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STABILIZED VIRTUAL ELEMENT METHOD FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In this work, we present a discretization for the incompressible Navier-Stokes equations based on the stabilized virtual element method (VEM). Basically, VEM can be considered a generalization of FEM that enables a polynomial decomposition of the domain. In this work, the concepts of stabilized methods are introduced in the VEM formulation. Thus, stabilization terms are included in the variational form to circumvent the Babuška-Brezzi condition and to stabilize the solution for convection dominated flows. Numerical examples are presented to show the behavior of the method.

Keywords: Virtual element methods, Navier-Stokes problem, stabilized methods.

AMS classification: 76D05, 65M60.

§1. Introduction

The virtual element method (VEM) can be considered a generalization of the finite element method (FEM) that allows a greater versatility in the partition of the domain. The basis of VEM was established in [4, 5, 12]. Many works related to VEM have been published both in the field of elasticity [6, 13, 19] and fluid mechanics [23, 10, 8, 9].

In this work, we address the stabilized VEM formulation for incompressible Navier-Stokes equations. The VEM has already been applied to the Stokes problem [2, 8] and the Navier-Stokes equations [9]. It is well known that the space for the velocity and pressure cannot be selected arbitrarily since the Babuška-Brezzi condition or inf-sup condition must be satisfied. However, stabilization terms can be introduced in the discretization in order to circumvent the inf-sup condition. In this work, the concepts of stabilized methods [15, 21, 22, 18, 26, 20, 16] are introduced in the VEM formulation for Navier-Stokes equations. This formulation enables to select the velocity and pressure spaces with equal order interpolation functions. Thus, stabilization terms are included in the variational form to circumvent the Babuška-Brezzi condition and to stabilize the solution for convection dominated flows. We consider the transient incompressible Navier-Stokes using a semi-discrete scheme (see, for instance, [18, 17]).

§2. The incompressible Navier-Stokes equations

The problem is defined on a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$. The boundary is partitioned into two non-overlapping zones Γ_g and Γ_h such that $\Gamma_g \cup \Gamma_h = \Gamma$ and $\Gamma_g \cap \Gamma_h = \oslash$.

Let us set up the unsteady incompressible Navier-Stokes equations, given by

$$\left\{\begin{array}{rlll} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\nabla \boldsymbol{u})\boldsymbol{u} - 2\nu\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}) + \dfrac{1}{\rho}\nabla p & = \boldsymbol{f} & \text{in} & \Omega\times(0,T) \\ \nabla\cdot\boldsymbol{u} & = 0 & \text{in} & \Omega\times(0,T) \\ \boldsymbol{u} & = \mathbf{g} & \text{on} & \Gamma_g\times(0,T) \\ 2\nu\boldsymbol{\varepsilon}(\boldsymbol{u})\boldsymbol{n} & = \boldsymbol{h} & \text{on} & \Gamma_h\times(0,T) \\ \boldsymbol{u} & = \boldsymbol{u}_0 & \text{in} & \Omega \text{ at } t=0 \end{array}\right. \tag{1}$$

where $\boldsymbol{u}$ and p are the unknown velocity and pressure, respectively. ρ is the fluid density, ν represents the kinematic viscosity, $\boldsymbol{f}$ is the source term.

The tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the symmetric part of the velocity gradient and is defined as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ for } i,j = 1,...,n_{sd} \tag{2}$$

where n_{sd} is the number of spatial dimensions, i.e., $n_{sd} = 2$ for 2D and $n_{sd} = 3$ for 3D.

2.1. Variational formulation

Firstly, we define the spaces for the test and trial functions,

$$\begin{aligned} \mathcal{V} &= \{\boldsymbol{v}(\cdot,t)\in H^1(\Omega)^{n_{\rm sd}}, t\in[0,T] \;\mid\; \boldsymbol{v}(\cdot,t) = \mathbf{0} \text{ on } \Gamma_g\} \\ \mathcal{S} &= \{\boldsymbol{u}(\cdot,t)\in H^1(\Omega)^{n_{\rm sd}}, t\in[0,T] \;\mid\; \boldsymbol{u}(\cdot,t) = \mathbf{g} \text{ on } \Gamma_g\} \\ \mathcal{P} = \mathcal{Q} &= \{q(\cdot,t)\in L^2(\Omega)\cap H^1(\Omega), t\in[0,T] \text{ s.t. } \int_\Omega q(\cdot,t)\mathrm{d}\Omega = 0\} \end{aligned}$$

