Monografías Matemáticas *García de Galdeano* **42**, 219–228 (2019)

Fractal Jackson approximation on the torus

María Antonia Navascués, Sangita Jha, María Victoria Sebastián, Arya Kumar Bedabrata Chand

Abstract. In this article we generalize an approximation formula for three dimensional periodic data on a grid using fractal techniques which helps us to construct both smooth and non-smooth approximants depending on the choice of scale factors. We obtain bounds of the approximation error and showed the convergence with very weak conditions, when the sampling frequency is indefinitely increased. The density of the mappings involved in the space of two-dimensional periodic and continuous functions is proved using certain ranges of the scaling factors. A numerical example is presented to illustrate the proposed approximation methods.

Keywords: Fractals, Fractal Interpolation Functions, Fractal Surfaces, Two Dimensional Approximation.

AMS classification: 28A80, 42A10, 42A15.

§1. Introduction

Current major investigations in the theory of approximation concern smooth approximation. However it would be good to have mathematical structures to describe real life models which are non-smooth in nature. Such a structure is provided, for instance, by the theory of fractal functions (see for instance cf. [1], [2], [3], [8], [9]). Barnsley (cf. [1], [2]) first introduced the concept of fractal interpolation functions (FIFs) using the theory of iterated function system (IFS) (cf. [5]). FIFs form the basis of iterative constructive approximation theory. Barnsley and Harrington (cf. [3]) derived the calculus of FIF and showed that depending on the parameters of the IFS, one can construct smooth or non-smooth FIFs. Adapting the notion of FIF, Navascués (cf. [10]) constructed an entire family of fractal functions f^α, parameterized by an appropriate vector α, beginning from a given continuous function f on a compact interval I. This type of maps tend to bridge the gap between the smoothness of the classical mathematical objects and the pseudo-randomness of experimental data.

In the theory of classical trigonometric approximation, D. Jackson (cf. [6], [7]) described the degree of approximation of a continuous function by means of algebraic trigonometric polynomials. For the one dimensional case, he introduced an approximation formula (cf. [6]) for 2π periodic continuous functions as

$$\Sigma_m f(x) = H_m \sum_{i=1}^{2m} f(x_i) \left(\frac{\sin\left(\frac{m(x_i-x)}{2}\right)}{m \sin\left(\frac{x_i-x}{2}\right)} \right)^4 , \tag{1.1}$$

where

$$x_{i+1} - x_i = \frac{\pi}{m}, i = 1, 2, \ldots, 2m-1 \text{ and } H_m^{-1} = \sum_{i=1}^{2m} \left(\frac{\sin\left(\frac{m(x_i - x)}{2}\right)}{m \sin\left(\frac{x_i - x}{2}\right)} \right)^4.$$

We generalize the previous formula (cf. [11]) using a positive exponent γ, and derive the convergence properties with very weak conditions on the original function. Recently, (cf. [14]), Navascués and Sebastián extended the approximation formula (1.1) for the two dimensional case. The formula proposed in [14] has an explicit representation in terms of the sample data on a two dimensional grid.

The approximation problem considered here is the representation of a prescribed periodic continuous and real-valued function of two variables using fractal techniques. In addition, we prove the density of the mappings involved in the space of two-dimensional periodic and continuous functions using certain ranges of the scaling factors. Numerical examples are given in the last section to illustrate the proposed process.

§2. Preliminaries

First we shall review the materials from the references (cf. [1], [7], [10], [13]) which will be used in the sequel.

2.1. Construction of fractal functions

Let us recall the construction of fractal interpolation functions in this section. Consider an interpolation data set $\{(x_i, y_i), i \in \mathbb{N}_N \cup \{0\}\}$, where $\mathbb{N}_N = \{1, 2, \ldots, N\}$. Let $\Delta := x_0 < x_1 < \cdots < x_N$ be a partition of the interval $I = [x_0, x_N]$. Let $L_i : I \to I_i = [x_{i-1}, x_i], i \in \mathbb{N}_N$ be contractive homeomorphisms such that

$$L_i(x_0) = x_{i-1}, \;\; L_i(x_N) = x_i. \tag{2.1}$$

Let $K = I \times \mathbb{R}$ and N continuous mappings, $F_i : K \to \mathbb{R}$ be satisfying

$$F_i(x_0, y_0) = y_{i-1}, F_i(x_N, y_N) = y_i, \;\; |F_i(x, y) - F_i(x, y')| \le |c_i||y - y'|, \tag{2.2}$$

where $(x, y), (x, y') \in K$, $c_i \in (-1, 1)$, $i \in \mathbb{N}_N$. Now define functions $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ as $w_i(x, y) = (L_i(x), F_i(x, y)) \; \forall \; i \in \mathbb{N}_N$.

