1991), 83–99.

[9] Oxley, J. *Matroid theory*, second ed., vol. 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2011.

[10] Shioda, T. On the Mordell-Weil lattices. *Comment. Math. Univ. St. Paul. 39*, 2 (1990), 211–240.

[11] Shioda, T. Plane quartics and Mordell-Weil lattices of type E_7. *Comment. Math. Univ. St. Paul. 42*, 1 (1993), 61–79.

[12] Shirane, T. A note on splitting numbers for Galois covers and π_1-equivalent Zariski k-plets. *Proc. Amer. Math. Soc. 145*, 3 (2017), 1009–1017.

[13] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.1)*. `https://www.sagemath.org`.

[14] Tokunaga, H.-o. Sections of elliptic surfaces and Zariski pairs for conic-line arrangements via dihedral covers. *J. Math. Soc. Japan 66*, 2 (2014), 613–640.

Sato Ryutaro and Shinzo Bannai
National Institute of Technology, Ibaraki College
866 Nakane, Hitachinaka-shi, Ibaraki-Ken 312-8508 JAPAN
`sbannai@ge.ibaraki-ct.ac.jp`