The variational formulation is defined as: Find $\boldsymbol{u}\in\mathcal{S}$ and $p\in\mathcal{P}$ such that

$$B(\boldsymbol{u},p;\boldsymbol{v},q) = F(\boldsymbol{v},q),\; (\boldsymbol{v},q)\in\mathcal{V}\times\mathcal{Q} \tag{4}$$

with

$$B(\boldsymbol{u},p;\boldsymbol{v},q) = d(\boldsymbol{u},\boldsymbol{v}) + a(\boldsymbol{u},\boldsymbol{v}) + c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + b_m(p,\boldsymbol{v}) + b_c(\boldsymbol{u},q) \tag{5}$$

where $d(\cdot,\cdot)$, $a(\cdot,\cdot)$, $b_m(\cdot,\cdot)$, $b_c(\cdot,\cdot)$ are bilinear forms and $c(\cdot;\cdot,\cdot)$ is the trilinear form that represents the convective term,

$$\begin{aligned} & d(\boldsymbol{u},\boldsymbol{v}) = \left(\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v}\right), \quad a(\boldsymbol{u},\boldsymbol{v}) = (\nu\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})), \quad c(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = ((\nabla\boldsymbol{u})\boldsymbol{w},\boldsymbol{v}) \\ & b_m(p,\boldsymbol{v}) = (\nabla p,\boldsymbol{v}), \quad b_c(\boldsymbol{u},q) = (\nabla\cdot\boldsymbol{u},q) \end{aligned} \tag{6}$$

and

$$F(\boldsymbol{v},q) = (f,\boldsymbol{v}) + (\boldsymbol{h},\boldsymbol{v})_{\Gamma_h} \tag{7}$$

§3. Virtual element method discretization

In this section, we show the VEM discretization of the variational form (4) using a first order approximation. The domain Ω is decomposed into a partition $\mathcal{T}_h$ composed of polygons K, and let $\mathcal{E}_h$ be the set of edges e of $\mathcal{T}_h$. Let $\widetilde{\Omega}$ denote the union of the polygons, $\widetilde{\Omega} = \bigcup_{e=1}^{n_{el}} K$ where n_{el} is the number of polygons. In this work, linear elements are employed. We define the following initial local space defined on each element:

$$\widetilde{V}_h(K) := \{v \in C^0(K) : \; v_{|e} \in \mathbb{P}_1(e) \; \forall \; e \subset \partial K, \Delta v \in \mathbb{P}_1(K)\},$$

where $\mathbb{P}_1(K)$ are the polynomials of degree 1 on the polygon K. In $\widetilde{V}_h(K)$, we can take the values of $v \in \widetilde{V}_h(K)$ at the vertices as degrees of freedom, *dof*. Then, the number of degrees of freedom in K is equal to the number of vertices N^V.

We define the following projectors in K:

- the H^1-seminorm projection $\Pi_1^{\nabla,K} : [\widetilde{V}_h(K)]^{n_{sd}} \; \rightarrow \; [\mathbb{P}_1(K)]^{n_{sd}}$,

$$\int_K \nabla(\Pi_1^\nabla \boldsymbol{v} - \boldsymbol{v}) : \nabla \boldsymbol{p}_1 \, dx = \mathbf{0} \;\text{ and }\; \int_{\partial K} (\Pi_1^\nabla \boldsymbol{v} - \boldsymbol{v}) \, \mathrm{d}s = \mathbf{0} \quad \forall p_1 \in \mathbb{P}_1, \tag{8}$$

- the L^2-projection for scalar functions $\Pi_k^{0,K}$: $\widetilde{V}_h \rightarrow \; \mathbb{P}_k(K)$ is defined locally as

$$\int_K (v - \Pi_k^0 v) \, p_k \, dx = 0 \quad \forall p_k \in \mathbb{P}_k \quad \text{for} \quad k = 0 \quad \text{and} \quad k = 1. \tag{9}$$

We can now introduce the local Virtual Element space:

$$V_h(K) := \{v \in \widetilde{V}_h(K) : \; \int_K v \, p_1 \, dx = \int_K \Pi_1^\nabla v \, p_1 \, dx \; \forall \; p_1 \in \mathbb{P}_1(K)\}. \tag{10}$$

The dimension of $V_h(K)$ is $N_{\text{dof}} = N^V$ as the same as the degrees of freedom which are unisolvent with respect to $V_h(K)$ [1].