Theorem 1. *The Iterated Function System (IFS) $\mathcal{I} = \{K; w_i, i = 1, 2, \ldots, N\}$ admits a unique attractor G, which is the graph of a continuous function $f : I \to \mathbb{R}$ which obeys $f(x_i) = y_i$ for $i = 0, 1, 2, \ldots, N$.*

The previous function is called a Fractal Interpolation Function (FIF) corresponding to the IFS $\mathcal{I} = \{L_i(x), F_i(x, y)\}_{i=1}^N$, and it satisfies the following functional equation:

$$f(x) = F_i(L_i^{-1}(x), f \circ L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_N. \tag{2.3}$$

In this paper we choose $L_i(x) = a_i x + b_i$ satisfying (2.1) and $F_i(x, y) = \alpha_i y + q_i(x)$, where $q_i : I \to \mathbb{R}$ are continuous functions verifying (2.2). The vector $\alpha = (\alpha_1, \ldots, \alpha_N)$ is called a vertical scaling factor and it must satisfy the inequality $|\alpha|_\infty = \max\{|\alpha_i|; i = 1, 2, \ldots, N\} < 1$.

2.2. α-fractal function

Let $f : I \to \mathbb{R}$ be a continuous function. Consider $q_i(x) = f \circ L_i(x) - \alpha_i b(x)$, where b is defined from f through a linear map L ($Lf = b$) satisfying $b(x_0) = f(x_0), b(x_N) = f(x_N)$. The fixed point function associated with the above IFS is known as the α-fractal function f^α, and it enjoys the following equation:

$$f^\alpha(x) = f(x) + \alpha_i(f^\alpha - b)(L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_N. \tag{2.4}$$

The previous equation provides the inequality

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|f - b\|_\infty = \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|f - Lf\|_\infty, \tag{2.5}$$

which bounds the uniform distance between f^α and f. Navascués (cf. [10]) proposed the linear and continuous operator $\mathcal{F}^\alpha$ defined by $\mathcal{F}^\alpha(f) = f^\alpha$.

§3. One dimensional fractal Jackson approximant

Let $C(T^1)$ denote the set of all continuous periodic function on $[-\pi, \pi]$. Let $\Delta_m : -\pi = x_0 < \cdots < x_{2m-1} < x_{2p} = \pi$ be such that $x_{i+1} = x_i + \frac{\pi}{m}$ for all $i = 0, 1, 2, \ldots, 2m - 1$. Let us consider the continuous and periodic basis $\left\{P_{mi\gamma}(x) = \left|\frac{\sin\left(\frac{m(x_i - x)}{2}\right)}{m \sin\left(\frac{x_i - x}{2}\right)}\right|^\gamma ; i = 0, 1, 2, \ldots, 2m\right\}$. Let us define the set $\tau_m = \text{span}\{P_{mi\gamma}\}_{i=0}^{2m}$. Let us consider a Jackson type operator $\mathcal{T}_{m\gamma} : C(T^1) \to \tau_m$ assigning a periodic approximant belonging to τ_m for every $g \in C(T^1)$ (with respect to the data $\{(x_i, g(x_i))\}_{i=0}^{2m}$), defined as

$$\mathcal{T}_{m\gamma}(g)(x) = H_{m\gamma}(x) \sum_{i=0}^{2m} g(x_i) P_{mi\gamma}(x),$$

where $(H_{m\gamma}(x))^{-1} = \sum\limits_{i=0}^{2m} \left|\frac{\sin\left(\frac{m(x_i - x)}{2}\right)}{m \sin\left(\frac{x_i - x}{2}\right)}\right|^\gamma$. It is easy to see that

$$\|\mathcal{T}_{m\gamma} g\|_{C(T^1)} \leq \|g\|_{C(T^1)}.$$

In fact, the equality holds if we choose $g(x) = 1$. In the one dimensional case, the error of discrete Jackson approximation was studied in cf. [12]. According to this reference, for $g \in C(T^1)$, and $\gamma > 2$, the error of the approximation can be bounded as

