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PERIODIC SOLUTIONS IN THE HÉNON-HEILES ROTATING SYSTEM

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Abstract. We consider a generalized Hénon-Heiles system in a rotating frame. Our aim is to prove the existence of periodic orbits in a neighborhood of the origin for appropriate values of the rotating frequency. To this end, we use classical averaging theory to demonstrate that the number of periodic orbits is in correspondence with the equilibrium solutions of the original system, with the same type of stability.

Keywords: Generalized Hénon-Heiles system, periodic orbits, averaging.

AMS classification: 70H08, 70H09, 70H12, 70H15, 34C25, 37C27.

§1. Introduction

Equilibrium points and periodic orbits of dynamical systems are of special interest to understand its dynamics. They organize the phase structure and, some times, the appearance of heteroclitic connections allows migration of orbits giving rise to a kind of transport phenomena. For instance, this is what happens in Celestial Mechanics in the framework of the three body problem [6, 10], but also in the context of galactic dynamics, where the existence of heteroclitic connections are proposed as a way to explain the formation of spiral arms [12]. The model considered in [12] is based on a logarithmic potential. However, many galactic models consider cubic or quartic polynomial potentials [3]. This is the case of the well known Hénon-Heiles system, used to describe stellar orbits under the action of the galaxy's core [7]. Although this model has been considered as a paradigmatic system to study chaos and other properties of planar dynamical systems in many different fields, it does not take into account the effect of a rotating framework. In this way, de Zeeuw & Merritt [5] consider the cubic potential of the Hénon-Heiles system for a rotating galaxy and other authors consider a similar model in the context of atomic physics [2, 9]. The presence of the rotating frequency makes the system more interesting, from a dynamics point of view, with the appearance of Lagrangian type equilibrium points. In [8], a detailed analysis of the stability of these points is performed. One of the remarkable facts of this system is the existence of a critical value of the rotating frequency in such a way that the nature of the critical points, as critical points of the effective potential, reverses. This is an interesting situation that deserves more insight. In particular, the existence of periodic orbits is the next step in understanding the dynamic of the system. To prove the existence of periodic orbits, we will use the classical averaging theory [13] used successfully to find periodic orbits in many different dynamical systems [1, 4, 11].

§2. The system

Let us consider the Hamiltonian system defined by

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2) - \omega(xY - yX) + \frac{1}{2}(x^2 + y^2) + ayx^2 + by^3, \tag{1}$$

which can be viewed as a generalized Hénon-Heiles system in a rotating reference frame with angular velocity ω, where we assume, without loss of generality, $a > 0$ and $\omega > 0$. The equations of the motion are given by

$$\begin{aligned} \dot{x} &= \frac{\partial \mathcal{H}}{\partial X} = X + \omega y, & \dot{X} &= -\frac{\partial \mathcal{H}}{\partial x} = -x + \omega Y - 2axy, \\ \dot{y} &= \frac{\partial \mathcal{H}}{\partial Y} = Y - \omega x, & \dot{Y} &= -\frac{\partial \mathcal{H}}{\partial y} = -y - \omega X - ax^2 - 3by^2. \end{aligned} \tag{2}$$

It is clear that the origin is always an equilibrium point. Moreover, three more equilibrium points can appear, depending on the values of the parameters a and b. An interesting fact is that if

$$E_0 \equiv (x_0, y_0, X_0, Y_0)$$

is an equilibrium point for $\omega = \omega_0$, then

$$\hat{E}_0 \equiv (-x_0/\omega_0^2, -y_0/\omega_0^2, -X_0/\omega_0^4, -Y_0/\omega_0^4)$$

is also a critical point for $\omega = 1/\omega_0$. In this way, there is a correspondence between the cases $0 < \omega < 1$ and $\omega > 1$. However, there is a slight difference. Indeed, equilibrium points are related to the critical points of the effective potential

$$\Phi_{\text{eff}} = \mathcal{H} - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}(x^2(2ay - \omega^2 + 1) + y^2(2by - \omega^2 + 1)), \tag{3}$$

in such a way that if E_0 is an equilibrium point of the system (1), then (x_0, y_0) is a critical point of the effective potential Φ_{eff}. In this way, if E_0 is a minimum (maximum) of the effective potential, then $\hat{E}_0$ is a maximum (minimum) of Φ_{eff}. In the case E_0 is a saddle point, the same happens for $\hat{E}_0$. As a consequence, linear stability properties cannot be extended directly from the case $0 < \omega < 1$ to the case $\omega > 1$ if the corresponding critical point is a minimum (maximum). While a minimum of Φ_{eff} is always a linear stable equilibrium, the same cannot be said for a maximum. Nevertheless, if the critical point is the origin, then it is always a linear stable equilibrium, it does not matter a minimum or a maximum. Indeed, the associated eigenvalues are

$$\lambda_{1,2} = \pm i(\omega - 1), \quad \lambda_{3,4} = \pm i(\omega + 1). \tag{4}$$

For a detailed study of equilibrium points and their stability properties the reader is referred to [8].