The global virtual spaces defined for the unknown variables of the discrete problem are

$$V_h^u := \{\boldsymbol{v} \in [H^1(\Omega)]^{n_{sd}} \; : \; \boldsymbol{v}_{|K} \in [V_h(K)]^{n_{sd}} \; \forall K \in \mathcal{T}_h\} \tag{11}$$

$$Q_h := \{q \in H^1(\Omega) \;\text{ s. t. }\; \int_\Omega q \; \mathrm{d}\Omega = 0 : \; q_{|K} \in V_h(K) \; \forall K \in \mathcal{T}_h\}. \tag{12}$$

The basis functions on each element K, $\varphi_i \in \widetilde{V}_h(K)$, are defined, as happens in FEM, as the canonical basis functions, $\text{dof}_i(\varphi_j) = \delta_{ij}$ for $i, j = 1, ..., N_{\text{dof}}$. We recall that the basis functions for the velocity and pressure are the same. Thus, the unknown variables $(\boldsymbol{u}_h, p_h)$ are expressed as a linear combination of these basis functions,

$$\boldsymbol{u}_h = \sum_{i=1}^{N_{\text{dof}}} \text{dof}_i(\boldsymbol{u}_h)\varphi_i \qquad p_h = \sum_{i=1}^{N_{\text{dof}}} \text{dof}_i(p_h)\varphi_i. \tag{13}$$

The Galerkin formulation reads: Find $(\boldsymbol{u}_h, p_h) \in V_h^u \times Q_h$ such that

$$B(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = F(\boldsymbol{v}_h, q_h), \text{ for all } (\boldsymbol{v}_h, q_h) \in \mathcal{V}_h \times \mathcal{P}_h \tag{14}$$

with

$$B(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = \quad d(\boldsymbol{u}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h, \boldsymbol{v}_h) + c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b_m(p_h, \boldsymbol{v}) + b_c(\boldsymbol{u}_h, q_h) \tag{15}$$

where the bilinear forms $d(\cdot,\cdot)$, $a(\cdot,\cdot)$, $b_m(\cdot,\cdot)$, $b_c(\cdot,\cdot)$ and the trilinear form $c(\cdot;\cdot,\cdot)$ are

$$\begin{aligned}
d(\boldsymbol{u}_h, \boldsymbol{v}_h) = & \sum_K d^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_K (\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h)_K \\
a(\boldsymbol{u}_h, \boldsymbol{v}_h) = & \sum_K a^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_K (\nu \boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_K \\
c(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = & \sum_K c^K(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_K ((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \boldsymbol{v}_h)_K \\
b_m(p_h, \boldsymbol{v}_h) = & \sum_K b_m^K(p_h, \boldsymbol{v}_h) = \sum_K (\nabla p_h, \boldsymbol{v}_h)_K \\
b_{c,h}(\boldsymbol{u}_h, q_h) = & \sum_K b_c^K(\boldsymbol{u}_h, q_h) = \sum_K (\nabla \cdot \boldsymbol{u}, q_h)_K.
\end{aligned} \tag{16}$$