$$\|\mathcal{T}_{m\gamma}(g) - g\|_{C(T^1)} \leq \left(\frac{\pi}{2}\right)^\gamma \omega_g\left(\frac{\pi}{4m}\right)\left(1 + 2^\gamma \zeta(\gamma - 1)\right), \tag{3.1}$$

where ζ is the Riemann zeta function. We define the α-fractal Jackson approximant of $g \in C(T^1)$ as

$$\mathcal{T}^\alpha_{m\gamma}(g)(x) = H_{m\gamma}(x)\mathcal{F}^\alpha\left(\sum_{i=0}^{2m} g(x_i) P_{mi\gamma}(x)\right) = H_{m\gamma}(x)\left(\sum_{i=0}^{2m} g(x_i) P^\alpha_{mi\gamma}(x)\right),$$

where $P^{\alpha}_{mi\gamma}(x)$ is the α-fractal function of $P_{mi\gamma}$ with respect to the partition Δ of $I = [-\pi, \pi]$ and a linear bounded operator L. Let us denote $\mathcal{P}_{m\gamma}(g)(x) = \sum_{i=0}^{2m} g(x_i)P_{mi\gamma}(x)$. Then

$$\|\mathcal{P}_{m\gamma}(g)\|_\infty \le \|g\|_\infty \|\sum_{i=0}^{2m} P_{mi\gamma}(x)\|_\infty. \tag{3.2}$$

Thus $\|\mathcal{P}_{m\gamma}(g)\|_\infty \le \|g\|_\infty \|H^{-1}_{m\gamma}\|_\infty$ which provides the inequality $\|\mathcal{P}_{m\gamma}\| \le \|H^{-1}_{m\gamma}\|_\infty$, where $\|\mathcal{P}_{m\gamma}\|$ represents the norm of the operator with respect to the supremum norm $\|.\|_\infty$ in $C(T^1)$. Here $H^{-1}_{m\gamma}$ represents the inverse with respect to the product. For the operator $\mathcal{T}^{\alpha}_{m\gamma}$,

$$\begin{aligned}\|\mathcal{T}^{\alpha}_{m\gamma}(g)\|_\infty \le& \|H_{m\gamma}\|_\infty \|\mathcal{F}^{\alpha}(\mathcal{P}_{m\gamma})\|_\infty \\ &\le \|\mathcal{F}^{\alpha}\| \|H_{m\gamma}\|_\infty \|H^{-1}_{m\gamma}\|_\infty \|g\|_\infty \\ &= R_{m\gamma\alpha}\|g\|_\infty,\end{aligned} \tag{3.3}$$

where $R_{m\gamma\alpha} = \|\mathcal{F}^{\alpha}\| \|H_{m\gamma}\|_\infty \|H^{-1}_{m\gamma}\|_\infty$. Then $\|\mathcal{T}^{\alpha}_{m\gamma}\| \le R_{m\gamma\alpha}$. Let us consider the error term $\mathcal{T}^{\alpha}_{m\gamma}(g) - g$:

$$\begin{aligned}\mathcal{T}^{\alpha}_{m\gamma}(g)(x) - g(x) =& H_{m\gamma}\sum_{i=0}^{2m} g(x_i)P^{\alpha}_{mi\gamma}(x) - H_{m\gamma}\sum_{i=0}^{2m} g(x_i)P_{mi\gamma}(x) + H_{m\gamma}\sum_{i=0}^{2m} g(x_i)P_{mi\gamma}(x) - g(x) \\ &= H_{m\gamma}\mathcal{P}^{\alpha}_{m\gamma}(g)(x) - H_{m\gamma}\mathcal{P}_{m\gamma}(g)(x) + H_{m\gamma}\mathcal{P}_{m\gamma}(g)(x) - g(x),\end{aligned}$$

where

$$\mathcal{P}^{\alpha}_{m\gamma}(g)(x) = \sum_{i=0}^{2m} g(x_i)P^{\alpha}_{mi\gamma}(x) = \mathcal{F}^{\alpha}(\mathcal{P}_{m\gamma}(g))(x).$$

Thus, using the above computations we obtain

$$\|\mathcal{T}^{\alpha}_{m\gamma}(g) - g\|_\infty \le \|H_{m\gamma}\|_\infty \|\mathcal{P}^{\alpha}_{m\gamma}(g) - \mathcal{P}_{m\gamma}(g)\|_\infty + \|\mathcal{T}_{m\gamma}(g) - g\|_\infty. \tag{3.4}$$