It is worth noting that in the transition case, $\omega = 1$, the origin loses its elliptic character, as two zero eigenvalues appear, precisely those coming from $\pm i(\omega - 1)$. Moreover, the origin is the unique equilibrium point of the system and a bifurcation occurs when all the equilibria come into coincidence. Thus, what happens in the vicinity of the origin as $\omega \to 1$ deserves some analysis. In particular, we focus on the existence of periodic orbits and their bifurcations, assuming that $ab \neq 0$, in order to avoid degenerate situations, when non isolated equilibria appear. To begin with, we observe that, being the origin an elliptic point with associated eigenvalues given by (4), the Hamiltonian function can be transformed into an equivalent one made of two coupled harmonic oscillators with frequencies $1-\omega$ and $1+\omega$. To this end, we transform the system by means of the canonical change of variables

$$\begin{aligned} x &= -\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}, \qquad & X &= -\frac{X_1}{\sqrt{2}} + \frac{X_2}{\sqrt{2}}, \\ y &= \frac{X_1}{\sqrt{2}} + \frac{X_2}{\sqrt{2}}, \qquad & Y &= -\frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}. \end{aligned} \tag{5}$$

The new Hamiltonian is given by

$$\mathcal{H}_2 = \frac{1}{2}(1-\omega)(x_1^2+X_1^2) + \frac{1}{2}(1+\omega)(x_2^2+X_2^2) + \frac{X_1+X_2}{2\sqrt{2}}(a(x_1-x_2)^2 + b(X_1+X_2)^2). \tag{6}$$

§3. Averaging and periodic orbits

Taking into account that $\omega \approx 1$, one of them oscillates with high frequency with respect to the other one and the theory of averaging is suitable to study the system. In particular, the following Theorem [13] can be applied

Theorem 1. *Let us consider the differential system*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), \tag{7}$$

with $x \in D \subseteq \mathbb{R}^n$, $t \geq 0$. Moreover f, g, $\partial f/\partial x$, $\partial^2 f/\partial x^2$, $\partial g/\partial x$ are defined, continuous and bounded by a constant M independent of ε in $[0,\infty) \times D$, $0 \leq \varepsilon \leq \varepsilon_0$. In addition f and g are T-periodic in t (T independent of ε). Then, if p is a non degenerate critical point of the system

$$\dot{y} = \varepsilon f^0(y),$$

where

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)\,dt,$$

there exists a T-periodic solution $\phi(t,\varepsilon)$ of (7) which is close to p such that

$$\lim_{\varepsilon \to 0} \phi(t,\varepsilon) = p.$$

The key point is to transform the Hamiltonian differential system defined by (6) into a system in the form (7). This can be done in several steps. First of all, taking into account that we are considering $\omega \approx 1$, we scale the variables and the frequency according to

$$x_j, X_j \to \varepsilon x_j, \varepsilon X_j, \quad j = 1,2, \qquad 1-\omega \to \varepsilon\nu.$$

Substituting into the Hamiltonian function, and taking out the common factor ε^2, we arrive to

$$\mathcal{H}_2 = x_2^2 + X_2^2 + \frac{\varepsilon}{2\sqrt{2}}\left[\sqrt{2}\nu(x_1^2 + X_1^2 - x_2^2 - X_2^2) + b(X_1 + X_2)^3 + a(X_1 + X_2)(x_1 - x_2)^2\right].$$

The equations of the motion are given by

$$\dot{x}_i = \frac{\partial \mathcal{H}_2}{\partial X_j}, \quad \dot{X}_j = -\frac{\partial \mathcal{H}_2}{\partial x_j}, \quad j = 1, 2.$$