The discrete terms belonging to $B(\cdot,\cdot)$ are computable using the projector operators and the degrees of freedom. Thus, we define the approximate bilinear and trilinear forms:

$$\begin{aligned}
a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = & \int_K \nu \Pi_0^0 \nabla \boldsymbol{u}_h : \Pi_0^0 \nabla \boldsymbol{v}_h \mathrm{d}\Omega + \mathcal{S}_\nu^K((I - \Pi_1^\nabla)\boldsymbol{u}_h, (I - \Pi_1^\nabla)\boldsymbol{v}_h) \\
d_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = & \int_K \frac{\partial}{\partial t} \Pi_1^0 \boldsymbol{u}_h \cdot \Pi_1^0 \boldsymbol{v}_h \mathrm{d}\Omega + \mathcal{S}_t^K((I - \Pi_1^0)\boldsymbol{u}_h, (I - \Pi_1^0)\boldsymbol{v}_h) \\
c_h^K(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = & \int_K [(\Pi_0^0 \nabla \boldsymbol{u}_h)(\Pi_1^0 \boldsymbol{u}_h)] \cdot \Pi_1^0 \boldsymbol{v}_h \,\mathrm{d}\Omega \\
b_{m,h}^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = & \int_K \Pi_0^0 \nabla p_h \cdot \Pi_1^0 \boldsymbol{v}_h \,\mathrm{d}\Omega \\
b_{c,h}^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = & \int_K (\Pi_0^0 \nabla \cdot \boldsymbol{u}_h)(\Pi_1^0 q_h) \,\mathrm{d}\Omega
\end{aligned} \tag{17}$$

where the VEM-stabilization terms $\mathcal{S}_\alpha^K$ are necessary for stability [1] and will be explained later.

In this work, the stabilized VEM that is proposed uses a linear approximation ($k = 1$) both for the velocity and the pressure. Thus, the degrees of freedom of pressure and velocity are the values at the vertices. We have followed the work of Franca et al. [18] to stabilize the VEM formulation. As it is well known, in stabilized methods additional terms are included in the Galerkin formulation that consist in weighting the residual by a determined differential operator (related to the differential equation) applied to the test functions. Besides, a generalized trapezoidal method is employed for the temporal term in order to reach the steady-state solutions and deal with the nonlinearity of the equations.

The stabilized VEM formulation includes additional terms to circumvent the Babuška-Brezzi condition and to obtain a stable solution for convection dominated flows. This formulation can be written as:

Find $(\boldsymbol{u}_h, p_h) \in V_h^u \times Q_h$ such that

$$B(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + B^\tau(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = F(\boldsymbol{v}_h, q_h) + F^\tau(\boldsymbol{v}_h, q_h) \quad \text{for all } (\boldsymbol{v}_h, q_h) \in \mathcal{V}_h \times \mathcal{P}_h \tag{18}$$

where

$$\begin{aligned} B^\tau(\boldsymbol{u}_h, p_h; \boldsymbol{v}, q) &= \sum_{K\in\widetilde{\Omega}} \Big((\frac{\partial \boldsymbol{u}_h}{\partial t} + (\nabla \boldsymbol{u}_h)\boldsymbol{u}_h + \nabla p_h - 2\nu\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \\ &\quad \tau((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h + \nabla q_h \pm 2\nu\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h))) + (\nabla\cdot\boldsymbol{u}_h, \delta\nabla\cdot\boldsymbol{v}_h)\Big)_K \end{aligned} \tag{19}$$

$$F^\tau(\boldsymbol{v}_h, q) = \sum_{K\in\widetilde{\Omega}} \Big(\boldsymbol{f}, \tau((\nabla \boldsymbol{v}_h)\boldsymbol{u}_h + \nabla q_h \pm 2\nu\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h))\Big)_K \tag{20}$$

where τ and δ are the stability parameters. They are taken from the work of Codina [16],

$$\tau = \left(\frac{c_1\nu}{h^2} + \frac{c_2\|\boldsymbol{u}_h\|_{L^\infty(K)}}{h}\right)^{-1} \qquad \delta = \frac{c_3 h^2}{\tau}. \tag{21}$$

The constants c_1, c_2 and c_3 are taken as $c_1 = 4$, $c_2 = 2$ and $c_3 = 1$. Other possibilities for non-regular elements can be found in [3]. The value of h (length of the element) is taken as $h = \sqrt{|K|}$, where $|K|$ is the area of the element.

We observe that the operators $B^\tau(\cdot,\cdot)$ and $F^\tau(\cdot)$ correspond to the stabilization terms. Since we only consider $k = 1$, the terms containing $\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$ disappear because $\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h) = 0$. The stabilized terms for the momentum and continuity equations are defined as follows.