Using (2.5), the first term of the above inequality can be bounded as

$$\begin{aligned}\|\mathcal{P}^{\alpha}_{m\gamma}(g) - \mathcal{P}_{m\gamma}(g)\|_\infty \le& \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|\mathcal{P}_{m\gamma}(g) - L\mathcal{P}_{m\gamma}(g)\|_\infty \\ &\le \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|I - L\|\|\mathcal{P}_{m\gamma}(g)\|_\infty \\ &\le \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|I - L\|\|H^{-1}_{m\gamma}\|_\infty\|g\|_\infty.\end{aligned} \tag{3.5}$$

Finally, using (3.1) and (3.5) in (3.4) we get

$$\|\mathcal{T}^{\alpha}_{m\gamma}(g) - g\|_{C(T^1)} \le \|H_{m\gamma}\|_\infty \frac{|\alpha|_\infty \|I - L\|}{1-|\alpha|_\infty}\|H^{-1}_{m\gamma}\|_\infty\|g\|_\infty + \left(\frac{\pi}{2}\right)^{\gamma}\omega_g\left(\frac{\pi}{4m}\right)(1 + 2^{\gamma}\zeta(\gamma-1)). \tag{3.6}$$

§4. Fractal Jackson approximation on T^2

In this section, the approximation process described above is extended to data on a two dimensional torus. Let be given two partitions $\Delta_m^1 : -\pi = x_0 < x_1 < \cdots < x_{2m-1} < x_{2m} = \pi$ and $\Delta_n^2 : -\pi = y_0 < y_1 < \cdots < y_{2n-1} < y_{2n} = \pi$ of the circle. Let us consider the grid $\Delta = \Delta_m^1 \times \Delta_n^2$ of $T^2 = T^1 \times T^1$ and data $\{(x_i, y_j, z_{ij}) : i = 0, 1, 2, \ldots, 2m; j = 0, 1, 2, \ldots, 2n\}$ with 2π-periodicity condition in both variables. Let $\alpha \in (-1, 1)^{2m}$ and $\beta \in (-1, 1)^{2n}$ be scale vectors for Δ_m^1 and Δ_n^2 respectively. Let us define the operator using different exponents γ_1, γ_2 for both single functions as

$$\mathcal{J}_{mn\gamma_1\gamma_2}(f)(x, y) = K_{mn\gamma_1\gamma_2}(x, y) \sum_{i=0}^{2m} \sum_{j=0}^{2n} f(x_i, y_j) P_{mi\gamma_1}(x) Q_{nj\gamma_2}(y),$$

where $x_{i+1} - x_i = \frac{\pi}{m}; i = 0, 1, 2, \ldots, 2m-1,\ y_{j+1} - y_j = \frac{\pi}{n}; j = 0, 1, 2, \ldots, 2n-1,$

$$P_{mi\gamma_1}(x) = \left| \frac{\sin\left(\frac{m(x_i - x)}{2}\right)}{m \sin\left(\frac{x_i - x}{2}\right)} \right|^{\gamma_1},$$

$$Q_{nj\gamma_2}(y) = \left| \frac{\sin\left(\frac{n(y_j - y)}{2}\right)}{n \sin\left(\frac{y_j - y}{2}\right)} \right|^{\gamma_2},$$

and

$$K^{-1}_{mn\gamma_1\gamma_2}(x, y) = \sum_{i=0}^{2m} \sum_{j=0}^{2n} \left| \frac{\sin\left(\frac{m(x_i - x)}{2}\right)}{m \sin\left(\frac{x_i - x}{2}\right)} \right|^{\gamma_1} \left| \frac{\sin\left(\frac{n(y_j - y)}{2}\right)}{n \sin\left(\frac{y_j - y}{2}\right)} \right|^{\gamma_2}.$$

Lemma 2. *(cf. [11]) For all $k = 1, 2, \ldots, \gamma$ and $z \in \mathbb{R}$:*

$$\left| \frac{\sin(kz)}{k \sin(z)} \right|^{\gamma} \leq 1.$$

Definition 1. (cf. [4]) Let f be a continuous function defined on T^2. The modulus of continuity of f is defined as

$$\omega_f(\delta) := \sup_{\|x_1 - x_2\| \leq \delta} \{|f(x_1) - f(x_2)| : x_1, x_2 \in T^2\}.$$

We will use the following properties of the modulus of continuity:

1. $\omega_f(\delta_1 + \delta_2) \leq \omega_f(\delta_1) + \omega_f(\delta_2)$.
2. $\omega_f(\lambda\delta) \leq \lambda\omega_f(\delta)$ for $\lambda \in \mathbb{N}$.