Now, we introduce polar coordinates for the pair of variables (x_2, X_2) in the form

$$x_2 = r\cos\theta, \quad X_2 = r\sin\theta.$$

Thus, the differential equations for r and θ turn to be

$$\dot{r} = \left(\frac{\partial \mathcal{H}_2}{\partial X}\cos\theta - \frac{\partial \mathcal{H}_2}{\partial x}\sin\theta\right), \quad \dot{\theta} = -\frac{1}{r}\left(\frac{\partial \mathcal{H}_2}{\partial x}\cos\theta + \frac{\partial \mathcal{H}_2}{\partial X}\sin\theta\right).$$

Explicitly, these equations read as

$$\begin{cases} \dot{r} = \dfrac{\varepsilon}{4\sqrt{2}}\,(a(x_1 - r\cos\theta)(r + 2x_1\cos\theta - 3r\cos 2\theta - 4X_1\sin\theta) + \\ \qquad\qquad 6b\cos\theta(X_1 + r^{1/2}\sin\theta)^2\Big), \\ \dot{\theta} = -2 + \dfrac{\varepsilon}{2\sqrt{2}r}\Big(2\sqrt{2}\nu r + a(x_1 - r\cos\theta)(2X_1\cos\theta - (x_1 - 3r\cos\theta)\sin\theta) - \\ \qquad\qquad 3b(X_1 + r\sin\theta)^2\sin\theta\Big), \end{cases} \tag{8}$$

whereas for the variables x_1, X_1 we obtain

$$\begin{cases} \dot{x}_1 = \dfrac{\varepsilon}{4}\left(4\nu X_1 + 3\sqrt{2}b(X_1 + r^{1/2}\sin\theta)^2 + \sqrt{2}a(y - r^{1/2}\cos\theta)^2\right), \\ \dot{X}_1 = \dfrac{\varepsilon}{2\sqrt{2}}\left(\sqrt{2}\nu x_1 - a(x_1 - r^{1/2}\cos\theta)(X_1 + r^{1/2}\sin\theta)\right). \end{cases} \tag{9}$$

On the other hand, the Hamiltonian function is expressed as

$$\mathcal{H}_2 = r^2 + \frac{\varepsilon}{2\sqrt{2}}\left(\sqrt{2}\nu(x_1^2 + X_1^2 - r^2) + a(X_1 + r\sin\theta)(x_1 - r\cos\theta)^2 + b(X_1 + r\sin\theta)^2\right). \tag{10}$$

From this expression, the radial variable can be obtained as a power series in ε. Up to the first order we get

$$\begin{aligned} r \approx \sqrt{h} + \frac{\varepsilon}{4\sqrt{2h}}\Big(&\sqrt{2}\nu(h - x_1^2 - X_1^2) \\ &- a(X_1 + h^{1/2}\sin\theta)(x_1 - h^{1/2}\cos\theta)^2 - b(X_1 + h^{1/2}\sin\theta)^3\Big). \end{aligned} \tag{11}$$

Now, we introduce the variable θ as a new time in the differential equations for x_1 and X_1. After replacing r by (11), and expanding in power series of ε up to the second order, we get

$$\frac{dx_1}{d\theta} = -\frac{\varepsilon}{4\sqrt{2}}\left(2\sqrt{2}\nu X_1 + a(x_1 - h^{1/2}\cos\theta)^2 + 3b(X_1 + h^{1/2}\sin\theta)^2\right) + \varepsilon^2 F(x_1, X_1, \theta; h, \varepsilon),$$
$$\frac{dX_1}{d\theta} = \frac{\varepsilon}{2\sqrt{2}}\left(\sqrt{2}\nu x_1 + a(x_1 - h^{1/2}\cos\theta)(X_1 + h^{1/2}\sin\theta)\right) + \varepsilon^2 G(x_1, X_1, \theta; h, \varepsilon),$$

where F and G are 2π-periodic in θ and satisfy the conditions of Theorem 1 for $h > 0$. Thus, according to this Theorem, the non degenerate equilibrium points of the averaged system give rise to periodic orbits. The equations of the averaged system are given by

$$\frac{dx_1}{d\theta} = -\frac{\varepsilon}{8\sqrt{2}}\left(4\sqrt{2}\nu X_1 + a(h + 2x_1^2) + 3b(h + 2X_1^2)\right),$$
$$\frac{dX_1}{d\theta} = \frac{\varepsilon}{2\sqrt{2}}\left(\sqrt{2}\nu x_1 + a x_1 X_1\right).$$