· *Stabilized terms for the momentum equations*

$$\begin{aligned}
\tau\left(\frac{\partial \boldsymbol{u}_h}{\partial t}, (\nabla \boldsymbol{v}_h)\boldsymbol{u}_h\right) &= \tau\int_K [(\Pi_0^0\nabla \boldsymbol{v}_h)(\Pi_1^0\boldsymbol{u}_h)]\cdot\Pi_1^0\frac{\partial \boldsymbol{u}_h}{\partial t}\,\mathrm{d}\Omega \\
\tau((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, (\nabla \boldsymbol{v}_h)\boldsymbol{u}_h) &= \tau\int_K [(\Pi_0^0\nabla \boldsymbol{v}_h)(\Pi_1^0\boldsymbol{u}_h)]\cdot[(\Pi_0^0\nabla \boldsymbol{v}_h)(\Pi_1^0\boldsymbol{v}_h)]\,\mathrm{d}\Omega \\
\tau(\nabla p_h, (\nabla \boldsymbol{v}_h)\boldsymbol{u}_h) &= \tau\int_K [(\Pi_0^0\nabla \boldsymbol{v}_h)(\Pi_1^0\boldsymbol{u}_h)]\cdot(\Pi_0^0\nabla q_h)\,\mathrm{d}\Omega \\
\tau(\frac{\partial \boldsymbol{u}_h}{\partial t}, \nabla q_h) &= \tau\int_K (\Pi_0^0\nabla q_h)\cdot\Pi_1^0\frac{\partial \boldsymbol{u}_h}{\partial t}\,\mathrm{d}\Omega \\
\tau((\nabla \boldsymbol{u}_h)\boldsymbol{u}_h, \nabla q_h) &= \tau\int_K (\Pi_0^0\nabla q_h)\cdot[(\Pi_0^0\nabla \boldsymbol{v}_h)(\Pi_1^0\boldsymbol{v}_h)]\,\mathrm{d}\Omega \\
\tau(\nabla p_h, \nabla q_h) &= \tau\int_K (\Pi_0^0\nabla q_h)\cdot(\Pi_0^0\nabla p_h) + \mathcal{S}_p^K((I-\Pi_1^\nabla)p_h, (I-\Pi_1^\nabla)q_h)\,\mathrm{d}\Omega
\end{aligned} \tag{22}$$

· *Stabilized terms for the continuity equation*

$$\delta(\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}_h) = \delta\int_K (\Pi_0^0\nabla\cdot\boldsymbol{v}_h)(\Pi_0^0\nabla\cdot\boldsymbol{u}_h)\,\mathrm{d}\Omega \tag{23}$$

We observe that in the presented VEM formulation, there appear some terms called $\mathcal{S}^K_\alpha(\cdot,\cdot)$ which are the VEM-stabilization part, for $\alpha = \nu, t, p$. These terms are a peculiarity of VEM and they emerge from the projection of the basis functions. In order to compute the above mentioned matrices, we decompose the basis functions as $\varphi = \Pi\varphi + (I - \Pi)\varphi$. Therefore, we project the basis functions, $\Pi\varphi$, from the virtual space to a determined polynomial space. Thus, these terms that involve the projection of the variables, both Π^∇_k and Π^0_k, can be computed exactly via numerical integration and they ensure consistency. However, a peculiarity of VEM is that the kernel of these projections, $(I - \Pi)\varphi$, must be considered for some terms to ensure the VEM stability [7, 4]. The terms $\mathcal{S}^K_\alpha(;\cdot)$ take into account the terms $(I - \Pi)\varphi$ which are not considered by the consistency part. The only condition is that $\mathcal{S}^K_\alpha(;\cdot)$ scales as the consistency part. In this case, it has been observed numerically that three stability terms must be considered.