Lemma 3. *For any $\gamma_1, \gamma_2 > 0$, the norm of $K_{mn\gamma_1\gamma_2}$ can be bounded as*

$$\|K_{mn\gamma_1\gamma_2}\|_\infty \leq \frac{1}{4} \left(\frac{\pi}{2}\right)^{2\gamma_{max}},$$

where $\gamma_{max} = \max\{\gamma_1, \gamma_2\}$.

Proof. From the definition of $K_{mn\gamma_1\gamma_2}$ we have

$$\begin{aligned}
K^{-1}_{mn\gamma_1\gamma_2}(x,y) &= \sum_{i=0}^{2m}\sum_{j=0}^{2n}\left|\frac{\sin(\frac{m(x_i-x)}{2})}{m\sin(\frac{x_i-x}{2})}\right|^{\gamma_1}\left|\frac{\sin(\frac{n(y_j-y)}{2})}{n\sin(\frac{y_j-y}{2})}\right|^{\gamma_2}\\
&= H^{-1}_{m\gamma_1}(x)H^{-1}_{n\gamma_2}(y)\\
&\le \frac{1}{2}\left(\frac{\pi}{2}\right)^{\gamma_1}\frac{1}{2}\left(\frac{\pi}{2}\right)^{\gamma_2}\\
&\le \frac{1}{4}\left(\frac{\pi}{2}\right)^{2\gamma_{max}},
\end{aligned}$$

where $H^{-1}_{m\gamma}(x)$ is defined in Section 3 and considering that $H^{-1}_{m\gamma}(x) \ge 2\left(\frac{2}{\pi}\right)^{\gamma}$ for any $\gamma > 0$ (cf. [12]). □

Theorem 4. *Let $f \in C(T^2)$. Then for any $\gamma_1, \gamma_2 > 2$, the approximant $\mathcal{J}_{mn\gamma_1\gamma_2}(f)$ converges uniformly to f whenever m, n tend to infinity.*

Proof. Consider the approximation error as $E_{mn\gamma_1\gamma_2}(f)(x,y) = \mathcal{J}_{mn\gamma_1\gamma_2}(f)(x,y) - f(x,y)$. Applying the definition of $K_{mn\gamma_1\gamma_2}$, modulus of continuity of f, and the changes $x_i - x = 2u_i, y_j - y = 2v_j$ we obtain

$$|E_{mn\gamma_1\gamma_2}(f)(x,y)| \le 2K_{mn\gamma_1\gamma_2}(x,y)\sum_{i=0}^{2m}\sum_{j=0}^{2n}(\omega_f(\bar{u}_i)+\omega_f(\bar{v}_j))\left|\frac{\sin m\bar{u}_i}{m\sin\bar{u}_i}\right|^{\gamma_1}\left|\frac{\sin n\bar{v}_j}{n\sin\bar{v}_j}\right|^{\gamma_2},$$

where $\bar{u}_i, \bar{v}_j$ are constructed as increasing order in $|u_i|, |v_j|$ respectively. From the inequalities (15) and (16) of the reference [14], for all $i, j \ge 2$,

$$\left|\frac{\sin(\frac{m(x_i-x)}{2})}{m\sin(\frac{x_i-x}{2})}\right|^{\gamma_1} \le \left(\frac{2}{i}\right)^{\gamma_1} \text{ and } \left|\frac{\sin(\frac{n(y_j-y)}{2})}{n\sin(\frac{y_j-y}{2})}\right|^{\gamma_2} \le \left(\frac{2}{j}\right)^{\gamma_2}. \tag{4.1}$$

Using (4.1) and similar lines as given in [14], we obtain an error bound as

$$|E_{mn\gamma_1\gamma_2}(f)(x,y)| \le \omega_f\left(\frac{\pi}{4m}+\frac{\pi}{4n}\right)F(\gamma_1,\gamma_2),$$

where $F(\gamma_1, \gamma_2)$ is independent of m, n. Thus the error term tends to zero when the partition is indefinitely refined. □

Definition 2. The fractal operator of Jackson approximation of a continuous f on the torus is defined as

$$\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f)(x,y) = K_{mn\gamma_1\gamma_2}(x,y)\sum_{i=0}^{2m}\sum_{j=0}^{2n} f(x_i,y_j)P^{\alpha}_{mi\gamma_1}(x)Q^{\beta}_{nj\gamma_2}(y).$$