By equating to zero these equations, we obtain the equilibrium points

$$E_{1,2} \equiv \left(\pm\sqrt{\frac{4\nu^2(2a-3b) - a^2h(1+3b)}{2a^3}}, -\frac{\sqrt{2}\nu}{a}\right),$$
$$E_{3,4} \equiv \left(0, \frac{-2\nu \pm \sqrt{4\nu^2 - 3bh(a+3b)}}{3\sqrt{2}b}\right).$$

Consequently, based on Theorem 1, we can establish the following result

Theorem 2. *For $\varepsilon \neq 0$ sufficiently small and at energy level $h > 0$ of the Hamiltonian $\mathcal{H}$ given in (1) and ω close to one, we find for its associated Hamiltonian system (2) periodic solutions bifurcating from the origin. The number of these periodic solutions depends on the parameters a, b, h and ν. Assuming $a > 0$*

1. *If $4\nu^2(2a-3b) - a^2h(1+3b) > 0$ and $4\nu^2 - 3bh(a+3b) > 0$, there are four periodic solutions.*
2. *If $(4\nu^2(2a-3b) - a^2h(1+3b))(4\nu^2 - 3bh(a+3b)) < 0$, there are two periodic solutions.*
3. *If $4\nu^2(2a-3b) - a^2h(1+3b) < 0$ and $4\nu^2 - 3bh(a+3b) < 0$, there are not periodic solutions.*

Even more, the linear stability of these orbits follows from the stability character of the equilibrium points, which is summarized in Figure 1. A remarkable fact is that, for h small enough, the number of periodic orbits, their bifurcations and stability match with the number of critical points, bifurcations and character of the critical points of the effective potential associated to the original Hamiltonian system given by (1).

The periodic orbits can be computed by inverting the process of averaging. Thus, starting with a value of h and ν and the coordinates of an equilibrium point, once fixed a and b, we recover r from (11) to obtain the original set of coordinates (x, y, X, Y), after using (5). As an example, we depict the four periodic orbits when $\omega = 0.9$, $h = 0.08$ and a and b are the

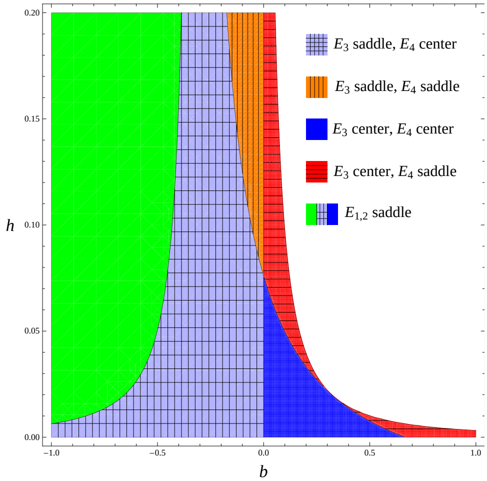

Figure 1: Stability character of the equilibrium points of the averaged system, when $a = 1$, in terms of b and h.

classical Hénon-Heiles parameters ($a = 1$, $b = -1/3$), that can be viewed in the left panel of Figure 2. There are four periodic orbits, a stable one centered at the origin and three unstable orbits that are at the same energy level. As a consequence, there is a heteroclitic connection between the three unstable orbits, allowing a mechanism of transport between different zones of the phase space (see the right panel of Figure 2).

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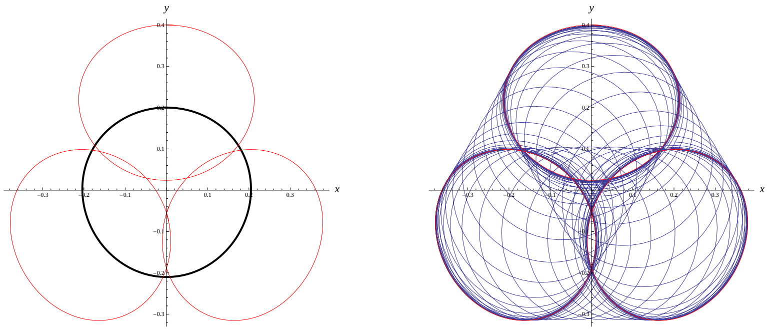

Figure 2: In the left panel, the four periodic orbits for the case $a = 1$, $b = -1/3$, $\omega = 0.9$ and $h = 0.08$. The red color indicates unstable orbits, whereas the black one stands for the stable orbit, centered at the origin. In the right panel, the heteroclitic connection between the three unstable orbits, at the same energy level, is depicted.

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