The term $\mathcal{S}^K_\alpha(;\cdot)$ can be selected in different ways. A rigorous work on the stability term can be found in [7, 11]. In [27, 19] are proposed different definitions for the VEM stabilization term $\mathcal{S}^K$. The authors exploited the flexibility of selecting this term in order to improve the characteristics of the method. Here, we define them as follows:

- The diffusion term,

$$\begin{aligned}\mathcal{S}^K_\nu((I - \Pi^\nabla_1)\boldsymbol{u}_h, (I - \Pi^\nabla_1)\boldsymbol{v}_h) \quad &\approx \nu[(I - \Pi^\nabla_1)\overrightarrow{u}_h]^T[(I - \Pi^\nabla_1)\overrightarrow{v}^{M_x}_h] \\ &+\nu[(I - \Pi^\nabla_1)\overrightarrow{v}_h]^T[(I - \Pi^\nabla_1)\overrightarrow{v}^{M_y}_h]\end{aligned} \tag{24}$$

- The temporal term,

$$\begin{aligned}\mathcal{S}^K_t((I - \Pi^0_1)\boldsymbol{u}_h, (I - \Pi^0_1)\boldsymbol{v}_h) \quad &\approx h^2_K[(I - \Pi^0_1)\overrightarrow{u}_h]^T[(I - \Pi^0_1)\overrightarrow{v}^{M_x}_h] \\ &+h^2_K[(I - \Pi^0_1)\overrightarrow{v}_h]^T[(I - \Pi^0_1)\overrightarrow{v}^{M_y}_h]\end{aligned} \tag{25}$$

- The stability term, $\tau(\nabla q_h, \nabla p_h)$

$$\mathcal{S}^K_p((I - \Pi^\nabla_1)p_h, (I - \Pi^\nabla_1)q_h) \approx [(I - \Pi^\nabla_1)\overrightarrow{p}_h]^T[(I - \Pi^\nabla_1)\overrightarrow{q}_h] \tag{26}$$

with $\overrightarrow{u}_h$, $\overrightarrow{u}_h$ and $\overrightarrow{p}_h$ being the vector containing the degrees of freedom of u_h, v_h and p_h in the element K, respectively. That is to say,

$$u_h|_K = \sum_{i=1}^{N_{\mathrm{dof,K}}} [\overrightarrow{u}_h]_i\varphi_i, \quad v_h|_K = \sum_{i=1}^{N_{\mathrm{dof,K}}} [\overrightarrow{v}_h]_i\varphi_i \quad \text{and} \quad p_h|_K = \sum_{i=1}^{N_{\mathrm{dof,K}}} [\overrightarrow{p}_h]_i\varphi_i, \tag{27}$$

where $N_{\mathrm{dof,K}}$ are the degrees of freedom in K. Similarly, we have that $\overrightarrow{v}^{M_x}_h$, $\overrightarrow{v}^{M_y}_h$ and $\overrightarrow{q}_h$ are the degrees of freedom for the test function in the x-momentum equation, y-momentum equation and continuity equation.

Whereas the VEM-stabilization term in the diffusion $\mathcal{S}^K_\nu$ is the classical choice in VEM, see for instance [4], we have considered two more VEM stabilizing terms. The temporal term stabilization is only necessary to be considered when the temporal term is dominant. However it has been observed that it improves considerably the condition number of the matrix. On the other hand, the term $\mathcal{S}^K_p$ is very important in order to obtain a proper solution since it helps to penalize those non-physical oscillations of the pressure.

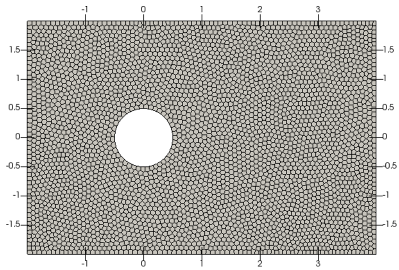


Figure 1: Domain dimensions and mesh.

As for the source term, it is approximated by

$$(\boldsymbol{f}_h, \boldsymbol{v}_h) = \sum_{K \in \mathcal{T}_h} (\boldsymbol{f}_h, \boldsymbol{v}_h)_K = \sum_{K \in \mathcal{T}_h} \int_K \Pi_1^0 \boldsymbol{f} \cdot \boldsymbol{v}_h \, \mathrm{d}\Omega = \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{f} \cdot \Pi_1^0 \boldsymbol{v}_h \, \mathrm{d}\Omega \tag{28}$$

where Eq. (28) expresses the RHS and it is computable using the degrees of freedom.

In the discrete problem, there are terms with derivatives of the velocities with respect to time that represent the evolution of the velocity field.