Theorem 5. *Let $f \in C(T^1 \times T^1)$ and $\gamma_1, \gamma_2 > 2$, then*

$$\|\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f)-f\|_\infty \le mn\left(\frac{\pi}{2}\right)^{2\gamma_{\max}}\left(\frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|I-L\|\|\mathcal{F}^\beta\|_\infty+\frac{|\beta|_\infty}{1-|\beta|_\infty}\|I-L^*\|\right)+\omega_f\left(\frac{1}{m}+\frac{1}{n}\right)F(\gamma_1,\gamma_2),$$

where α, β are suitable scaling vectors used to construct the fractal perturbation of the basis functions $P_{mi\gamma_1}$ and $Q_{nj\gamma_2}$.

Proof. To attain the prescribed upper bound we will use

$$\|\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f) - f\|_\infty \leq \|\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f) - \mathcal{J}_{mn\gamma_1\gamma_2}(f)\|_\infty + \|\mathcal{J}_{mn\gamma_1\gamma_2}(f) - f\|_\infty.$$

According to the definition of $\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f)$ and $K_{mn\gamma_1\gamma_2}$,

$$\|\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f) - \mathcal{J}_{mn\gamma_1\gamma_2}(f)\|_\infty \leq \|K_{mn\gamma_1\gamma_2}\|_\infty \left\| \sum_{i=0}^{2m} \sum_{j=0}^{2n} f(x_i, y_j) \left(P^\alpha_{mi\gamma_1}(x) Q^\beta_{nj\gamma_2}(y) - P_{mi\gamma_1}(x) Q_{nj\gamma_2} \right) \right\|_\infty .$$

The norm of the sum in the previous expression can be bounded as

$$\begin{aligned} &\left\| \sum_{i=0}^{2m} \sum_{j=0}^{2n} f(x_i, y_j) \left(P^\alpha_{mi\gamma_1} Q^\beta_{nj\gamma_2} - P_{mi\gamma_1} Q_{nj\gamma_1} \right) \right\|_\infty \\ &\leq \|f\|_\infty \sum_{i=0}^{2m} \sum_{j=0}^{2n} \|P^\alpha_{mi\gamma_1} Q^\beta_{nj\gamma_2} - P_{mi\gamma_1} Q_{nj\gamma_2}\|_\infty \\ &\leq \|f\|_\infty \sum_{i=0}^{2m} \sum_{j=0}^{2n} \left(\|P^\alpha_{mi\gamma_1} Q^\beta_{nj\gamma_2} - P_{mi\gamma_1} Q^\beta_{nj\gamma_2}\|_\infty + \|P_{mi\gamma_1} Q^\beta_{nj\gamma_2} - P_{mi\gamma_1} Q_{nj\gamma_2}\|_\infty \right). \end{aligned} \tag{4.2}$$

Now the first norm of the expression (4.2) in the parenthesis can be bounded as

$$\begin{aligned} \|P^\alpha_{mi\gamma_1} Q^\beta_{nj\gamma_2} - P_{mi\gamma_1} Q^\beta_{nj\gamma_2}\|_\infty \leq& \|P^\alpha_{mi\gamma_1} - P_{mi\gamma_1}\|_\infty \|Q^\beta_{nj\gamma_2}\|_\infty \\ &\leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|P_{mi\gamma_1} - LP_{mi\gamma_1}\|_\infty \|\mathcal{F}^\beta\| \, \|Q_{nj\gamma_2}\|_\infty \\ &\leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\| \|P_{mi\gamma_1}\|_\infty \|\mathcal{F}^\beta\| \, \|Q_{nj\gamma_2}\|_\infty, \end{aligned} \tag{4.3}$$

where we have assumed $b_{mi\gamma_1} = LP_{mi\gamma_1}$ for a bounded linear operator L. But $\|P_{mi\gamma_1}\|_\infty \leq 1$, $\|Q_{nj\gamma_2}\|_\infty \leq 1$ due to Lemma 2. Similarly, the second norm of (4.2) in the parenthesis can be bounded as

$$\begin{aligned} \|P_{mi\gamma_1} Q^\beta_{nj\gamma_2} - P_{mi\gamma_1} Q_{n,j,\gamma_2}\|_\infty \leq& \|P_{mi\gamma_1}\|_\infty \|Q^\beta_{nj\gamma_2} - Q_{n,j,\gamma_2}\|_\infty \\ &\leq \frac{|\beta|_\infty}{1 - |\beta|_\infty} \|I - L^*\|, \end{aligned} \tag{4.4}$$

where $b^*_{nj\gamma_2} = L^* Q_{nj\gamma_2}$ for a bounded linear operator L^*. Finally, using (4.3), (4.4) in (4.2) we obtain

$$\left\| \sum_{i=0}^{2m} \sum_{j=0}^{2n} f(x_i, y_j) \left(P^\alpha_{m,i,\gamma_1} Q^\beta_{n,j,\gamma_2} - P_{m,i,\gamma_1} Q_{n,j,\gamma_2} \right) \right\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\| \, \|\mathcal{F}^\beta\| + \frac{|\beta|_\infty}{1 - |\beta|_\infty} \|I - L^*\|.$$

Using Lemma 3, Theorem 4 and the above expression, the final bound for the error is

$$\|\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f) - f\|_\infty \leq mn \left(\frac{\pi}{2} \right)^{2\gamma_{\max}} \left(\frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\| \, \|\mathcal{F}^\beta\| + \frac{|\beta|_\infty}{1 - |\beta|_\infty} \|I - L^*\| \right) +$$

$$\omega_f \left(\frac{1}{m} + \frac{1}{n} \right) F(\gamma_1, \gamma_2).$$

$\square$

Corollary 6. *If $f \in C([-\pi, \pi] \times [-\pi, \pi]), \gamma_1, \gamma_2 > 2$ and if we choose scaling vectors α, β such that $mn|\alpha|_\infty$, $mn|\beta|_\infty$ have the same rate of convergence as that of ω_f, then the discrete fractal approximant $\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f)$ converges uniformly to f as m, n tend to infinity. The order of convergence does not depend on γ_1, γ_2.*

Remark 1. The present approach may be extended to high-dimensional settings, for functions defined on hypertori. The convergence results would remain qualitatively equal to those exposed in this paper.

§5. Example

In this section we give the numerical explanation of the proposed approximants for different exponents and scale vectors. Figure 1(a) represents the graph of the smooth function $f(x, y) = 2\sin^2(x) + 3\cos^2(y)$ over the interval $[-\pi, \pi] \times [-\pi, \pi]$. Figure 1(b) represents the surface corresponding to the discrete approximant $\mathcal{J}_{mn\gamma_1\gamma_2}(f)$ for the values of $m = n = 10$ and $\gamma_1 = \gamma_2 = 4$. In order to get the fractal surface $\mathcal{J}^{\alpha\beta}_{mn\gamma_1\gamma_2}(f)$ corresponding to the discrete surface data, we consider a uniform partition of $[-\pi, \pi]$ in both directions with $M = N = 10$. Figure 1(c) depicts the fractal surface corresponding to $\alpha_i = \beta_i = 0.12$ for $i = 1, 2, \ldots, N$ and $\gamma_1 = \gamma_2 = 4$. Figure 1(d) represents another periodic fractal surface for $\alpha_i = 0.08, \beta_i = 0.1$ for $i = 1, 2, \ldots, N$ and $\gamma_1 = 3, \gamma_2 = 4$. Usually, we tend to think that the sample points come from a smooth function, but in practice this is not always the case. Thus for non-smooth periodic surface data, these procedures may help to provide a better approximation.

Acknowledgements

This work has been partially supported by the Projects CUD-ID: 2015-05 and CUD-ID: 2017-03 of the Centro Universitario de la Defensa de Zaragoza.

References

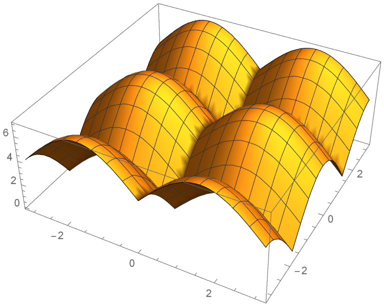
[1] Barnsley, M. F. Fractal functions and interpolation. *Constr. Approx. 2*, 1 (1986), 303–329. Available from: `https://doi.org/10.1007/BF01893434`.

[2] Barnsley, M. F. *Fractals Everywhere*. Academic Press, Inc., Boston, MA, 1988.