We consider the generalized trapezoidal rule given by the following predictor multi-corrector algorithm [18]. We name $\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t}$ the acceleration. For the purpose of integrating in time, we write Eq. (18) separating the terms that include the acceleration $\boldsymbol{a}$ and the others,

$$\boldsymbol{M}(\boldsymbol{a}_h) + \boldsymbol{K}(\boldsymbol{u}_h, p_h) = \boldsymbol{F} + \boldsymbol{F}^{\tau} \tag{29}$$

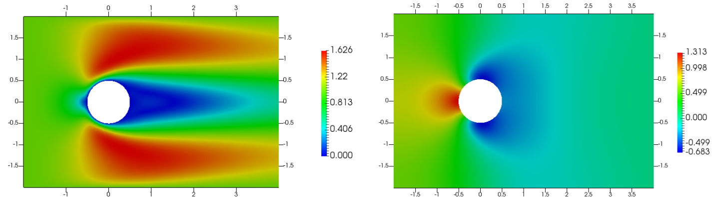
where, for the sake of simplicity, we use now $\boldsymbol{a}_h$, $\boldsymbol{u}_h$ and p_h to denote the vectors including the global degrees of freedom of the acceleration, velocity and pressure, respectively. In [24] the time integration algorithm is explained in more detail.

§4. Numerical examples: Flow around a circular cylinder

This problem has been studied extensively in the literature, see for instance [14] and its references. The flow around the circular cylinder depends on the Reynolds number which is defined as $Re = \frac{U \cdot D}{\nu}$, where U is the incoming flow velocity, D is the diameter, and ν is the kinematic viscosity. The domain is depicted in Fig. 1 and the mesh consists of hexagonal elements generated by *PolyMesher*, [25]. We have employed 8000 elements. We impose the velocity $(u, v) = (1, 0)$ on the outer boundary except on the right boundary where natural outflow boundary conditions are set. The no-slip boundary condition is applied on the cylinder surface. Fig. 2 represents the velocity and pressure magnitudes for $Re = 25$.

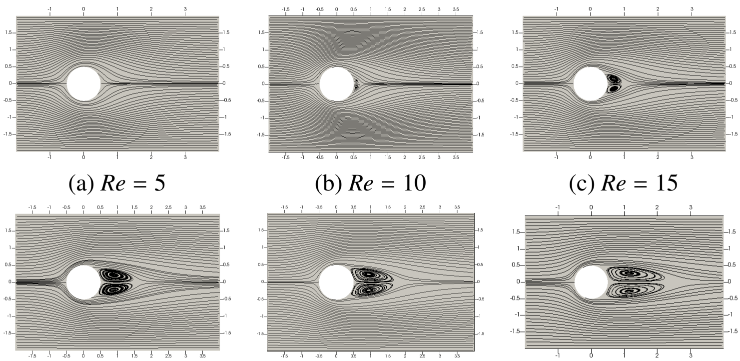
We have simulated this problem for Reynolds number up to 45 in which the steady flow becomes unstable. It is well-known that for Re that range from 6 to 45 approx. the flow is symmetric with two vortices behind the cylinder, see Fig. 3. In contrast, for higher Re, a Hopf bifurcation arises producing unstable flow.

As we can observed the numerical solution is stable and similar to the expected one for this problem. Also, in comparison with the use of FEM and stabilized methods, the solution



(a) Velocity field (b) Pressure

Figure 2: Velocity field and pressure. $Re = 25$.



(a) $Re = 5$ (b) $Re = 10$ (c) $Re = 15$

(d) $Re = 25$ (e) $Re = 30$ (f) $Re = 45$

Figure 3: Streamlines for several Reynolds numbers.

is close to the VEM solution we have presented [18, 24]. In [24], there are more numerical examples related to this work.

§5. Conclusions

In this work, the Navier-Stokes equations are discretized using VEM. The numerical method is based on the theory of stabilized methods. Thus, this method enables to select the velocity and pressure spaces with equal order interpolation functions circumventing the Babuška-Brezzi condition. Also, it can be applied to convection dominated flows since stabilization terms are considered. Numerical examples show the good performance of the method.

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