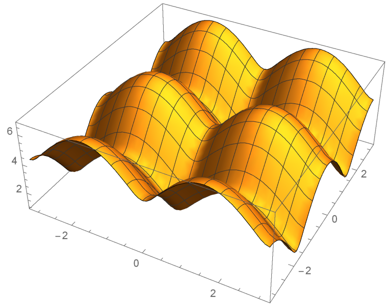
[3] Barnsley, M. F., and Harrington, A. N. The calculus of fractal interpolation functions. *J. Approx. Theory 57*, 1 (1989), 14–34. Available from: `https://doi.org/10.1016/0021-9045(89)90080-4`.

[4] Cheney, E. W. *Introduction to Approximation Theory*. AMS Chelsea Publishing, Providence, RI, 1998.

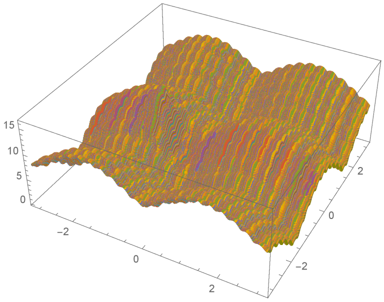
[5] Hutchinson, J. E. Fractals and self-similarity. *Indiana Univ. Math. J. 30*, 5 (1981), 713–747. Available from: `https://doi.org/10.1512/iumj.1981.30.30055`.



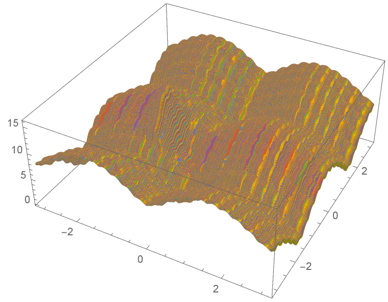
(a) Graph of f.



(b) Graph of classical discrete approximant.



(c) Graph of the fractal discrete approximant for $\gamma_1 = \gamma_2$.



(d) Graph of the fractal discrete approximant for different γ_1, γ_2.

Figure 1: Graph of f, its discrete classical and fractal approximants for different values of γ_1, γ_2, α and β.

[6] Jackson, D. On approximation by trigonometric sums and polynomials. *Trans. Amer. Math. Soc. 13*, 4 (1912), 491–515. Available from: `https://doi.org/10.2307/1988583`.

[7] Jackson, D. On the accuracy of trigonometric interpolation. *Trans. Amer. Math. Soc. 14*, 4 (1913), 453–461. Available from: `https://doi.org/10.2307/1988698`.

[8] Massopust, P. R. *Interpolation and Approximation with Splines and Fractals*. Oxford University Press, Oxford, 2010.

[9] Massopust, P. R. *Fractal Functions, Fractal Surfaces, and Wavelets*, second ed. Elsevier/Academic Press, London, 2016.

[10] Navascués, M. A. Fractal polynomial interpolation. *Z. Anal. Anwendungen 24*, 2 (2005), 401–418. Available from: `https://doi.org/10.4171/ZAA/1248`.

[11] Navascués, M. A., Jha, S., Chand, A. K. B., and Sebastián, M. V. Fractal approximants on the circle. *Chaotic Modeling and Simulation 3* (2018), 343–353.

[12] Navascués, M. A., Jha, S., Chand, A. K. B., and Sebastián, M. V. Fractal approximation of Jackson type for periodic phenomena. *Fractals 26*, 5 (2018), 1850079. Available from: `https://doi.org/10.1142/S0218348X18500792`.

[13] Navascués, M. A., Jha, S., Chand, A. K. B., and Sebastián, M. V. Generalized trigonometric interpolation. *J. Comput. Appl. Math. 354* (2019), 152–162. Available from: https://doi.org/10.1016/j.cam.2018.08.003.

[14] Navascués, M. A., and Sebastián, M. V. Fitting functions of Jackson type for three-dimensional data. *Int. J. Comput. Math.* (2018), 1–18. Available from: https://doi.org/10.1080/00207160.2018.1458099.

M. A. Navascués
Departamento de Matemática Applicada
Universidad de Zaragoza
500018 Zaragoza, Spain
manavas@unizar.es

S. Jha and A. K. B. Chand
Department of Mathematics
Indian Institute of Technology Madras
600036 Chennai, India
sangitajha285@gmail.com and chand@iitm.ac.in

M. V. Sebastián
Centro Universitario de la Defensa de Zaragoza
Academia General Militar
50090 Zaragoza, Spain
msebasti@unizar.es